Instituto de Física Teórica
Universidade Estadual Paulista

TESE DE DOUTORAMENTO IFT–T.002/2012

DE SITTER RELATIVITY: FOUNDATIONS AND SOME PHYSICAL IMPLICATIONS

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Fevereiro de 2012

Agradecimentos

En primer lugar agradezco a mi mamá y a mi papá, que desde el comienzo de mis estudios siempre creyeron en mi, y no existió un solo momento en el que no sintiera sus presencias en mi corazón.
À Samira, que esteve comigo sempre nos bons e maus momentos, que cada dia faz minha vida mais feliz. Eu não consigo imaginar como teriam sido estes anos sem ela do meu lado.
Ao professor José Geraldo, pela sua orientação, assim como a todos os professores do IFT, com os quais aprendi muito.
I'm very grateful to Prof. Ugo Moschella e Prof. Sergio Cacciatori for helping me during my visit to Como - Italy. I had very enlightening discussions with them.
A Juan Pablo por la ayuda en el desenvolvimiento de este trabajo, así como a su gran amistad. Es difícil encontrar personas con tanta rectitud, ademas de inteligentes y a las vez divertidas.
A mis mejores amigos Humberto y Carlos, con los cuales me conozco hace ya muchos años y llevamos juntos esta aventura de estudiar física, todo es mas fácil de superar teniendo amigos de gran calidad con los que siempre puedo contar.
A Alexis, Cristhiam, David y Oscar, con los que vivir en la "casa" fue una muy buena experiencia.
Per Kazz, Chito, Jovana e Giada, che hanno fatto il mio soggiorno all'Italia un ricordo molto bello.
À FAPESP pelo apoio financeiro.

Resumo

Na presença de uma constante cosmológica, interpretada como uma entidade puramente geométrica, a ausência de matéria é representada pelo espaço de de Sitter. Como consequência, a relatividade especial de Poincaré não é mais válida e deve ser substituída por uma relatividade especial baseada no grupo de de Sitter, o que produz modificações em todas as áreas da Física. Nesta tese, vamos estudar os fundamentos da relatividade especial de de Sitter, bem como algumas implicações desta mudança. Em particular, estudaremos as implicações para o problema da energia escura, para a propagação de raios gama de altíssimas energias e para o espalhamento Compton. Como um subproduto desses estudos, desenvolvemos um novo método de se obter equações de campo com invariância conforme, o qual faz uso dos invariantes de Casimir do grupo de de Sitter. Usando esse novo método, fazemos um estudo crítico das equações que descrvem um campo fundamental de spin-2.

Palavras Chaves: Constante cosmológica; Relatividade de de Sitter, Equações de campo com invariância conforme

Áreas do conhecimento: Gravitação e cosmologia; Teoria de campos e partículas elementares.

Abstract

In the presence of a cosmological constant, interpreted as a purely geometric entity, absence of matter is represented by a de Sitter spacetime. As a consequence, ordinary Poincaré special relativity is no longer valid and must be replaced by a special relativity based on the de Sitter group, which produces concomitant changes in all areas of Physics. In this thesis, we are going to explore the implications for the dark energy problem, for the propagation of ultra high-energy gamma rays, and for the Compton scattering formula. As a byproduct of these studies, we developed a new method for obtaining conformal invariant field equations, which makes use of the Casimir operators of the de Sitter group. Using this new method, we make a critical review of the field equations describing a fundamental spin-2 field.

Contents

Notation

The notation introduce here will be used along all the text, unless it is otherwise specified.

Capital latin indices $A, B, C, M, N, O, ...$ run over five space-time coordinates $(0, ..., 4)$.

Latin indices $a, b, c, i, j, k, ...$ run over three spatial coordinates $(1, 2, 3)$.

Greek indices $\alpha, \beta, \gamma, \mu, \nu, \rho, ...$ run over four space-time coordinates $(0, ..., 3)$.

The metric signature in four dimension is $(+1, -1, -1, -1)$.

Repeated indices are summed.

The speed of light is taken to be unity.

Chapter 1

Introduction

Our understanding of elementary particle physics is intimately related to both group representations and special relativity. In fact, all particles of Nature can be classified according to the irreducible representations of the Poincaré group $\mathcal{P}$, the kinematic group of special relativity. This property suggests that the symmetry of special relativity must be considered as an exact symmetry of Nature, a fact corroborated by many experiments. In principle, therefore, there is no reason to replace Poincaré as the kinematic group of spacetime. However, there are theoretical and experimental evidences that, at ultra-high energies, Poincaré relativity fails to be true.

The theoretical indications are related to the physics at the Planck scale, where a fundamental length parameter (the Planck length l_P) naturally shows up. This scale, as is well known, shows up as the threshold of a new physics, represented by quantum gravity. Now, whether gravity is classical or quantum cannot depend on the observer. This means that the Planck length, or some fundamental scale related to it, must remain invariant under the relevant kinematics ruling the high-energy physics near the Planck scale.[1] Since in ordinary special relativity a length will contract under a Lorentz boost, the invariance requirement of such length scale seems to indicate that either the Lorentz symmetry must be broken down, or the action of the Lorentz group $\mathcal{L}$, and in particular the boosts, must somehow be modified [1, 2, 3, 4, 5].

The experimental evidences come basically from the propagation of very-high energy photons, which seems to violate ordinary special relativity. More precisely, very-high energy extragalactic gamma-ray flares seem to travel slower than lower energy ones [6]. If this comes to be confirmed, it will constitute a clear violation of special relativity. The superluminal neutrinos found recently by the OPERA collaboration may becomes another evidence of breakdown of special relativity [7]. The above indications suggest that one should look for another special relativity, which would rule the kinematics at

[1] It is important to observe that this argument is true independently of whether spacetime at the Planck scale is curved or not. In fact, even if spacetime is curved, there is always a subjacent special relativity that holds locally. If the ensuing local kinematics does not allow the existence of an invariant length, it will necessarily conflict with the existence of the invariant Planck length.

ultra-high energies. That is to say, near the Planck scale, the Poincaré group must be replaced by a more general group, which will preside over the high-energy kinematics. In fact, several alterative special relativities have been proposed recently, as for example those based on a κ-deformed Poincaré group (see Ref. [8, 9] and references therein).

Then comes the point: *the de Sitter group has Lorentz as subgroup and at the same time it leaves invariant the length parameter related to the non-gravitational curvature of the de Sitter spacetime.* This means essentially that the de Sitter group preserves the relation between inertial frames of ordinary special relativity, and is able to deal with the existence of an invariant length-scale, provided this length-scale is assumed to define the (non-gravitational) curvature of spacetime at the Planck scale.[2] When one does that, a new special relativity emerges, which is called "de Sitter special relativity". Its kinematic group is the de Sitter group, and it takes place in de Sitter spacetime.

Like Minkowski, the de Sitter spacetime is a maximally symmetric homogeneous space. Whereas Minkowski is defined as the quotient between Poincaré and the Lorentz groups,

$$M = \mathcal{P}/\mathcal{L},$$

the de Sitter spacetime is defined as the quotient space between the de Sitter and the Lorentz groups, [10]:

$$dS(4,1) = SO(4,1)/\mathcal{L}.$$

Differently from Minkowski, which is transitive under spacetime translations, de Sitter is transitive under a combination of translations and proper conformal transformation. Since the invariance under translations are related, through Noether's theorem, to the conservation of energy and momentum, the very notions of conserved energy and momentum will be changed. As ordinary spacetime translations are no longer a symmetry of spacetime, the old notions of energy and momentum will not be conserved.

To get some insight on how a de Sitter special relativity might be thought of, let us briefly recall the relationship between the de Sitter and the Galilei groups, which comes from the Wigner–Inönü processes of group contraction and expansion [11, 12]. Ordinary Poincaré special relativity can be viewed as describing the implications to Galilei's relativity of introducing a fundamental velocity-scale into the Galilei group. Conversely, the latter can be obtained from the special-relativistic Poincaré group by taking the formal limit of the velocity scale going to infinity (non-relativistic limit). We can, in an analogous way, say that de Sitter relativity describes the implications to Galilei's relativity of introducing both a velocity and a length scales in the Galilei group. In the formal limit of the length-scale going to infinity, the de Sitter groups contract to the Poincaré group, in which only the velocity scale is present. It is interesting to

[2]Observe that, although the de Sitter spacetime is curved, its metric tensor does not depend on the gravitational constant, which means that its curvature does not have a gravitational origin.

observe that the order of the group expansions (or contractions) is not important. If we introduce in the Galilei group a fundamental length parameter, we end up with the Newton-Hooke group [13, 14, 15], which describes a (non-relativistic) relativity in the presence of a cosmological constant [16]. Adding to this group a fundamental velocity scale, we end up again with the de Sitter group, whose underlying relativity involves both a velocity and a length scales. Conversely, the low-velocity limit of the de Sitter group yields the Newton-Hooke group, which contracts to the Galilei group in the limit of a vanishing cosmological constant.

Considering that the de Sitter group naturally incorporates an invariant length-parameter, in addition to the speed of light, de Sitter relativity can be interpreted as a new example of the so called *doubly special relativity* [1, 2, 8, 9, 17]. It is important to mention that one drawback of the usual doubly special relativity models is that they are valid only at the energy scales where ordinary special relativity is supposed to break down, giving rise to a kind of patchwork relativity. This restriction is known as the "soccer-ball problem" [18]. On the other hand, de Sitter relativity is invariant under a simultaneous re-scaling of mass, energy and momentum [19], and is consequently valid at all energy scales, from cosmology to quantum gravity. This is a very important property shared by all fundamental theories.

As already mentioned, the de Sitter special relativity changes the notions of energy and momentum [20]. As a consequence, all areas of physics will change accordingly. Of course, these changes are significant only for very high energies. In this thesis, we are going to explore the basic foundations of a special relativity based on the de Sitter group, as well as to look for possible signs of violation of ordinary special relativity in order to verify whether the de Sitter special relativity could provide a reasonable explanation for these violations.

We are going to proceed according to the following scheme. In the second chapter, we study the basic elements of the de Sitter space and group. In chapter 3, we introduce the fundamentals of de Sitter special relativity, and discuss its main properties. In chapter 4, by projecting Einstein equation along the Killing vectors of the de Sitter spacetime, we are able to find a relation between the local value of the cosmological term and the energy density of a given physical system. Chapter 5 is dedicated to some applications of the de Sitter relativity. First, we consider a possible solution for the dark energy problem; then we study possible implications for the propagation of high energy gamma rays; and finally, we explore implications for the Compton effect. In chapter 6, review the properties of all spin-2 Lorentz representations, clearly distinguishing the representations with helicity $\sigma = 0$, $\sigma = \pm 1$ and $\sigma = \pm 2$. In chapter 7, by using the machinery developed in the study of the de Sitter special relativity, we introduce a new method to obtain conformal invariant field equations. This new method is then used to critically review the consistency problems of the field equations for a fundamental spin-2 field in the presence of gravitation. Finally, in chapter 8, we present the conclusions and draw some perspectives for future developments.

Chapter 2

The de Sitter Spacetime and Group

In this chapter, we are going to study the basic properties of the geometry and algebra of the de Sitter space and group. First we will introduce the most used coordinates systems of de Sitter space and explain some of its characteristics. Then, we will explore the algebraic properties of de Sitter space, first the contractions of its kinematical group,[1] and second as a quotient space. This last section will be of great importance to make contact with Lorentz and Galilei Groups. The basic references for this chapter are [21, 22, 23].

2.1 Classical Geometry

The de Sitter space is a solution to the Einstein equations without any source and with a cosmological term:[2]

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 0, \tag{2.1}$$

an important fact is that this solution can be interpreted as the hyperboloid

$$\eta_{AB}\chi^A\chi^B = -l^2, \tag{2.2}$$

embedded in the ambient space $\mathbf{E}^{4,1}$ with line element

$$ds^2 = \eta_{AB}d\chi^A d\chi^B = (d\chi^0)^2 - (d\chi^1)^2 - ... - (d\chi^4)^2 \tag{2.3}$$

where the de Sitter radius l is related to the cosmological constant by

$$\Lambda = \frac{3}{l^2}. \tag{2.4}$$

A graphical representation of this hyperboloid can be seen in figure 2.1.

Now, let us introduce some different parametrizations for the hyperboloid (2.2). In

[1] Usually called the de Sitter group.

[2] Note that de Sitter Space can be constructed using algebraic properties and without using any reference to Einstein equations or gravitational systems, this means, that this space is more elemental and it must be treated at the same level as Minkowski space.

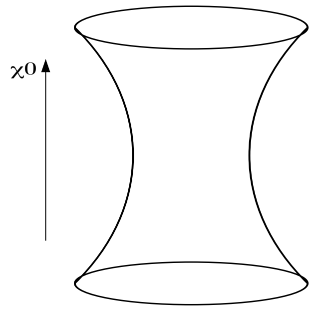

Figure 2.1: Graphical Representation of the Hyperboloid (2.2).

some of the coordinates systems that we are going to use, we will refer to the angular coordinates ω^a of the four-dimensional sphere S^4:

$$\begin{aligned} \omega^1 &= \cos\theta_1 \\ \omega^2 &= \sin\theta_1\cos\theta_2 \\ \omega^3 &= \sin\theta_1\sin\theta_2\cos\theta_3 \\ \omega^4 &= \sin\theta_1\sin\theta_2\sin\theta_3 \end{aligned} \tag{2.5}$$

where the different angles θ_a can take the following values: $0 \leq \theta_k < \pi$ for $k = 1, 2$, and $0 \leq \theta_3 < 2\pi$. This can be easily generalized to an arbitrary number of dimensions.

2.1.1 Global Coordinates (τ, θ_i)

The first parametrization that we are going to define is

$$\chi^0 = l\sinh(\tau/l) \;\; ; \;\; \chi^p = \omega^p l\cosh(\tau/l) \;\; ; \;\; p = 1, ..., 4, \tag{2.6}$$

with $-\infty < \tau < \infty$. This parametrization completely covers the de Sitter Space (2.2). For this reason the coordinates (τ, θ_i) are called Global Coordinates. Replacing in (2.3), the line element in this coordinates takes the form:

$$ds^2 = d\tau^2 - l^2\cosh^2(\tau/l)d\Omega_3^2, \tag{2.7}$$

where

$$d\Omega_3^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1\sin^2\theta_2 d\theta_3^2.$$

Note that in the last equation the sections $\tau = constant$ are spheres, so we can guess that the topology of the de Sitter space is the Cartesian product $R \times S^3$. We can say that in $\tau = -\infty$ the spheres have an infinite radius, then, they contract until the radius is l at $\tau = 0$ and expand again to the infinity at $\tau = \infty$. The asymptotic regions $\mathcal{I}^{\pm}$ characterized by $\tau = \pm\infty$, are called of past null infinity and future null infinity, respectively. To compute the Penrose diagram of de Sitter space, let us make the transformation

$$\cosh(\tau/l) = \frac{1}{\cos(T/l)}. \tag{2.8}$$

This transformations projects the unbounded interval $-\infty < \tau < \infty$ into the bounded interval $-\pi/2 < T/l < \pi/2$, making possible to describe de Sitter space with coordinates that have a finite domain. With this new coordinates the line element takes the form

$$ds^2 = \frac{1}{\cos^2(T/l)}\left[dT^2 - l^2 d\Omega_3^2\right]. \tag{2.9}$$

The equation above is particularly useful if we are interested in the causal structure, because the null geodesics [3] with respect to the interval (2.7) are also null geodesics with respect to the interval

$$d\tilde{s}^2 = dT^2 - l^2 d\Omega_3^2. \tag{2.10}$$

This last equation describes equally as (2.7) the causal structure of the space. We present the Penrose diagram in figure 2.2, the asymptotic regions $\mathcal{I}^{\pm}$ are represented by $T = \pm\pi/2$.

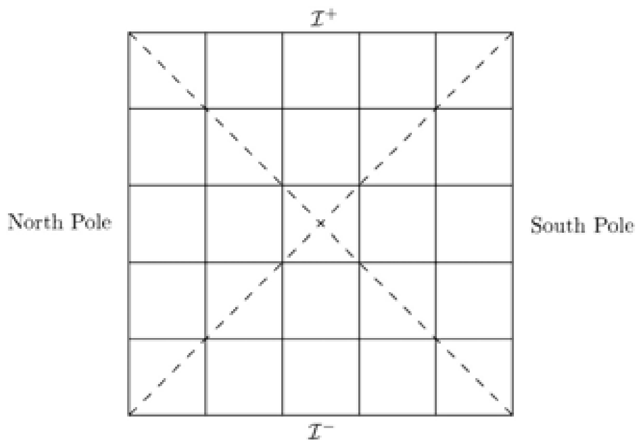

Figure 2.2: Penrose diagram for global coordinates. The vertical lines are $\theta_1 = const.$ The north and south pole are represented by $\theta_1 = \pi$ and $\theta_1 = 0$ respectively. The horizontal lines are $\tau = const.$ The diagonals are the null geodesics that connect the regions $\mathcal{I}^+$ where $\tau = \infty$, and $\mathcal{I}^-$ where $\tau = \infty$. All null geodesics are parallel to this diagonals.

At this point, we begin to see a fundamental difference between the de Sitter space and the Minkowski space. The asymptotic structure of Minkowski space is such that an inertial observer eventually would have causal access to the whole space, provided that its world line reachs the future null infinity $\mathcal{I}^+$. This possibility is excluded in de Sitter space, since an inertial observer in $\mathcal{I}^+$ will always have inaccessible regions from its past (look at fig. 2.2).

[3] These define the limits between the regions causally connected or, let us say, the light cone.

2.1.2 Planar Coordinates (t, x^i)

The planar coordinates are defined as

$$\begin{aligned} \chi^0 &= l \sinh \frac{t}{l} - \frac{1}{2}\frac{(x^i)^2}{l} e^{-\frac{t}{l}} \\ \chi^i &= x^i e^{-\frac{t}{l}}, \qquad i = 1, ..., 3. \\ \chi^4 &= l \cosh \frac{t}{l} - \frac{1}{2}\frac{(x^i)^2}{l} e^{-\frac{t}{l}} \end{aligned} \tag{2.11}$$

with $-\infty < t < \infty$, $-\infty < x^i < \infty$. In this coordinates the line element takes the form

$$ds^2 = dt^2 - l^2 e^{-2\frac{t}{l}} \delta_{ij} dx^i dx^j. \tag{2.12}$$

Note that in this coordinates, the sections $t = const.$ are flat and have the contraction factor $\exp(-2t/l)$. Making the transformation $t \to -t$ the interval changes to

$$ds^2 = dt^2 - l^2 e^{2\frac{t}{l}} \delta_{ij} dx^i dx^j, \tag{2.13}$$

which describes a space in expansion with exponential factor $\exp(2t/l)$, just as the inflationary period. Opposed to the global coordinates, the planar coordinates only cover the low half of the diagonal $\chi^0 = \chi^4$ of the hyperboloid (2.2), this is due to the fact that

$$-\chi^0 + \chi^4 = l \exp(-t/l) < 0. \tag{2.14}$$

To calculate the Penrose diagram, we first introduce the Kruskal-like coordinates[4]

$$U = \frac{r/l - \exp(t/l)}{2} \quad ; \quad V = \frac{2}{\exp(t/l) + r/l} \quad ; \quad (r^2 = \delta_{ij} x^i x^j), \tag{2.15}$$

their inverse relations are

$$r/l = 1/V + U \quad ; \quad \exp(t/l) = 1/V - U. \tag{2.16}$$

We can see that this coordinates describe the region $V > 0$. The origin $r = 0$ correponds to $UV = -1$, the past null infinity and the future null infinity $t \to \pm\infty$, are $V = 0$ and $UV = -1$, respectively. The spatial infinity $r \to \infty$ corresponds to $UV = 1$. The metric in this coordinates is:

$$ds^2 = \frac{4l^2}{(1 - UV)^2} \left[dUdV - (1 + UV)^2 d\Omega_2^2 \right], \tag{2.17}$$

[4]The motivation for using here the Kruskal coordinates is similar to that in Schwarzchild solution, where they permit to make an analytic continuation of the solution into the inner region of the event horizon.

where

$$d\Omega_2^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2.$$

Under the transformations $U \to -U$ e $V \to -V$ this metric remains invariant, what allows us to make an analytic continuation to the region $V < 0$. With this metric is possible to describe the space dS_4 completely.

The next step is to put this region into a bounded domain. We use the transformations:

$$U = \tan\left(\frac{\varphi - \zeta}{2}\right) \quad ; \quad V = \tan\left(\frac{\varphi + \zeta}{2}\right), \tag{2.18}$$

which takes the de Sitter space into the compact domain $|\zeta \pm \varphi| < \pi$. The conformal diagram is presented in figure 2.3.

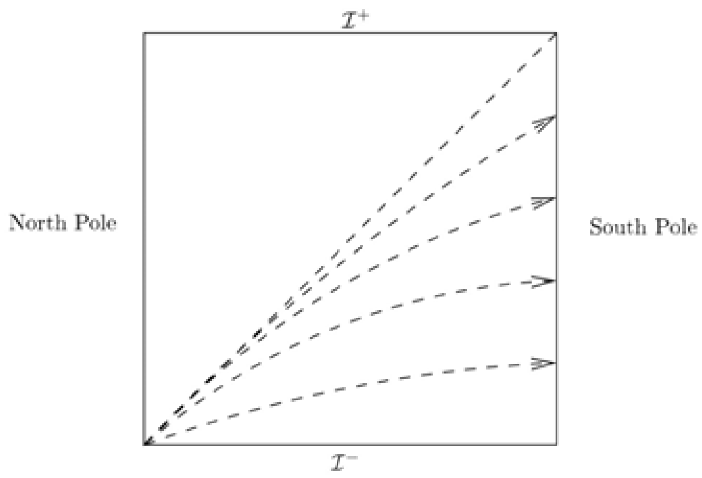

Figure 2.3: The dashed lines are $t = const$. The asymptotic region $\mathcal{I}^-$ corresponds to $t = -\infty$. The diagonal is $t = \infty$. The south pole is in $r = 0$ and the north pole in $r = \infty$.

2.1.3 Hyperbolic Coordinates $(\bar{\tau}, \psi, \theta_a)$

The hyperbolic coordinates are described by

$$\begin{aligned} \chi^0 &= l \sinh(\bar{\tau}/l) \cosh\psi \\ \chi^i &= l\omega^i \sinh(\bar{\tau}/l) \sinh\psi \\ \chi^4 &= l \cosh(\bar{\tau}/l), \end{aligned} \tag{2.19}$$

where $-\infty < \bar{\tau} < \infty$ and $0 \leq \psi < \infty$. This coordinates parametrize the region bounded by the condition

$$\cos\theta_1 \cosh(\bar{\tau}/l) \geq 1. \tag{2.20}$$

The line element takes the form

$$ds^2 = -d\bar{\tau}^2 + \sinh^2(\bar{\tau}/l)[d\psi^2 + \sinh^2\psi d\Omega_{d-2}^2]. \tag{2.21}$$

The sections $\bar{\tau} = const.$ are open 3-dimensionals hyperboloids H^3. The Penrose diagram for this coordinates is shown in figure 2.4.

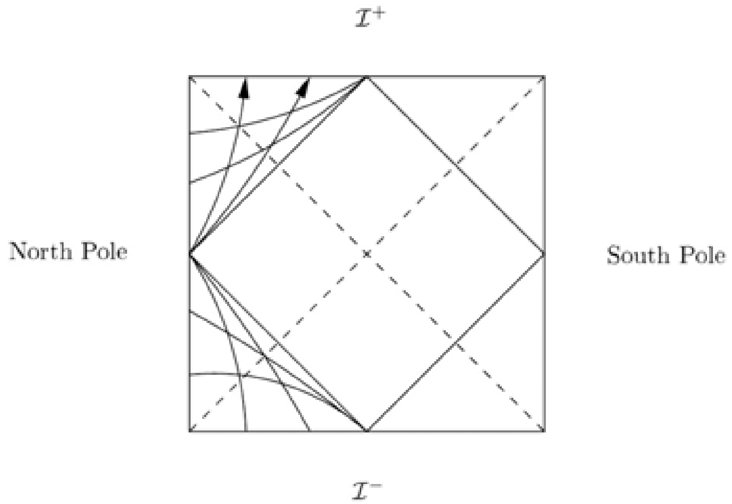

Figure 2.4: The Hyperbolic coordinates only describe the part where $\bar{\tau} = const.$, these are represented here by the lines with arrows.

2.1.4 Stereographic Coordinates (x^μ)

This set of coordinates are obtained by performing the stereographic projection of the hyperboloid, represented by the equation (2.2), to the Minkowski space. This projections is given by

$$\chi^\mu = \Omega x^\mu \quad ; \quad \chi^4 = -l\Omega\left(1 + \frac{\sigma^2}{4l^2}\right) \tag{2.22}$$

where

$$\Omega = \frac{1}{1 - \sigma^2/4l^2}, \qquad \sigma^2 = \eta_{\mu\nu}x^\mu x^\nu \tag{2.23}$$

and x^μ takes values in the Minkowski space where the projection is made. In this coordinates, the line element takes the conformal form:

$$ds^2 = \Omega^2 \eta_{\mu\nu} dx^\mu dx^\nu. \tag{2.24}$$

These coordinates will be used along the following chapters. The importance of this set is that the symmetry generators written in this coordinates are very similar to the generators of the Poincaré group in Cartesian coordinates. For this reason, they are ideal to make the contraction process that we are going to define in the following section.

2.2 Contractions of the de Sitter Group

The de Sitter space is similar to Minkowski space in the sense that they both are homogeneous spaces and maximally symmetric. Now, we are going to explore this similarity and show some of the consequences of the presence of the cosmological constant in the symmetry group. First, we are going to review the process of Lie group contraction [11, 12, 24], then we will apply it to the de Sitter group $SO(4,1)$ and obtain the possible limits for the different parameters, this also include the non-relativistic limits [15, 25] and the formal limit $\Lambda \to \infty$ whose result is called *The Second Poincaré Group* [26].

2.2.1 Lie Group Contractions

One of the original motivations to study contractions was to see how the Galilei group can be obtained as a non relativistic limit of Lorentz group. It is natural to think that the Galilei group appears as the limit of infinite light speed, because the Galilei transformations are obtained using this limit from Lorentz transformations. However, the process of "taking the limit" on the Lorentz group must be developed in a more precise way because the result can depend of the initial chosen representation. To get an idea of how this can happen, let us take a matrix representation of the Poincaré Group in two dimensions

$$\mathbf{\Lambda} = \begin{pmatrix} \cosh\lambda & \sinh\lambda & a_x \\ \sinh\lambda & \cosh\lambda & a_t \\ 0 & 0 & 1 \end{pmatrix},$$

where[5]

$$\sinh\lambda = \frac{1}{[1 - v^2/c^2]^{1/2}} \quad ; \quad \cosh\lambda = \frac{v/c}{[1 - v^2/c^2]^{1/2}}, \tag{2.25}$$

v is the relative velocity between two reference frames, c is the speed of light and a_x, a_t are the components of a vector representing the translations in spacetime. Taking the non relativistic limit $c \to \infty$, the previous representation transforms to

$$\mathbf{\Lambda} \to \begin{pmatrix} 1 & 0 & a_x \\ 0 & 1 & a_t \\ 0 & 0 & 1 \end{pmatrix},$$

which is a representation of the translation group in two dimensions but it is not a representation of the Galilei group. However, if applied a similarity transformation that

[5]Here we reintroduce the light speed c, which is one of the parameters that will allow us to make different contractions in Poincaré and de Sitter groups.

depends on c

$$\mathbf{C} = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

redefine the parameters to $b_x = ca_x$, and then take the limit $c \to \infty$, the answer is

$$\mathbf{C\Lambda C^{-1}} \to \begin{pmatrix} 1 & v & b_x \\ 0 & 1 & a_t \\ 0 & 0 & 1 \end{pmatrix},$$

which is the appropriate representation of the Galilei group!. This is a good example where if we don't take the appropriate parametrization of the group G, we won't obtain a representation of the desire group G' after the limit. For example, in the de Sitter case, we have to make some manipulations in the parameters and generators to obtain the desired answer. In the following we will see the precise definition of a contraction.

Definition

Let G be a Lie group with X_i $(i = 1, ..., n)$ the generators and b^i the parameters of the group. The generators satisfy the commutation relations:

$$[X_i, X_j] = c_{ij}^k X_k \,, \tag{2.26}$$

with c_{ij}^k the structure constants. Now, make a non singular linear transformation to the generators using the matrix U_i^j, to obtain the new generators:

$$Y_i = U_i^j X_j \,. \tag{2.27}$$

In this process we have to redefine the parameters, the commutation relations and the structure constants

$$a^i = U_j^i b^j \quad ; \quad [Y_i, Y_j] = C_{ij}^k Y_k \quad ; \quad C_{ij}^k = U_i^l U_j^m c_{lm}^n (U^{-1})_n^k \,. \tag{2.28}$$

If the matrix U is not singular the group structure remains unchanged, but if U is singular the group structure will change and we will obtain a new group as result of the transformation. Consider the matrix U depending linearly on one parameter $\epsilon > 0$, in the following way

$$U_i^j = u_i^j + \epsilon w_i^j \,, \tag{2.29}$$

and suppose that the matrix U is not singular when ϵ goes to infinity, but it is singular when ϵ is zero. Additionally, we are going to assume that the matrices u and w can be written as

$$u = \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0_{p\times p} \end{pmatrix} \quad ; \quad w = \begin{pmatrix} v_{r\times r} & 0 \\ 0 & I_{p\times p} \end{pmatrix}, \qquad n = r + p \tag{2.30}$$

where v is some matrix. This is a necessary condition, because it is not always possible to write the matrices u and w in the previous form. In order to see the difference between the generators of the subgroup where the contractions are made, let us define additional sub-indices for the generators X and Y. Then, applying the transformation (2.29) and using the form above, we can split the generators as

$$Y^1_\alpha = X_\alpha + \epsilon \sum_{\beta=1}^{r} v^\beta_\alpha X_\beta, \tag{2.31}$$

$$Y^2_\mu = \epsilon X_\mu \tag{2.32}$$

where the first half of the greek alphabet $\alpha, \beta, ... = 1, ..., r$ is used to count the elements of the first block in (2.30) and, the second half $\mu, \nu, ... = r+1, ..., p$ is used for the second block. The parameters of the group also change to

$$a^\alpha_1 = b^\alpha + \epsilon \sum_{\beta=1}^{r} b^\beta v^\alpha_\beta, \tag{2.33}$$

$$a^\mu_2 = \epsilon b^\mu, \tag{2.34}$$

note that if the limit $\epsilon \to 0$ is well defined, then the group G will contract to the subgroup define by the generators X_α. This is why it is called a contraction.

Now, we are going to study the conditions when this limit is well defined. As we are interested in the limit $\epsilon \to 0$, we are going to keep the singular part in ϵ. The commutation relations can be separated as

$$[Y^1_\alpha, Y^1_\beta] = \sum_{\gamma=1}^{r} c^\gamma_{\alpha,\beta} Y^1_\gamma + \frac{1}{\epsilon} \sum_{\mu=r+1}^{n} c^\mu_{\alpha,\beta} Y^2_\mu + O(\epsilon), \tag{2.35}$$

where $O(\epsilon)$ represents the linear terms and the superior orders in ϵ. In order to obtain a finite limit in the commutation relation when ϵ goes to zero, we must have that

$$c^\gamma_{\alpha,\beta} = 0, \tag{2.36}$$

which means that the generators X_α form a subgroup in G. It can be verified that the condition (2.36) is the only condition we need for the limit $\epsilon \to 0$ to exist. With this condition, the new structure constants in the limit take the values

$$C^{\gamma}_{\alpha,\beta} = c^{\gamma}_{\alpha,\beta},\, C^{\mu}_{\alpha,\beta} = c^{\mu}_{\alpha,\beta} = 0, \tag{2.37}$$

$$C^{\gamma}_{\alpha,\mu} = 0,\, C^{\mu}_{\alpha,\nu} = c^{\mu}_{\alpha,\nu}, \tag{2.38}$$

$$C^{\gamma}_{\mu,\nu} = C^{\rho}_{\mu,\nu} = 0 \tag{2.39}$$

and also they satisfy the Jacobi identity

$$C^{l}_{ij}C^{m}_{kl} + C^{l}_{jk}C^{m}_{il} + C^{l}_{ki}C^{m}_{jl} = 0, \tag{2.40}$$

which is the condition to be a Lie group. At the end of the process we have a new group G' which is not isomorph to the initial group G, but, with the same dimension. The group G was contracted with respect to a subgroup $S \subset G$ that is spanned by the generators X_α. The relations (2.37) show that it exists a new subgroup $S' \subset G'$ that is isomorph to S. The relations (2.38) and (2.39) also show that it exists a new group $A \subset G'$ which is invariant and abelian.

Now, we will make a couple of examples with the objective of making clear the definition of the contraction process. The two examples are: the contraction of the rotation group to the translation group and the contraction of the Lorentz group to the Galilei group.

Rotations

The generators of the rotations group in 3 dimension J_i $(i = 1, 2, 3)$ satisfy the algebra

$$[J_i, J_j] = \epsilon_{ijk} J_k. \tag{2.41}$$

To make everything more explicit, let us write the algebra in the form

$$[J_1, J_2] = J_3 \quad ; \quad [J_2, J_3] = J_1 \quad ; \quad [J_3, J_1] = J_2. \tag{2.42}$$

We are going to contract the group with respect to the subgroup generated by J_3. First, redefine the other two generators as

$$I_3 = J_3 \quad ; \quad I_1 = \varepsilon J_1 \quad ; \quad I_2 = \varepsilon J_2. \tag{2.43}$$

In the limit $\varepsilon \to 0$, the algebra (2.42) transform to

$$[I_1, I_2] = 0 \quad ; \quad [I_2, I_3] = I_1 \quad ; \quad [I_3, I_1] = I_2. \tag{2.44}$$

This is the algebra of translation and rotations in the Euclidean plane. What happens geometrically in this example can be understood in the following way: The generator for the rotations around the z axis is unchanged, but, the generators for the x and y axes are restricted to act infinitesimally, so they look like translations in the plane $x-y$. The parameter used in this contraction was the angle between the axes x and y.

Lorentz

The second example is the contraction of the Lorentz group with respect to the rotations. This group is spanned by the generators $L_{\alpha\beta}$ $(\alpha, \beta = 0, ..., 3)$ that satisfy the algebra

$$\begin{aligned} [L_{ab}, L_{cd}] &= \delta_{bc}L_{ad} + \delta_{ad}L_{bc} - \delta_{bd}L_{ac} - \delta_{ac}L_{bd} \qquad (2.45) \\ [L_{a0}, L_{bc}] &= \delta_{ab}L_{0c} - \delta_{ac}L_{0b} \qquad (2.46) \\ [L_{a0}, L_{b0}] &= L_{ba}, \qquad (2.47) \end{aligned}$$

where $(a, b, ... = 1, 2, 3)$. Redefine the generators to

$$L_{ab} = L_{ab} \quad ; \quad T_a = \varepsilon L_{a0} \qquad (2.48)$$

with $\varepsilon = 1/c$. Now take the limit $\varepsilon \to 0$ to obtain the algebra

$$[T_a, L_{bc}] = \delta_{ac}T_b - \delta_{ab}T_c \quad ; \quad [T_a, T_b] = 0 \qquad (2.49)$$

which is the algebra of the Galilei group containing the 3-dimensional rotations and the transformations between reference frames in relative movement in three dimensions. In this way, we see that taking the limit where the velocity of light goes to infinity the symmetry group of special relativity contracts to the symmetry group of the Newtonian mechanics. A similar contraction takes the Poincaré group to the non homogeneous Galilei group that includes the translations in spacetime.

2.3 de Sitter Space and Group

In this section, we will focus in some of algebraic properties of the de Sitter space. The stereographic coordinates, defined in a former section, are of particular interest. In this coordinates is possible to highlight the structure similarities between the Poincaré and the de Sitter groups, which we are going to study simultaneously. The preliminary works on fundamental properties of de Sitter group, as also the representations and contraction limit $\Lambda \to 0$, can be found in [12].

2.3.1 de Sitter Space

Spaces with constant scalar curvature R are maximally symmetric, this means that they can lodge the maximum possible number of Killing vectors (generators of the symmetry group). Given a signature for the metric, there is only one space for each value of R [27]. Minkowski space M with zero scalar curvature, is the more simple example. Its symmetry group is the Poincaré group $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$, the semi-direct product between the Lorentz group $\mathcal{L} = SO(3,1)$ and the abelian group of translations $\mathcal{T}$. The latter acts transitively in M and its group manifold can be identified with M. In fact, the Minkowski space is an homogeneous space under $\mathcal{P}$, actually is the quotient space

$$M = \mathcal{P}/\mathcal{L}.$$

Now, from all curved spaces, de Sitter and anti-de Sitter spaces are the only ones with constant scalar curvature (positive and negative curvature[6]). As we have said before, these spaces can be defined as hyper-surfaces in the pseudo-euclidean spaces $\mathbf{E}^{4,1}$ e $\mathbf{E}^{3,2}$. Using coordinates $(\chi^A) = (\chi^0, \chi^1, \chi^2, \chi^3, \chi^4)$ they must satisfy

$$\eta_{AB}\chi^A\chi^B \equiv (\chi^0)^2 - (\chi^1)^2 - (\chi^2)^2 - (\chi^3)^2 - (\chi^4)^2 = -\, l^2$$

and

$$\eta_{AB}\chi^A\chi^B \equiv (\chi^0)^2 - (\chi^1)^2 - (\chi^2)^2 - (\chi^3)^2 + (\chi^4)^2 = l^2.$$

Taking the Minkowski metric to be $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$ and taking $s = \eta_{44}$, we can study both cases simultaneously. For $s = -1$ we have de Sitter space dS_4 whose metric is induced from the pseudo-euclidean metric $\eta_{AB} = \text{diag}\ (+1, -1, -1, -1, -1)$. It has for symmetry group the orthonormal group $SO(4,1)$. The case $s = +1$ corresponds to anti-de Sitter space AdS_4. Its metric is induced from the metric $\eta_{AB} = (+1, -1, -1, -1, +1)$ and its symmetry group is $SO(3,2)$. Both spaces are homogeneous in the sense that [10]:

$$dS_4 = SO(4,1)/\mathcal{L} \quad \text{e} \quad AdS_4 = SO(3,2)/\mathcal{L}.$$

Besides, each group manifold is a principal bundle with the correspondent de Sitter or anti-de Sitter space as base space, and the Lorentz group $\mathcal{L}$ as the fibre. These spaces are solutions to the Einstein equations without any source, provided that the cosmological constant Λ and the de Sitter radius l are related by

$$\Lambda = -\frac{3\mathsf{s}}{l^2}. \tag{2.50}$$

In stereographic coordinates (2.22) with metric (2.24), the Christoffel connection takes the form

$$\Gamma^\lambda{}_{\mu\nu} = \left[\delta^\lambda{}_\mu \delta^\sigma{}_\nu + \delta^\lambda{}_\nu \delta^\sigma{}_\mu - \eta_{\mu\nu}\eta^{\lambda\sigma}\right] \partial_\sigma[\ln \Omega(x)], \tag{2.51}$$

[6]The sign of the curvature depends on conventions

and the Riemann tensor is

$$R^{\mu}{}_{\nu\rho\sigma} = -\,\frac{\Lambda}{3}\,\left[\delta^{\mu}{}_{\rho} g_{\nu\sigma} - \delta^{\mu}{}_{\sigma} g_{\nu\rho}\right]. \tag{2.52}$$

Finally, the Ricci tensor and the scalar curvature are

$$R_{\mu\nu} = -\,\Lambda\, g_{\mu\nu} \quad \text{and} \quad R = -\,4\Lambda. \tag{2.53}$$

The convention used here is, de Sitter (anti-de Sitter) space has negative (positive) scalar curvature.

2.3.2 The Kinematical Groups

The isometry group of a spacetime will always have a subgroup that accounts for the isotropy of the space and the equivalence between reference frames in relative movement. Also, there is a part of the isometry group that is responsible for the homogeneity of the spacetime. This part is usually called the *translations* and can be either commutative or non commutative. This happens in Galilean relativity and any others relativities that we can think of [13]. The more known case of relativistic kinematics is the Poincaré group $\mathcal{P}$. It is naturally associated with Minkowski space M, being its symmetry group. The Poincaré group is constructed by the semi-direct product of Lorentz group $\mathcal{L} = SO(3,1)$ and the translations group $\mathcal{T}$. The last one acts transitively in M. In fact, Minkowski space is an homogeneous space under $\mathcal{P}$, actually it is the quotient $M \equiv \mathcal{T} = \mathcal{P}/\mathcal{L}$. The invariance of M under $\mathcal{P}$ reflects the *uniformity*. The Lorentz subgroup gives locally the notion of isotropy and the invariance under translations takes the isotropy to any other point of the space. This is the usual notion of *uniformity* where $\mathcal{T}$ is responsible for the equivalence between all points in the spacetime. Once established these notions, we are going to study the symmetry group of de Sitter and anti-de Sitter spaces. In Cartesian coordinates χ^A, the generators of infinitesimal transformations are

$$J_{AB} = \eta_{AC}\,\chi^C\,\frac{\partial}{\partial\chi^B} - \eta_{BC}\,\chi^C\,\frac{\partial}{\partial\chi^A}, \tag{2.54}$$

these satisfy the commutation relations

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC} - \eta_{AC} J_{BD}. \tag{2.55}$$

In terms of stereographic coordinates $\{x^{\mu}\}$, they assume the form

$$J_{\mu\nu} \equiv L_{\mu\nu} = \eta_{\mu\rho}\, x^{\rho}\, P_{\nu} - \eta_{\nu\rho}\, x^{\rho}\, P_{\mu} \tag{2.56}$$

and

$$J_{4\mu} = -\mathsf{s}\left(l\,P_\mu + \frac{\mathsf{s}}{4l}\,K_\mu\right), \tag{2.57}$$

where

$$P_\mu = \frac{\partial}{\partial x^\mu} \quad \text{and} \quad K_\mu = \left(2\eta_{\mu\nu}x^\nu x^\rho - \sigma^2\delta_\mu{}^\rho\right)\frac{\partial}{\partial x^\rho} \tag{2.58}$$

are respectively the generators of translations and proper conformal transformations. For $\mathsf{s} = -1(+1)$, they give rise to the (anti-)de Sitter group $SO(4,1)(SO(3,2))$. The generators $J_{\mu\nu}$ refers to the Lorentz subgroup $SO(3,1)$ and $J_{\mu 4}$ defines the transitivity of the corresponding homogeneous spaces. In terms of the previous decomposition, we can see explicitly the last properties in the commutation relations:

$$[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\rho}J_{\nu\sigma}, \tag{2.59}$$

$$[J_{4\mu}, J_{\nu\rho}] = \eta_{\mu\nu}J_{4\rho} - \eta_{\mu\rho}J_{4\nu}, \tag{2.60}$$

$$[J_{4\mu}, J_{4\nu}] = -\mathsf{s}J_{\mu\nu}. \tag{2.61}$$

From equation (2.57), it is obvious that theses spaces are transitive under a mix of translations and proper conformal transformation. The relative importance of these two components is weighted by the value of the cosmological constant. Particularly, for a zero cosmological constant, the two spaces (de Sitter and anti-de Sitter spaces) transform to Minkowski space M with Poincaré $\mathcal{P}$ as symmetry group.

2.4 Contration Limits

In this section, we will present the different contraction limits of the two de Sitter groups. To this end, it is necessary to make some adequate redefinitions of the parameters and generators. In these redefinitions , we are going to take into account the two constants c and l, and see their importance on the redefinitions and on the meaning of the limits that we are going to take. For more details in the algebraic and geometrical aspects of the contraction for the non relativistic limit see [15].

2.4.1 Zero Cosmological Constant

To study the limit ($l \to \infty$) is convenient to write the generators in the following way

$$J_{\mu\nu} \equiv L_{\mu\nu} = \eta_{\mu\rho}\,x^\rho\,P_\nu - \eta_{\nu\rho}\,x^\rho\,P_\nu, \tag{2.62}$$

and

$$\Pi_\mu \equiv \frac{J_{\mu 4}}{l} = s\left(P_\mu + \mathsf{s}\,\frac{1}{4l^2}K_\mu\right). \tag{2.63}$$

The generators $L_{\mu\nu}$ span the Lorentz group acting on Minkowski space and they satisfy the commutation relation

$$[L_{\mu\nu}, L_{\rho\sigma}] = \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma}. \tag{2.64}$$

The other commutation relations, that describe the transitivity sector, are written as

$$[\Pi_\mu, L_{\nu\rho}] = \eta_{\mu\nu}\Pi_\rho - \eta_{\mu\rho}\Pi_\nu, \tag{2.65}$$
$$[\Pi_\mu, \Pi_\nu] = -\frac{s}{l^2}L_{\mu\nu}. \tag{2.66}$$

For $l \to \infty$, the generators Π_μ reduce to the ordinary translations and the de Sitter group contracts to Poincaré group $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$. Together, with the modifications of the algebra and group, the de Sitter space transforms to Minkowski space

$$M = \mathcal{P}/\mathcal{L},$$

which is transitive under ordinary translations.

2.4.2 Non Relativistic Limit: Newton–Hooke Space

The Newton-Hooke space can be considered as the non relativistic limit of de Sitter space. The principal characteristic of this space is that the curvature is non zero because it is inherited from de Sitter space. The effect of the curvature appears explicitly in the translations generators. The adequate redefinition of the generators in this case is:

$$\mathsf{L}_{ab} \equiv J_{ab}\ ,\ \mathsf{L}_{a0} \equiv J_{a0}/c\ ,\ \mathsf{T}_a \equiv sJ_{a4}/c\tau\ ,\ \mathsf{T}_0 \equiv sJ_{04}/\tau, \tag{2.67}$$

where $a, b, ... = 1, 2, 3$ are indices in the algebra and $\tau = l/c$ is keep constant during the process. These redefinitions in the generators corresponds to modify the parameters $\omega^{\alpha\beta}$, so that, $\omega^{ab} \to \omega^{ab}$; $\omega^{a0} \to c\omega^{a0}$; $\omega^a \to \epsilon c\tau\omega^a$, e $\omega^0 \to \epsilon\tau\omega^0$. The factors are absorbed into the redefined group parameters, and these get a dimensionality. The algebra in terms of these redefined generators is

$$
\begin{aligned}
[\mathsf{L}_{ab}, \mathsf{L}_{de}] &= \delta_{bd}\mathsf{L}_{ae} + \delta_{ae}\mathsf{L}_{bd} - \delta_{be}\mathsf{L}_{ad} - \delta_{ad}\mathsf{L}_{be}, && (2.68)\\
[\mathsf{L}_{ab}, \mathsf{L}_{d0}] &= \delta_{bd}\mathsf{L}_{a0} - \delta_{ad}\mathsf{L}_{b0}, && (2.69)\\
[\mathsf{L}_{0b}, \mathsf{L}_{0e}] &= \frac{1}{c^2}\mathsf{L}_{be}, && (2.70)\\
[\mathsf{L}_{ab}, \mathsf{T}_{d}] &= \delta_{bd}\mathsf{T}_{a} - \delta_{ad}\mathsf{T}_{b}, && (2.71)\\
[\mathsf{L}_{a0}, \mathsf{T}_{b}] &= \frac{1}{c^2}\delta_{ab}\mathsf{T}_{0}, && (2.72)\\
[\mathsf{L}_{a0}, \mathsf{T}_{0}] &= -\mathsf{T}_{a}, && (2.73)\\
[\mathsf{L}_{ab}, \mathsf{T}_{0}] &= 0, && (2.74)\\
[\mathsf{T}_{a}, \mathsf{T}_{b}] &= -\frac{\mathsf{s}}{\tau^2 c^2}\mathsf{L}_{ab}, && (2.75)\\
[\mathsf{T}_{a}, \mathsf{T}_{0}] &= -\frac{\mathsf{s}}{\tau^2}\mathsf{L}_{a0}, && (2.76)\\
[\mathsf{T}_{0}, \mathsf{T}_{0}] &= 0\ . && (2.77)
\end{aligned}
$$

This is the adequate parametrization to obtain the algebra of the Newton-Hooke group. Taking the limit $c \to \infty$, we get

$$
\begin{aligned}
[\mathsf{L}_{ab}, \mathsf{L}_{de}] &= \delta_{bd}\mathsf{L}_{ae} + \delta_{ae}\mathsf{L}_{bd} - \delta_{be}\mathsf{L}_{ad} - \delta_{ad}\mathsf{L}_{be}, && (2.78)\\
[\mathsf{L}_{ab}, \mathsf{L}_{d0}] &= \delta_{bd}\mathsf{L}_{a0} - \delta_{ad}\mathsf{L}_{b0}\ ; && (2.79)\\
[\mathsf{L}_{0b}, \mathsf{L}_{0e}] &= 0\ , && (2.80)\\
[\mathsf{L}_{ab}, \mathsf{T}_{d}] &= \delta_{bd}\mathsf{T}_{a} - \delta_{ad}\mathsf{T}_{b}\ ; && (2.81)\\
[\mathsf{L}_{a0}, \mathsf{T}_{b}] &= 0\ , && (2.82)\\
[\mathsf{L}_{a0}, \mathsf{T}_{0}] &= -\mathsf{T}_{a}\ , && (2.83)\\
[\mathsf{L}_{ab}, \mathsf{T}_{0}] &= 0\ , && (2.84)\\
[\mathsf{T}_{a}, \mathsf{T}_{b}] &= 0\ , && (2.85)\\
[\mathsf{T}_{a}, \mathsf{T}_{0}] &= -\frac{\mathsf{s}}{\tau^2}\mathsf{L}_{a0}\ , && (2.86)\\
[\mathsf{T}_{0}, \mathsf{T}_{0}] &= 0, && (2.87)
\end{aligned}
$$

which span the algebra of the Galilei group, with an important difference in equation (2.86), where can be seen the non commutativity in the translations of spacetime. This effect is originated in the non zero scalar curvature of de Sitter space [15]. It is important to note that τ must be keep constant during the contraction process, so, the result is well defined.

2.4.3 Infinite Cosmological Constant

It is important to emphasize that the limit $\Lambda \to \infty$ must be understood as purely classic, because taking this limit means that the values of the de Sitter radius l are arbitrarily small, and at this length scale or equivalently this energy scale, we expect important effects of quantum nature that forbids l to take these small values. The fundamental interest of this limit resides in that it gives a consistent algebraic structure of the symmetry group when a small length scale is introduced, for example the Plank scale, and eventually it can have relevance in the study of physics at that scale.

For this case, let us write the generators in the form

$$L_{\mu\nu} = \eta_{\mu\rho}\, x^\rho\, P_\nu - \eta_{\nu\rho}\, x^\rho\, P_\mu \tag{2.88}$$

and

$$\bar{\Pi}_\mu \equiv 4l\, L_{\mu 4} = \mathsf{s}(4l^2 P_\mu + \mathsf{s}K_\mu). \tag{2.89}$$

With these new definitions the algebra can be written as

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma} & (2.90)\\ \left[\bar{\Pi}_\mu, \bar{\Pi}_\nu\right] &= -\mathsf{s}l^2 L_{\mu\nu} & (2.91)\\ \left[\bar{\Pi}_\mu, \bar{L}_{\nu\rho}\right] &= \eta_{\mu\nu}\bar{\Pi}_\rho - \eta_{\mu\rho}\bar{\Pi}_\nu. & (2.92) \end{aligned}$$

In the limit $l \to \infty$, the generators $\bar{\Pi}_\mu$ convert to the proper conformal transformations K_μ, and the commutation relations take the form

$$\left[\bar{\Pi}_\mu, \bar{\Pi}_\nu\right] = 0 \quad ; \quad \left[\bar{\Pi}_\mu, \bar{L}_{\nu\rho}\right] = \eta_{\mu\nu}\bar{\Pi}_\rho - \eta_{\mu\rho}\bar{\Pi}_\nu, \tag{2.93}$$

maintaining the part that correspondent to the Lorentz group equal. The group contracts to the *second Poincaré group* $\bar{\mathcal{P}}$ [26]. Consequently , with the group contraction, the de Sitter space transforms to

$$N = \bar{\mathcal{P}}/\mathcal{L}.$$

This new space is also maximally symmetric [26] and it is called the *Cone* space.

2.5 de Sitter as a Quotient Space

As we have said before, the de Sitter space is more fundamental than a simple solution of Einstein equations, and can be constructed from an algebraic point of view. Right now we will see how this can be done. In the previous section we write that

$$dS_4 = SO(4,1)/\mathcal{L}. \tag{2.94}$$

To prove the last equation, take a representation of $SO(4,1)$, this representations acts naturally on a 5-dimensional Minkowski space. We can find the matrix elements of the generators using that

$$\delta x^C = L_{AB}\, x^C = (L_{AB})^C_D\, x^D \tag{2.95}$$

where

$$L_{AB} = \eta_{AC} X^C \partial_B - \eta_{BC} X^C \partial_C. \tag{2.96}$$

With the two equation above, we find:
The 5-dimensional boosts

$$L_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_{04} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the 3-dimensional rotations

$$L_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$L_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the rotation along the 4 dimension

$$L_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \qquad L_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$L_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We can see that the generators L_{ij} and L_{0i} are the generators of the Lorentz group. The remaining generators we will call them the "de Sitter translations" and they are

$$L_{A4} : (L_{04}, L_{i4}).$$

Now, an element P in the quotient space $SO(4,1)/SO(3,1)$ has the form

$$P = \exp p = \exp \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & d \\ a & -b & -c & -d & 0 \end{pmatrix}, \qquad p = E^A L_{4A}, \qquad E^A = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \tag{2.97}$$

We need to compute the exponential of this matrix. By definition, we have that

$$P \equiv \exp\ p = \sum_{n=0}^{\infty} \frac{p^n}{n!}, \tag{2.98}$$

define

$$E^\mu = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \qquad E_\mu \equiv \eta_{\mu\nu} E^\nu = \begin{pmatrix} a & -b & -c & -d \end{pmatrix} \tag{2.99}$$

then write the matrix p as

$$p = \begin{pmatrix} 0 & E^\mu \\ E_\nu & 0 \end{pmatrix}. \tag{2.100}$$

The powers of the matrix p have the following properties

$$p^{2n} = \begin{pmatrix} (E^\mu E_\nu)^n & 0 \\ 0 & (E^\rho E_\rho)^n \end{pmatrix} \qquad p^{2n+1} = (E^\rho E_\rho)^n \, p. \tag{2.101}$$

We can separate the sum (2.98) in two terms

$$P = \sum_{n=o}^{\infty} a \frac{(E^\rho E_\rho)^n}{(2n+1)!} + \sum_{n=o}^{\infty} \frac{a^{2n}}{(2n)!}, \tag{2.102}$$

which in the limit is equal to

$$P = \begin{pmatrix} \cosh(E^\mu E_\nu)^{1/2} & A^a \frac{\sinh(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}} \\ E_\nu \frac{\sinh(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}} & \cosh(E^\rho E_\rho)^{1/2} \end{pmatrix}. \tag{2.103}$$

Define

$$\chi^\mu = E^a \frac{\sinh(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}}, \tag{2.104}$$

$$\chi^4 = \cosh(E^\rho E_\rho)^{1/2}. \tag{2.105}$$

It can be easily show that these coordinates satisfy the following identity

$$\eta_{\mu\nu}\chi^\mu\chi^\nu - (\chi^4)^2 = -1 \tag{2.106}$$

which is the equation of a 4-dimensional de Sitter space embedded in a 5-dimensional Minkowski space. This shows that the quotient space $SO(4,1)/SO(3,1)$ is in fact the de Sitter space dS_4. It is possible to construct the geodesics and the metric using the same algebraic approach. The straight lines that go through the origin of the algebra are mapped to geodesics in the quotient space, these geodesics have the form

$$P = \exp\left[t \begin{pmatrix} 0 & E^\mu \\ E_\nu & 0 \end{pmatrix} \right] = \begin{pmatrix} \cosh t(E^\mu E_\nu)^{1/2} & A^a \frac{\sinh t(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}} \\ E_\nu \frac{\sinh t(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}} & \cosh t(E^\rho E_\rho)^{1/2} \end{pmatrix} \tag{2.107}$$

or in 5-dimensional coordinates

$$\chi^\mu = E^\mu \frac{\sinh t(E^\rho E_\rho)^{1/2}}{(E^\sigma E_\sigma)^{1/2}} \tag{2.108}$$

$$\chi^4 = \cosh t(E^\rho E_\rho)^{1/2}. \tag{2.109}$$

Note that all the information about the geodesic is in χ^μ, this means that χ^4 is irrevelant because of the equation (2.106). Now, with the notion of geodesics is possible to define the distance between two points in the quotient space. First, let us define the distance between a point c in the quotient space and the identity. To do this, define the curve $g(t)$ that connects c to the origin (the identity)

$$
\begin{aligned}
g(t) &= \exp\,[t\,p], \qquad (2.110)\\
g(0) &= Id, \qquad (2.111)\\
g(1) &= c, \qquad (2.112)
\end{aligned}
$$

then, we identify the distance between these two points using the norm of the matrix p

$$D(c, Id) = ||p||. \qquad (2.113)$$

To find the norm of this matrix, we need to calculate the Cartan-Killing metric for $SO(4,1)$, which is done more easily using the adjoint representation. First, remember that

$$[M, M_j] = \sum_{k=1}^{n} \{ad(M)\}_j^k \, M_k \qquad \text{where} \qquad \{ad(M_p)\}_j^k = C_{pj}^k, \qquad (2.114)$$

therefore, if we know the structure constants we can obtain the metric. The commutation relations for $SO(4,1)$ are

$$[L_{AB}, L_{CD}] = \eta_{AD}L_{BC} - \eta_{BC}L_{AD} + \eta_{AC}L_{BD} - \eta_{BD}L_{AC} \qquad (2.115)$$

then

$$C_{(AB)(CD)}^{(MN)} = \eta_{AD}\delta_B^M\delta_C^N - \eta_{BC}\delta_A^M\delta_D^N + \eta_{AC}\delta_B^M\delta_D^N - \eta_{BD}\delta_A^M\delta_C^N. \qquad (2.116)$$

The base for elements of the algebra in the quotient space is $L_{\mu 4}$, this implies that the Cartan-Killing metric is

$$h_{\mu\nu} = tr(ad(L_{4\mu})ad(L_{4\nu}) = C_{(4\mu)(CD)}^{(MN)} \; C_{(4\nu)(MN)}^{(CD)} = \eta_{\mu\nu} \qquad (2.117)$$

With this result, we can write the distance between the point c and the origin

$$D(c, Id) \equiv ||p|| \equiv \langle E^\mu L_{4\mu}, E^\nu L_{4\nu}\rangle = \eta_{\mu\nu}E^\mu \; E^\nu \qquad (2.118)$$

Let us compute the metric tensor $g_{\mu\nu}(c)$ at the point c in the quotient space using the Cartan-Killing metric. For this, observe that the metric at the origin of $SO(4,1)/SO(3,1)$ can be identified with the Cartan-Killing metric

$$g_{\mu\nu}(Id) = \eta_{\mu\nu}. \qquad (2.119)$$

The distance ds between the identity and an element dP that is infinitesimally close is given by

$$\begin{aligned} ds^2(0) &= |D(dP, Id)|^2 && (2.120)\\ g_{\mu\nu}(0)d\chi^\mu d\chi^\nu &= \eta_{\mu\nu}dE^\mu dE^\nu, && (2.121)\end{aligned}$$

where

$$dP = \exp(dE^\mu L_{4\mu}), \tag{2.122}$$

but

$$d\chi^\mu = dE^\mu \frac{\sinh(dE^\rho dE_\rho)^{1/2}}{(dE^\sigma dE_\sigma)^{1/2}} = dE^\sigma,$$

therefore, the metric at the origin is

$$g_{\mu\nu}(0) = \eta_{\mu\nu}. \tag{2.123}$$

Observe that the infinitesimal distance is invariant under translations

$$\begin{aligned} ds^2(0) &= ds^2(c) && (2.124)\\ g_{\mu\nu}(0)d\chi^\mu(0)d\chi^\nu(0) &= g_{\mu\nu}(c)d\chi^\mu(c)d\chi^\nu(c) && (2.125)\end{aligned}$$

then, the metric in a point c must be

$$g_{\mu\nu}(c) = g_{\rho\sigma}(0)\frac{\partial\chi^\rho(0)}{\partial\chi^\mu(c)}\frac{\partial\chi^\sigma(0)}{\partial\chi^\nu(c)}. \tag{2.126}$$

We need to find the infinitesimals in the point c, so, we use the following property

$$gc = c'k. \tag{2.127}$$

The above equations mean that if we act with an element $g \in SO(4,1)$ on a point $c \in SO(4,1)/SO(3,1)$, the result is a new point $c' \in SO(4,1)/SO(3,1)$ times an element $k \in SO(3,1)$ of the isotropy group. To find the infinitesimals at the point c, we use this last property, but taking the same point c as an element of the group acting on the infinitesimals, in this way

$$\begin{pmatrix} W^\mu{}_\nu & \chi^\mu \\ \chi_\nu & \chi^4 \end{pmatrix}\begin{pmatrix} 1 & d\chi^\rho(0) \\ d\chi_\nu(0) & 1 \end{pmatrix} = \begin{pmatrix} W'^\mu_\nu & \chi'^\mu \\ \chi'_\nu & \chi'^4 \end{pmatrix}\begin{pmatrix} k^\rho{}_\nu & 0 \\ 0 & 1 \end{pmatrix} \tag{2.128}$$

where

$$
\begin{aligned}
W^{\mu}{}_{\nu} &= \cosh(E^{\rho}E_{\rho})^{1/2} && (2.129)\\
k^{\mu}{}_{\nu} &\in SO(3,1) && (2.130)
\end{aligned}
$$

if we define $d\chi^{\mu}(c) \equiv \chi'^{\mu} - \chi^{\mu}$, we can write

$$
d\chi^{\mu}(c) = W^{\mu}{}_{\nu} d\chi^{\nu}(0) \tag{2.131}
$$

or

$$
W^{\mu}{}_{\nu} = \frac{\partial\chi^{\mu}c)}{\partial\chi^{\nu}(0)}. \tag{2.132}
$$

To put the matrix W in more friendly way, we will use the following series representation

$$
\sinh^2 x = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n)!} x^{2n}, \tag{2.133}
$$

then

$$
\begin{aligned}
W^{\mu}{}_{\nu} &\equiv \left(\delta^{\mu}{}_{\nu} + \sinh^2(E^{\mu}E_{\nu})^{1/2}\right)^{1/2} = \left(\delta^{\mu}{}_{\nu} + \sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n)!}(E^{\rho}E_{\rho})^{n}\right)^{1/2}\\
W^{\mu}{}_{\nu} &= \left(\delta^{\mu}{}_{\nu} + E^{\mu}E_{\nu} \sum_{n=1}^{\infty} \frac{2^{2n-1}}{(2n)!}(E^{\rho}E_{\rho})^{n}\right)^{1/2} = \left(\delta^{\mu}{}_{\nu} + \frac{\sinh^2{(E^{\rho}E_{\rho})^{1/2}}}{(E^{\sigma}E_{\sigma})^{1/2}}\right)^{1/2}\\
W^{\mu}{}_{\nu} &= \left(\delta^{\mu}{}_{\nu} + \chi^{\mu}\chi_{\nu}\right)^{1/2}.
\end{aligned}
$$

Finally, the metric in a point c in the quotient space is

$$
g_{\mu\nu}(c) = \eta_{\rho\sigma}(W^{-1})^{\rho}{}_{\mu}(W^{-1})^{\sigma}{}_{\nu}, \tag{2.134}
$$

Remember that the coordinates χ^{A} are the 5-dimensional coordinates with the constraint (2.2), this means they are restricted to the de Sitter hyper-surface.

Chapter 3

de Sitter Special Relativity

3.1 Kinematics in de Sitter Spacetime

In this section, we are going to study the kinematics of de Sitter space [28, 29]. First, we will explore the kinematics of Minkowski space and write some familiar concepts that can be easily generalized to de Sitter space and group. Then, we will study the equations of motion of a particle in de Sitter space.

3.1.1 Minkowski Spacetime Revisited

The action functional describing a free particle of mass m moving in a Minkowski spacetime is

$$S = -m \int_a^b ds, \tag{3.1}$$

where

$$ds = (\eta_{\mu\nu}\, dx^\mu dx^\nu)^{1/2} \tag{3.2}$$

is the Lorentz invariant interval. Now, the kinematic group of Minkowski is the Poincaré group $\mathcal{P} = \mathcal{L} \oslash \mathcal{T}$, the semi-direct product of the Lorentz $\mathcal{L}$ and the translation group $\mathcal{T}$. The first Casimir invariant of the Poincaré group, on the other hand, is

$$\mathcal{C}_P = \eta_{\mu\nu}\, p^\mu p^\nu = m^2 \tag{3.3}$$

where $p^\mu = mu^\mu$ is the particle four-momentum, with $u^\mu = dx^\mu/ds$ the four–velocity. Considering that the action S and the Lagrangian L are related by

$$S = \frac{1}{c} \int_a^b L\, ds, \tag{3.4}$$

the corresponding Lagrangian can then be written in the form

$$L = -\,(\eta_{\mu\nu}\, p^\mu p^\nu)^{1/2} = -\,\sqrt{\mathcal{C}_P}. \tag{3.5}$$

The identity $\eta_{\mu\nu}\, p^\mu p^\nu = m^2$ is a weak constraint in the sense that it can be used only after the variational calculus is performed. The resulting equation of motion is

$$\frac{dp^\mu}{ds} = 0. \tag{3.6}$$

The equation of motion, therefore, coincides with the conservation of the particle four-momentum, which follows from the invariance of the system under spacetime translation. Its solution determines the geodesics of the Minkowski spacetime. The invariance of the system under Lorentz transformations yields the conservation of the particle angular momentum $l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$, that is,

$$\frac{dl^{\mu\nu}}{ds} = 0. \tag{3.7}$$

3.1.2 Casimir Invariant

For a spinless particle of mass μ, the first Casimir invariant of the de Sitter group is given by [12]

$$\mathcal{C}_{dS} = -\frac{1}{2l^2}\, \eta_{AC}\, \eta_{BD}\, L^{AB}\, L^{CD}, \tag{3.8}$$

where

$$L^{AB} = \mu \left(\chi^A \, \frac{d\chi^B}{d\tau} - \chi^B \, \frac{d\chi^A}{d\tau} \right) \tag{3.9}$$

is the conserved five-dimensional angular momentum. In terms of the stereographic coordinates $\{x^\mu\}$, the Casimir invariant (3.8) assumes the form

$$\mathcal{C}_{dS} = \eta^{\mu\nu}\, \pi_\mu\, \pi_\nu - \frac{1}{2l^2}\, \eta^{\mu\rho}\, \eta^{\nu\sigma}\, l_{\mu\nu}\, l_{\rho\sigma}, \tag{3.10}$$

where[1]

$$\pi_\mu \equiv \frac{\mathcal{L}_{\mu 4}}{l} = p_\mu - \frac{k_\mu}{4l^2} \tag{3.11}$$

represents the de Sitter momentum, with

$$p^\nu = \mu\, \frac{dx^\nu}{d\tau} \qquad \text{and} \qquad k^\nu \equiv \bar{\delta}^\nu{}_\sigma\, p^\sigma = \mu\, (2\eta_{\rho\sigma}\, x^\rho\, x^\nu - \sigma^2\, \delta^\nu{}_\sigma)\, \frac{dx^\sigma}{d\tau}, \tag{3.12}$$

respectively, the linear and the conformal momentum,[2] and

$$l_{\mu\nu} = \eta_{\mu\rho}\, x^\rho\, p_\nu - \eta_{\nu\rho}\, x^\rho\, p_\mu \tag{3.13}$$

[1] Analogously to the generators, we use a parameterization appropriate for a small cosmological constant.

[2] Similarly to the identification $p^\mu = T^{\mu 0}$, with $T^{\mu\nu}$ the energy-momentum tensor, the conformal momentum k^a is defined by $k^a = K^{a0}$, with K^{ab} the conformal current [30].

represents the orbital angular momentum. In the above expression,

$$\bar{\delta}^{\mu}{}_{\nu} = 2\eta_{\nu\rho}\, x^{\rho}\, x^{\nu} - \sigma^2\, \delta^{\mu}{}_{\nu} \tag{3.14}$$

is a kind of conformal Kroenecker delta. Since λ^{AB} is conserved, we have also

$$\frac{dl^{\mu\nu}}{d\tau} = 0 \qquad \text{and} \qquad \frac{d\pi^{\mu}}{d\tau} = 0. \tag{3.15}$$

We remark that $l^{\mu\nu}$ is the Noether conserved momentum related to the invariance of the system under the transformations generated by $\hat{L}_{\mu\nu}$, whereas π^{μ} is the Noether conserved momentum related to the invariance of the system under the transformations generated by $\hat{\Pi}_{\mu}$.[3]

3.1.3 Equations of Motion

Relying on the Minkowski case, the Lagrangian of a spinless particle of mass m in de Sitter spacetime can be assumed to be given by $-c\sqrt{\mathcal{C}_{dS}}$. In the five–dimensional spacetime, however, it is necessary to add a constraint restricting the movement to the de Sitter hyperboloid. In this case, therefore, the Lagrangian turns out to be

$$L = -c\left[\left(\mathcal{C}_{dS}\right)^{1/2} + \;\beta\left(\eta_{AB}\,\chi^{A}\chi^{B} + l^2\right)\right], \tag{3.16}$$

where β is a Lagrange multiplier. Using Eq. (3.8), the corresponding action is written as

$$S = -\int_a^b \left[\left(-\frac{1}{2l^2}\,\eta_{AC}\,\eta_{BD}\,\lambda^{AB}\,\lambda^{CD}\right)^{1/2} + \beta\left(\eta_{AB}\,\chi^{A}\chi^{B} + l^2\right)\right] ds, \tag{3.17}$$

with ds the de Sitter invariant interval (2.3). Performing a functional variation, and neglecting the surface term coming from an integration by parts, the invariance of the action yields the equation of motion

$$\frac{d^2\chi^{A}}{d\tau^2} + \left(\frac{1}{l^2} - 2\,\beta\right)\chi^{A} = 0. \tag{3.18}$$

Using the constraints

$$\eta_{AB}\,\chi^{A}\chi^{B} = -\,l^2 \qquad \text{and} \qquad \eta_{AB}\,\frac{d\chi^{A}}{d\tau}\frac{d\chi^{B}}{d\tau} = 1, \tag{3.19}$$

the value of the Lagrange multiplier is found to be

$$\beta = \frac{1}{l^2}, \tag{3.20}$$

[3] The same conservation laws, but in different coordinates, were studied in ref. [31].

and the equation of motion reduces to to

$$\frac{d^2\chi^A}{d\tau^2} - \frac{\chi^A}{l^2} = 0. \tag{3.21}$$

In terms of the stereographic coordinates, it becomes

$$\frac{d}{d\tau}\left(p^\nu + \frac{1}{4l^2}\,k^\nu\right) + \frac{x^\rho\,u_\rho}{l^2\,\Omega}\left(p^\nu + \frac{1}{4l^2}\,k^\nu\right) - \frac{\mu\,c}{l^2\,\Omega}\,x^\nu = 0. \tag{3.22}$$

The corresponding equation for the covariant components of the momentum is

$$\frac{d}{d\tau}\left(p_\nu + \frac{1}{4l^2}\,k_\nu\right) - \frac{\mu\,c\,\Omega}{l^2}\,\eta_{\nu\rho}\,x^\rho = 0. \tag{3.23}$$

Of course, due to the universality of gravitation, this equation is independent of the mass when written in terms of the four velocity. In fact, it is the same as

$$\frac{d}{d\tau}\left(u_\mu + \frac{1}{4l^2}\,\bar{\delta}_\mu{}^\rho\,u_\rho\right) - \frac{\Omega}{l^2}\,\eta_{\mu\rho}\,x^\rho = 0, \tag{3.24}$$

with $\bar{\delta}_\mu{}^\rho$ given by Eq. (3.14). This is the equation of motion of a spinless particle of mass μ in a de Sitter spacetime. Its solutions determine the geodesics of this spacetime.

Using the second of the conservation laws (3.15), it is possible to obtain separate evolution equations for p^μ and k^μ. For example, the equation of motion for the linear momentum p^μ is found to be

$$\frac{dp_\nu}{d\tau} - \frac{\mu\,c\,\Omega}{2\,l^2}\,\eta_{\nu\rho}\,x^\rho = 0, \tag{3.25}$$

or equivalently,

$$\frac{dp^\nu}{d\tau} + \frac{x^\rho\,u_\rho}{l^2\,\Omega}\,p^\nu - \frac{\mu\,c}{2l^2\,\Omega}\,x^\nu = 0. \tag{3.26}$$

This equation is nothing but the geodesic equation

$$\frac{dp^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho}\,p^\nu\,u^\rho = 0, \tag{3.27}$$

with $\Gamma^\mu{}_{\nu\rho}$ the Levi-Civita connection of the de Sitter metric (2.24). On the other hand, the equation of motion for the conformal momentum k^ν assumes the form

$$\frac{dk^\nu}{d\tau} + \frac{x^\rho\,u_\rho}{l^2\,\Omega}\,k^\nu - \frac{2\,\mu\,c}{\Omega}\,x^\nu = 0. \tag{3.28}$$

Differently from the ordinary momentum p^ν, which is conserved with a covariant derivative, the conformal momentum k^ν is not covariantly conserved. In fact, it is found to

satisfy
$$\frac{dk^\nu}{d\tau} + \Gamma^\nu{}_{\rho\sigma}\, k^\rho\, u^\sigma = \frac{2\mu c}{\Omega}\left[1 - \frac{1}{4l^2}\left(\frac{2}{\Omega^2}\, u_\rho\, u_\sigma\, x^\rho\, x^\sigma - \sigma^2\right)\right] x^\nu. \tag{3.29}$$
Put together, however, these two momenta yield the truly conserved total momentum π^ν.

3.2 Dispersion Relation

In this last section, we are going to obtain the dispersion relation of de Sitter special relativity. To this end, we will use again the notion of Casimir invariant of the de kinematics groups, Minkowski and de Sitter groups, respectively.

The first Casimir operator of the de Sitter group is [12]

$$\hat{\mathcal{C}}_{dS} = -\frac{1}{2l^2}\, \eta^{AC}\, \eta^{BD}\, \hat{\mathcal{L}}_{AB}\, \hat{\mathcal{L}}_{CD}, \tag{3.30}$$

In terms of the generators $\hat{L}_{\mu\nu}$ and $\hat{\Pi}_\mu$, it assumes the form

$$\hat{\mathcal{C}}_{dS} = \eta^{\alpha\beta}\, \hat{\Pi}_\alpha\, \hat{\Pi}_\beta - \frac{1}{2l^2}\, \eta^{\alpha\beta}\, \eta^{\gamma\delta}\, \hat{L}_{\alpha\gamma}\, \hat{L}_{\beta\delta}. \tag{3.31}$$

In the contraction limit $l \to \infty$, the de Sitter generators reduce to the Poincaré generators, and the de Sitter Casimir reduces to the corresponding Poincaré Casimir.

The kinematic group of ordinary special relativity is the Poincaré group. The first Casimir operator of the Poincaré group is

$$\hat{\mathcal{C}}_P = \eta^{\alpha\beta}\, \hat{P}_\alpha \hat{P}_\beta, \tag{3.32}$$

where $\hat{P}_\alpha$ are the translation generators. For a particle of mass m belonging to an irreducible representation of the Poicaré group, its eigenvalue is $\mathcal{C}_P = m^2$. From the identity $\hat{\mathcal{C}}_P = \mathcal{C}_P$, the dispersion relation of ordinary special relativity is found to be

$$\eta_{\alpha\beta}\, p^\alpha p^\beta = m^2, \tag{3.33}$$

where p^α is the conserved charge related to the translation symmetry generated by $\hat{P}_\alpha$.

In the same token, for particles belonging to representations of the *principal series*, the eigenvalue $\mathcal{C}_{dS}$ of the first Casimir operator of the de Sitter group is [32]

$$\mathcal{C}_{dS} = m^2 - \frac{1}{l^2}\,[\mathsf{s}(\mathsf{s}+1) - 2] \equiv \mu^2, \tag{3.34}$$

with m the mass and s the spin of the field under consideration. From the identity

$\hat{\mathcal{C}}_{dS} = \mathcal{C}_{dS}$, the dispersion relation of the de Sitter relativity is consequently

$$\eta_{\alpha\beta}\, \pi^\alpha \pi^\beta - \frac{1}{2l^2}\, \eta_{\alpha\beta}\, \eta_{\gamma\delta}\, l^{\alpha\gamma}\, l^{\beta\delta} = \mu^2, \tag{3.35}$$

where π_α and $l_{\alpha\gamma}$ are, respectively, the de Sitter momentum and the angular momentum of the particle. In stereographic coordinates $\{x^\alpha\}$, π_α issues the form

$$\pi^\alpha \equiv p^\alpha - \frac{1}{4l^2} k^\alpha, \tag{3.36}$$

where p^α is the ordinary momentum, and

$$k^\alpha = \left(2\eta_{\gamma\delta}\, x^\gamma x^\alpha - \sigma^2 \delta^\alpha_\delta\right) p^\delta \tag{3.37}$$

is the proper conformal momentum. The angular momentum, on the other hand, has the usual form

$$l^{\alpha\gamma} = x^\alpha p^\gamma - x^\gamma p^\alpha. \tag{3.38}$$

Now, due to the fact that de Sitter spacetime is isotropic and homogeneous, it has no preferred origin, and the angular momentum $l^{\alpha\beta}$ vanishes for free particles, that is, for particles moving along a geodesic of the de Sitter spacetime. This is similar to what happens in Minkowski spacetime, which is also isotropic and homogeneous. The de Sitter dispersion relation, therefore, is actually given by

$$\eta_{\alpha\beta}\, \pi^\alpha \pi^\beta = \mu^2. \tag{3.39}$$

In addition to the change in the definition of momentum, the mass in de Sitter special relativity turns out to be replaced by the eigenvalue μ, which depends on the spin of the particle. This is related to the fact that the de Sitter group is essentially a “rotation” group.

Chapter 4

de Sitter General Relativity

One of the principal ideas of this work is that the de Sitter group generalizes the Poincaré group for *high–energy* kinematics, just in the same way that the Poincaré group generalizes the Galilei group for *high–velocity* kinematics. In other words, ordinary special relativity is here replaced by de Sitter relativity.

Now, we make a step ahead and stop thinking that the cosmological constant Λ is a free parameter, instead, it can be determined in terms of other physical quantities that describes the system. When we apply this idea in some particular examples, we obtain very interest result. It can, for this reason, be considered a new paradigm to approach the quantum gravity problem.

The first questions that appears is: given a physical system, how to obtain the associated cosmological term? As a motivation, we can make an approach on how we can do this, then, in the following sections we will develop a mechanism to find the Cosmological constant of a system.

Fist remember that according to quantum mechanics, there is a lower limit for all of the physical quantities of a system, also, the uncertainty principle doesn't allows to take arbitrary values in this quantities. For example, the smallest amount of an electromagnetic field, a photon, is determined by the Planck constant as a quantum of the field. In a similar fashion, the smallest possible length for the de Sitter radius l is the Planck length $l_P = \sqrt{G\hbar/c^3}$.[1] Let us then consider a de Sitter spacetime with $l = l_P$, for which the corresponding cosmological term is

$$\Lambda_P = \frac{3}{l_P^2}. \tag{4.1}$$

Considering that a cosmological term represents ultimately an energy density, we define the Planck energy density

$$\varepsilon_P = \frac{m_P\, c^2}{(4\pi/3) l_P^3}\,, \tag{4.2}$$

[1] From here on, we will use $c \neq 1$ for the speed of light.

with $m_P = \sqrt{c\hbar/G}$ the Planck mass. In terms of ε_P, Eq. (4.1) assumes the form

$$\Lambda_P = \frac{4\pi G}{c^4}\,\varepsilon_P\,. \tag{4.3}$$

Now, the very definition of Λ_P is very particular and is for the extremal case of systems with such a high energy density. However, we can try to generalize this result to physical systems with energy density ε. Accordingly, the associated "cosmological" term is

$$\Lambda = \frac{4\pi G}{c^4}\,\varepsilon. \tag{4.4}$$

It is important to reinforce that the ε appearing in this equation is not the dark energy density and it has nothig to with it, it is simply the matter energy density of the system itself. For small values of ε, the local cosmological term Λ will be small, spacetime will approach Minkowski, and de Sitter special relativity will approach ordinary special relativity, whose kinematics is governed by the Poincaré group.[2]

In the following, we will develop a mechanism to find the cosmological constant for some systems (gravitational systems).

4.1 Conserved Source Currents

In order to comply with de Sitter relativity, any physical system must engender on spacetime a local cosmological term. This means that spacetime must present a local kinematic-related curvature. We have then to verify whether the presence of this kinematic curvature is consistent with general relativity, the theory that governs the spacetime dynamics.

Due to the transitivity properties of the de Sitter spacetime, de Sitter relativity naturally incorporates the proper conformal generators in the definition of spacetime transitivity. As a consequence, a conformal current will appear as part of the Noether conserved current, producing a change in the very notions of energy and momentum [29]. To see that, let us consider a general matter field with Lagrangian $\mathcal{L}_m$. Its action integral is

$$S_m = \frac{1}{c}\int \mathcal{L}_m\, d^4x. \tag{4.5}$$

Under a local spacetime transformation δx^ρ, the change in S is

$$\delta S_m = -\frac{1}{2c}\int T^{\mu\nu}\,\delta g_{\mu\nu}\,\sqrt{-g}\,d^4x, \tag{4.6}$$

[2]We remark that the relation (4.4) between the local value of Λ and the energy density ε is the same as that appearing in Einstein universe. See, for example, Ref. [33], page 104.

where

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}} \tag{4.7}$$

is the symmetric energy-momentum tensor. Although the coefficient of the variation is the energy-momentum tensor, the conserved quantity depends on the transformation δx^ρ. For example, invariance of the action under translations $\delta x^\rho = \xi^\rho(x)$ leads to the conservation of $T^{\mu\nu}$ itself, whereas the invariance under a Lorentz transformation $\delta x^\rho = \omega^\rho{}_\lambda(x)\, x^\lambda$ leads to the conservation of the *total* angular momentum tensor [27]

$$J^{\rho\mu\nu} = x^\mu\, T^{\rho\nu} - x^\nu\, T^{\rho\mu}. \tag{4.8}$$

Now, when the local kinematics is assumed to be ruled by the de Sitter group, the underlying spacetime is necessarily a de Sitter spacetime. As already said, that spacetime is not transitive under ordinary translations, but under the so called de Sitter "translations", whose infinitesimal version is

$$\delta x^\rho = \omega^\alpha(x) \xi_\alpha{}^\rho, \tag{4.9}$$

where $\omega^\alpha(x)$ is the transformation parameter and[3]

$$\xi_\alpha{}^\rho = \delta^\rho_\alpha - \frac{1}{4l^2}\left(2\eta_{\alpha\nu}\, x^\nu x^\rho - \sigma^2 \delta^\rho_\alpha\right) \equiv \delta^\rho_\alpha - \frac{1}{4l^2} \bar{\delta}_\alpha{}^\rho \tag{4.10}$$

represent the Killing vector components, with $\sigma^2 = \eta_{\mu\nu}\, x^\mu x^\nu$. Under such a transformation, the metric tensor changes according to

$$\delta g_{\mu\nu} = -\nabla_\nu[\omega^\alpha(x)\xi_{\alpha\mu}] - \nabla_\mu[\omega^\alpha(x)\xi_{\alpha\nu}], \tag{4.11}$$

with ∇_ν a covariant derivative in the spacetime metric. Using the fact that the $\xi_{\alpha\mu}$'s are Killing vectors, this can be rewritten in the form

$$\delta g_{\mu\nu} = -\xi_{\alpha\mu} \nabla_\nu \omega^\alpha(x) - \xi_{\alpha\nu} \nabla_\mu \omega^\alpha(x). \tag{4.12}$$

Substituting in Eq. (4.6), the invariance of the action yields the conservation law

$$\nabla_\mu \Pi^{\mu\nu} = 0, \tag{4.13}$$

where

$$\Pi^{\mu\nu} \equiv T^{\mu\alpha} \xi_\alpha{}^\nu = T^{\mu\nu} - \frac{1}{4l^2}\, K^{\mu\nu}, \tag{4.14}$$

with $T^{\mu\nu}$ the symmetric energy–momentum tensor, and $K^{\mu\nu}$ the proper conformal current [30]

$$K^{\mu\nu} \equiv T^{\mu\alpha} \bar{\delta}_\alpha{}^\nu = T^{\mu\alpha}\left(2\eta_{\alpha\rho}\, x^\rho x^\nu - \sigma^2 \delta_\alpha{}^\nu\right). \tag{4.15}$$

[3]For ordinary translations, as is well known, the Killing vectors reduce to δ^ρ_α.

When the underlying spacetime is the de Sitter spacetime, the covariant conserved source is the projection of the energy–momentum tensor $T^{\mu\alpha}$ along the Killing vector ${\xi_\alpha}^\mu$.

In general, neither $T_{\mu\nu}$ nor $K_{\mu\nu}$ is conserved separately. In fact, as an explicit calculation shows,

$$\nabla_\mu T^{\mu\nu} = \frac{2\,{T^\rho}_\rho\, x^\nu}{4l^2 - \sigma^2} \qquad \text{and} \qquad \nabla_\mu K^{\mu\nu} = \frac{2\,{T^\rho}_\rho\, x^\nu}{1 - \sigma^2/4l^2}. \tag{4.16}$$

Only when the trace of the energy-momentum tensor vanishes are the currents $T^{\mu\nu}$ and $K^{\mu\nu}$ separately conserved, which is something expected because the trace of energy-momentum tensor is proportional to the mass, then, if the mass is zero conformal symmetry must appear. In the formal limit of a vanishing cosmological term (corresponding to $l \to \infty$), we obtain

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \text{and} \qquad \nabla_\mu K^{\mu\nu} = 2\,{T^\rho}_\rho\, x^\nu. \tag{4.17}$$

On the other hand, in the formal limit of an infinite cosmological term (corresponding to $l \to 0$), we get

$$\nabla_\mu T^{\mu\nu} = -\,2\,{T^\rho}_\rho\, \frac{x^\nu}{\sigma^2} \qquad \text{and} \qquad \nabla_\mu K^{\mu\nu} = 0. \tag{4.18}$$

In this limit, physics becomes conformally invariant, and the proper conformal current turns out to be conserved.

4.2 Second Bianchi Identity

Let us consider the gravitational action functional

$$S_g = -\,\frac{c^3}{16\pi G}\int R\,\sqrt{-g}\;d^4x. \tag{4.19}$$

Up to a surface term, the variation of this action is

$$\delta S_g = -\,\frac{c^3}{16\pi G}\int G^{\mu\nu}\,\delta g_{\mu\nu}\,\sqrt{-g}\,d^4x, \tag{4.20}$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2}\,g^{\mu\nu}\,R \tag{4.21}$$

is Einstein's tensor. For the specific case of de Sitter "translations", in which the metric tensor transforms according to Eq. (4.12), we get

$$\delta S_g = -\,\frac{c^3}{16\pi G}\int \nabla_\mu \left(G^{\mu\alpha}{\xi_\alpha}^\nu\right)\xi_\nu(x)\,\sqrt{-g}\,d^4x. \tag{4.22}$$

From the invariance of the action, and considering the arbitrariness of the local parameters $\xi_\nu(x)$, we obtain

$$\nabla_\mu \left(G^{\mu\alpha} \xi_\alpha{}^\nu\right) = 0. \tag{4.23}$$

In the case of ordinary general relativity, whose underlying kinematics is ruled by the Poincaré group, the Killing vectors are simply δ^ν_α, and Eq. (4.23) reduces to the usual contracted form of the second Bianchi identity [34]. When the underlying kinematics is ruled by the de Sitter group, however, what is covariantly conserved is the projection of $G^{\mu\alpha}_{(\mathrm{T})}$ along the Killing vector $\xi_\alpha{}^\nu$. This projection represents Einstein's tensor of the *total* spacetime curvature: the dynamic curvature generated by $T^{\mu\nu}$, and the kinematic curvature associated to the local de Sitter spacetime. It is, for this reason, the Bianchi identity consistent with de Sitter special relativity.

4.3 Cosmological Term as a Kinematic Effect

In ordinary general relativity, spacetime is assumed to reduce locally to Minkowski, whose kinematics is ruled by the Poincaré group, and whose homogeneity is related to the translational Killing vectors δ^ρ_α. As a consequence, Einstein equation is written in the form

$$\delta^\rho_\alpha \, G^{\alpha\mu} = \frac{8\pi G}{c^4} \, \delta^\rho_\alpha \, T^{\alpha\mu}, \tag{4.24}$$

with $G^{\alpha\mu} = R^{\alpha\mu} - \frac{1}{2} h^{\alpha\mu} R$ the Einstein tensor. If instead of Poincaré, local kinematics is to be ruled by the de Sitter group, spacetime must reduce locally to de Sitter, whose homogeneity is defined by the "translational" Killing vectors ξ^α_μ. In this case, Einstein equation must be generalized to

$$\xi^\rho_\alpha \, G^{\alpha\mu} = \frac{8\pi G}{c^4} \, \xi^\rho_\alpha \, T^{\alpha\mu}. \tag{4.25}$$

Using the Killing vectors (4.10), as well as the definition (4.14), the field equation (4.25) can be rewritten in the form

$$G^{\rho\mu}_T - \frac{1}{4l^2} \, G^{\rho\mu}_K = \frac{8\pi G}{c^4} \left(T^{\rho\mu} - \frac{1}{4l^2} \, K^{\rho\mu}\right), \tag{4.26}$$

where we have introduced the notations

$$G^{\rho\mu}_T = \delta^\rho_\alpha \, G^{\alpha\mu} \qquad \text{and} \qquad G^{\rho\mu}_K = \bar{\delta}^\rho_\alpha \, G^{\alpha\mu}. \tag{4.27}$$

In the limit $l \to \infty$, it reduces to the usual Einstein equation, which is consistent with ordinary special relativity.

It is important to note that the replacement of ordinary (Poincaré-based) special relativity by de Sitter relativity changes only the local spacetime symmetry, not the dynamics of general relativity. As a matter of fact, it changes only the strong equivalence principle, which states now that in any locally inertial frame, the non-gravitational laws of physics are those of de Sitter special relativity. This means essentially that Einstein equation

$$G_T^{\rho\mu} = \frac{8\pi G}{c^4}\, T^{\rho\mu}, \tag{4.28}$$

which describes the dynamic spacetime curvature, and the kinematic equation

$$G_K^{\rho\mu} = \frac{8\pi G}{c^4}\, K^{\rho\mu}, \tag{4.29}$$

which is an equation describing the de Sitter effects on the spacetime geometry, can be solved separately. Of course, the total spacetime curvature will always be composed of two parts: a dynamic part $G_T^{\rho\mu}$, whose source is the energy-momentum tensor of matter, and a kinematic part $G_K^{\rho\mu}$, whose source is the proper conformal current of matter. The importance of each part depends on the value of the pseudo-radius l.

Now, since in the present context $G_K^{\rho\mu}$ represents the curvature of the underlying de Sitter spacetime, it depends necessarily on l. As a matter of fact, if we recall that, for a de Sitter spacetime,

$$G^{\alpha\mu} = \frac{3}{l^2}\, g^{\alpha\mu}, \tag{4.30}$$

and taking into account that $\Lambda = 3/l^2$, with Λ the cosmological term, we see from Eq. (4.27) that

$$G_K^{\rho\mu} = -\Lambda\, \bar{\delta}^\rho_\alpha\, g^{\alpha\mu}. \tag{4.31}$$

Substituting into Eq. (4.29), and taking the trace on both sides, we get

$$\Lambda = \frac{4\pi G}{c^4} \frac{K_\mu{}^\mu}{\sigma^2}. \tag{4.32}$$

Using Eq. (4.15), the cosmological term is then found to be given by

$$\Lambda = \frac{4\pi G}{c^4} \left(2\, T_{\rho\mu}\, u^\rho u^\mu - T^\mu{}_\mu\right), \tag{4.33}$$

where

$$u^\mu = \frac{x^\mu}{s} \tag{4.34}$$

is the four-velocity in *comoving coordinates*, with s the invariant form $s = (g_{\mu\nu} x^\mu x^\nu)^{1/2} \equiv \Omega(x)\, \sigma$. In fact, observe that the variation of the four-velocity of a given point of spacetime depends only on the change of the invariant form s.

According to de Sitter special relativity, therefore, the source of the local de Sitter spacetime, or equivalently, of the local value of the cosmological term, is not the energy-momentum current, but the (trace of the) proper conformal current of ordinary matter. It should be noticed, however, that there is a crucial difference between the local de Sitter spacetime underlying all physical systems, and the usual notion of de Sitter spacetime: since equation (4.33) is purely kinematics, and considering that the conformal current $K^{\rho\mu}$ vanishes outside the source, in this region the local de Sitter spacetime becomes the flat Minkowski space. This is quite similar to a proposal made by Mansouri in a different context [35].

Chapter 5

Some Application of de Sitter Special and General Relativity

In the last two chapter, we introduce the de Sitter special relativity and its extension to a de Sitter general relativity. We are going to use the proposals of these theories in some applications at cosmological level and a high energy scales.

5.1 The de Sitter Scenario for a Variable Λ

As we have seen, what is conserved in de Sitter special relativity is a combination of energy-momentum and proper conformal currents. Considering that neither of these currents is individually conserved, energy-momentum can be transformed into proper conformal current, and vice-versa. Since the proper conformal current appears as the source of Λ, this means that ordinary matter can be transformed into dark energy, and vice versa. This mechanism provides a natural scenario for an evolving Λ. When applied to the universe as a whole, it predicts the existence of a pervading cosmological term whose value, at each time, is determined by the matter content of the universe [36].

Let us begin by considering a universe with an infinite cosmological term. Such universe is obtained in the limit $l \to 0$,[1] under which spacetime becomes a flat, singular cone-spacetime, transitive under proper conformal transformations [19]. Its kinematic group is the conformal Poincaré group, the semi-direct product between Lorentz and the proper conformal groups [26]. In this limit, physics becomes conformal invariant, and the proper conformal current turns out to be conserved (see section 4.1). Observe that, since this spacetime is transitive under proper conformal transformations only, the usual notions of space and time will not be present. As a consequence, the energy-momentum current cannot be defined, which means that the degrees of freedom of gravity are turned off. Only the degrees of freedom associated with the proper conformal transformations are active.[2] It is also interesting to observe that, under such extreme situation, the

[1] We remark that, in order to consider the limit $l \to 0$, a different parametrization of the generators must be used. In fact, instead of the parametrization (2.63), which is appropriate to study the limit $l \to \infty$, it would be necessary to use $\hat{\Pi}_\rho \equiv l\hat{L}_{\rho 4}$ [19].

[2] This is somewhat similar to the "Weyl Curvature Hypothesis" of Penrose [37].

thermodynamic properties of the de Sitter horizon fit quite reasonably to what one would expect for an initial condition for the universe: it has an infinite temperature, vanishing entropy, and vanishing energy; the energy density, however, is infinite [38].

It is important to remark that the limit $l \to 0$ is purely formal in the sense that quantum effects preclude it to be fully performed. The outcome of this limit, that is, the cone spacetime, must be thought of as the frozen geometrical structure behind the quantum spacetime fluctuations taking place at the Planck scale. A more realistic situation is to assume that the de Sitter pseudo-radius l is of the order of the Planck length l_P, in which case the cosmological term would have the value

$$\Lambda_P \equiv \frac{3}{l_P^2} \simeq 1.15 \times 10^{66}\,\mathrm{cm}^{-2}. \tag{5.1}$$

Notice that, due to the fact that spacetime becomes locally a high-Λ de Sitter spacetime, it will naturally be endowed with a causal de Sitter horizon, and consequently with a holographic structure [39]. As the Λ term decays and the universe expands, the proper conformal current is gradually transformed into energy-momentum current, giving rise to a Friedmann-Robertson-Walker universe. Concomitantly, spacetime becomes transitive under a combination of translations and proper conformal transformations. This means that our usual (translational) notion of space and time begin to emerge, though at this stage spacetime is still preponderantly transitive under proper conformal transformation.

As the proper conformal current is continuously transformed into energy-momentum current, the matter content of the universe turns out to be described by a sum of many different components. As a simple example, we consider here a one-component perfect fluid, whose energy-momentum tensor is of the form

$$T^\rho{}_\mu = (\varepsilon + p)\, u^\rho u_\mu - p\, \delta^\rho{}_\mu, \tag{5.2}$$

with ε the energy density, and p the pressure. Substituting in Eq. (4.33), it yields the cosmological term

$$\Lambda = \frac{4\pi G}{c^4}\,(\varepsilon + 3p). \tag{5.3}$$

Observe that, even though Λ can decay to small values, it cannot vanish.[3] This means that, similar to the limit $l \to 0$, the limit $l \to \infty$ cannot be accomplished either because it would lead to an empty universe represented by Minkowski spacetime. At the present time, the matter content of the universe can be accurately described by dust, which is characterized by $p = 0$. In this case, expression (5.3) yields

$$\Lambda = \frac{4\pi G}{c^4}\,\varepsilon. \tag{5.4}$$

[3] It is interesting to note that, in order to have a vanishing Λ, it would be necessary to suppose the existence of an exotic fluid satisfying the unphysical equation of state $\varepsilon = -3p$.

Using the current value for the energy density of the universe, we get

$$\Lambda \simeq 10^{-56}\ \mathrm{cm}^{-2}, \tag{5.5}$$

which is consistent with recent observations [40, 41, 42, 43]. For such value of Λ, spacetime becomes preponderantly transitive under ordinary spacetime translations.

Now, as is well known, in order to allow the formation of the cosmological structures we observe today, the universe must necessarily have passed through a period of non-accelerating expansion, which means that the cosmological term must have decayed to a tiny value during some cosmological period in the past. The currently observed value (5.5), however, indicate that the universe is presently entering an accelerated expansion era. Even though the reason why the universe is entering this accelerated expansion is an open question, the de Sitter special relativity leaves room for some speculations. For example, although the total energy $E = E_T - (1/4l^2)E_K$ is conserved, the translational energy E_T alone is not, in the sense that it can flow to E_K, and vice versa. As discussed above, at the initial instant of the universe, characterized by the huge cosmological term (5.1), most of the energy was in the form of conformal energy E_K, and E_T was very small. As the Λ term decays and the universe expands, E_K flows to E_T, and the entropy experience a continuous increase. This process can proceed until most of the energy is in the form of translational energy, and the cosmological term acquires a very small value.

Then comes the point: the existence of the conformal degrees of freedom allows the inverse process to occur without violating the energy conservation. In this process, E_T flows back to E_K, entropy begins decreasing, and the value of Λ would concomitantly grow up. In fact, considering that the entropy S associated with the de Sitter horizon is given by [44]

$$S = \frac{k_B\, A_h}{4\, l_P^2} \sim k_B\, \frac{l^2}{l_P^2}, \tag{5.6}$$

with k_B the Boltzmann constant and $A_h = 4\pi l^2$ the area of the horizon, a decrease in the entropy would imply a decrease in the de Sitter pseudo-radius l, and consequently an increase in the value of $\Lambda = 3/l^2$.[4] A possible source for this mechanism could be a loss of information occurring in black holes, which could result in a decrease of the universe entropy [46]. We reinforce that this mechanism turns out to possible just because there are two notions of energy, which can be transformed into each other. Of course, the details of these qualitative speculations have yet to be worked out, but they are anyway useful for getting new insights on the physics underlying the de Sitter special relativity.

An interesting example of such new insight refers to the fact that an accelerated expansion does not mean necessarily that the universe is breaking apart, but rather that it is moving back towards its initial state, characterized by an infinite cosmological term.

[4]Notice that, in this context, the Boltzmann constant represents the entropy of a de Sitter spacetime with $l = l_P$. It can be considered, in this sense, a quantum of entropy [45].

As we have already discussed, such state is represented by a flat, singular cone-spacetime, transitive under proper conformal transformations, in which all energy is in the form of proper conformal current (or equivalently, in the form of dark energy). According to the de Sitter relativity, therefore, in order to return to its initial state, the universe expansion does not need to stop, and then contracts back. Instead, it is through an accelerated expansion that the universe will be lead back to its initial state. We notice in passing that this picture points to a cyclic view of the universe, with the timelike future singular, conformal spacetime being identified with the initial state of the next era.

5.2 Photon Kinematics in de Sitter Relativity

According to quantum gravity considerations, high energies might cause small–scale fluctuations in the texture of spacetime. These fluctuations could, for example, act as small–scale lenses, interfering in the propagation of ultra–high energy photons. The higher the photon energy, the more it changes the spacetime structure, the larger the interference will be. This kind of mechanism could be the cause of the recently observed delay in high energy gamma–ray flares from the heart of the galaxy Markarian 501 [6]. Those observations compared gamma rays in two energy ranges, from 1.2 to 10 TeV, and from 0.25 to 0.6 TeV. The first group arrived on Earth four minutes later than the second. Since de Sitter relativity gives a precise meaning to these local spacetime fluctuations, it provides a precise high energy phenomenology, opening up the door for experimental predictions.

With this in mind, let us consider a photon of wavelength λ and energy $E = hc/\lambda$. Although the photons in a gamma–ray beam are not necessarily in thermal equilibrium, we are going to use the thermodynamic expression [47]

$$\varepsilon = \frac{\pi^2}{15} \frac{(kT)^4}{(\hbar c)^3} \tag{5.7}$$

to estimate the photons energy density. Setting $kT = E$, it becomes

$$\varepsilon = \frac{\pi^2}{15} \frac{E^4}{(\hbar c)^3}. \tag{5.8}$$

Substituting in Eq. (5.3) and using the fact that for a photon gas $\varepsilon = 3p$, we obtain

$$\Lambda \simeq \frac{8\pi^3}{15\hbar^2 c^2} \frac{E^4}{E_P^2}, \tag{5.9}$$

where $E_P = \sqrt{c^5\hbar/G}$ is the Planck energy. The corresponding de Sitter length parameter is given by

$$l = \sqrt{3/\Lambda}. \tag{5.10}$$

To get an idea of the order of magnitude, we give in Table 1 the local values of l and Λ for several photons with different wavelength λ.[5] In the first line are the values for a photon with energy of the order of the Planck energy. Gamma-rays (1) and (2) correspond to the two observed gamma-ray flares from Markarian 501. For comparison purposes, we give also the values for a visible (red) photon.

	E (GeV)	λ (cm)	l (cm)	Λ (cm^{-2})
Planck photon	1.2×10^{19}	1.0×10^{-32}	9.7×10^{-34}	5.9×10^{66}
Gamma-ray (1)	1.0×10^{4}	1.2×10^{-17}	1.4×10^{-3}	2.8×10^{6}
Gamma-ray (2)	0.6×10^{3}	2.1×10^{-16}	3.8×10^{-1}	3.7×10^{1}
Red light	1.8×10^{-9}	7.0×10^{-5}	4.5×10^{22}	2.9×10^{-45}

Table 5.1: *Local values of l and Λ for several different photons.*

Since the photons produce such Λ in the place they are located, we can assume that they are always propagating in a de Sitter spacetime with that cosmological term.

5.2.1 Geometric Optics Revisited

In flat spacetime, the condition for geometric optics to be applicable is that

$$\lambda \ll l, \tag{5.11}$$

where l is the typical dimension of the physical system. Since the physical system is now the local de Sitter spacetime produced by the photon, that dimension is given by the de Sitter length parameter l. From Table 1 we see that, for a photon with wavelength of the order of the Planck length, this condition is not fulfilled. However, for gamma-rays (1) and (2), as well as for red light, condition (5.11) is fulfilled, which means that we can use geometric optics to study their propagation.

In the geometric optics domain, any wave-optics quantity A which describes the wave field is given by an expression of the type

$$A = b\, \mathrm{e}^{i\phi}, \tag{5.12}$$

where the amplitude b is a slowly varying function of the coordinates and time, and the phase ϕ, the eikonal, is a large quantity which is *almost linear* in the coordinates and the time. The time derivative of ϕ yields the angular frequency of the wave,

$$\frac{\partial \phi}{\partial t} = \omega, \tag{5.13}$$

[5]The values given in this table where update with respect to the ones given in [20], because the new formula (5.9).

whereas the space derivative gives the wave vector

$$\frac{\partial \phi}{\partial \boldsymbol{r}} = -\,\boldsymbol{k}. \tag{5.14}$$

The characteristic equation for Maxwell's equations in an isotropic (but not necessarily homogeneous) medium of refractive index $n(r)$ is

$$\left(\frac{\partial \phi}{\partial \boldsymbol{r}}\right)^2 - \frac{n^2(r)}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 = 0\ , \tag{5.15}$$

which implies the usual relation

$$\boldsymbol{k}^2 = n^2(r)\,\frac{\omega^2}{c^2}. \tag{5.16}$$

Now, as is well known, there exists a deep relationship between optical media and metrics [48]. This relationship allows to reduce the problem of the propagation of electromagnetic waves in a gravitational field to the problem of wave propagation in a refractive medium in flat spacetime. Let us then consider the specific case of a de Sitter spacetime, for which the quadratic line element ds^2 can be written in the form [21]

$$ds^2 = d\tau^2 - n^2(E)\,\delta_{ij}\,dx^i dx^j, \tag{5.17}$$

where

$$n(E) \equiv \exp\left[\sqrt{\Lambda/3}\;\tau\right], \tag{5.18}$$

with $\tau = ct$. In these coordinates, the metric components are

$$g_{00} = g^{00} = 1, \quad g_{ij} = -\,n^2(E)\,\delta_{ij}, \tag{5.19}$$

and the components of the "conformal" Ricci tensor is

$$R^{\mu}_{(\mathrm{K})\nu} = -\,\Lambda\,\delta^{\mu}{}_{\nu}, \tag{5.20}$$

with Λ given by Eq. (5.9). It is then easy to see that, with the metric components (5.19), the curved spacetime eikonal equation for a $n = 1$ refractive medium,

$$g^{\mu\nu}\,\frac{\partial \phi}{\partial x^{\mu}}\,\frac{\partial \phi}{\partial x^{\nu}} = 0, \tag{5.21}$$

coincides formally with the flat-spacetime eikonal equation (5.15), valid in a medium of refractive index $n(r)$. For this reason, g_{ij} is usually called the *refractive metric*, with $n(E)$ playing the role of refractive index [10].

5.2.2 Electromagnetic Waves in the Geometric Optics Limit

As already remarked, according to de Sitter relativity the photons produce a local de Sitter spacetime in the place they are located. We can then assume that they are always propagating in a de Sitter spacetime, with Λ given by Eq. (5.9). Let us then consider the electromagnetic field equations in a de Sitter spacetime, restricting ourselves to the domain of geometric optics. Denoting the electromagnetic gauge potential by A_μ, and assuming the generalized Lorenz gauge $\nabla_\mu A^\mu = 0$, Maxwell's equation reads

$$\Box A^\mu - R^\mu_{(\mathrm{K})\nu} \, A^\nu = 0, \tag{5.22}$$

where $\Box = g^{\lambda\rho}\nabla_\lambda \nabla_\rho$. Substituting the Ricci tensor components (5.20), Maxwell equation (5.22) becomes

$$\Box A^\mu + \Lambda \, A^\mu = 0. \tag{5.23}$$

Although the term involving the cosmological constant looks like a background-dependent mass for the photon field, this interpretation leads to properties which are physically unacceptable [49]. In fact, as Maxwell equations in four dimensions are conformally invariant, and de Sitter spacetime is conformally flat, the electromagnetic field must propagate on the light–cone [50, 51], which implies a vanishing mass for the photon field.

Assuming a massless photon field, therefore, we take the monochromatic plane-wave solution to the field equation (5.23) to be

$$A_\mu = b_\mu \exp[i \, k_\alpha \, x^\alpha], \tag{5.24}$$

where b_μ is a polarization vector, and $k_\alpha = (\omega(|\boldsymbol{k}|)/c, -\boldsymbol{k})$ is the wave-number four-vector, with $\omega(|\boldsymbol{k}|)$ the angular frequency. In order to be a solution of equation (5.23), the following dispersion relation must be satisfied,

$$\omega(k) = \frac{c}{n(E)} \left[k^2 + n^2(E) \, \Lambda\right]^{1/2}, \tag{5.25}$$

where we used the notation $k = |\boldsymbol{k}|$. Considering that

$$\frac{1}{n(E) \, \Lambda^{1/2}} \sim l, \tag{5.26}$$

with l the dimension of the local de Sitter spacetime, and remembering that $k \sim \lambda^{-1}$, the condition (5.11) for geometric optics to be applicable turns out to be

$$k \gg n(E) \, \Lambda^{1/2}. \tag{5.27}$$

In this domain, therefore, the dispersion relation (5.25) assumes the form

$$\omega(k) = c \, \frac{k}{n(E)}, \tag{5.28}$$

and the corresponding velocity of propagation of an electromagnetic wave, given by the group velocity, is [52]

$$v \equiv \frac{d\omega(k)}{dk} = \frac{c}{n(E)}. \tag{5.29}$$

In the limit $\Lambda \to 0$, which corresponds to a contraction from de Sitter to ordinary special relativity, $n(E) \to 1$, and there will be no effect on the photon propagation.

5.2.3 Application to the Gamma–Ray Flares

Let us consider now the propagation of gamma–rays. Substituting the local cosmological term (5.9) into the refractive index (5.18), we obtain

$$n(E) \simeq \exp\left[\sqrt{\frac{8\pi^3}{45\hbar^2c^2}\,\frac{E^2}{E_P}\,\tau}\right]. \tag{5.30}$$

For the local de Sitter spacetime produced by a photon, the length τ can be identified with its own wavelength $\lambda = hc/E$. Hence, we get

$$n(E) \simeq \exp\left[\sqrt{\frac{32\pi^5}{45}\,\frac{E}{E_P}}\right]. \tag{5.31}$$

For energies small compared to E_P, we can write

$$n(E) \simeq 1 + \sqrt{\frac{32\pi^5}{45}\,\frac{E}{E_P}}. \tag{5.32}$$

For a visible (red) electromagnetic radiation,

$$n_{(\mathrm{red})} \simeq 1 + 2.2 \times 10^{-27}. \tag{5.33}$$

For gamma–rays (1) and (2), we get, respectively,

$$n_{(1)} \simeq 1 + 1.2 \times 10^{-14} \quad \text{and} \quad n_{(2)} \simeq 1 + 7.2 \times 10^{-16}. \tag{5.34}$$

Taking into account that the velocity of each photon is given by Eq. (5.29), the time difference Δt to travel a distance d will be

$$\Delta t = \frac{d}{c}\left[n_{(1)} - n_{(2)}\right]. \tag{5.35}$$

Using the refractive indices (5.34), we see that, for a distance of 500 millions light–year, which corresponds to $d = 4.7 \times 10^{26}$ cm, the time difference will be

$$\Delta t \simeq 178 \text{ s} = 2.96 \text{ min}. \tag{5.36}$$

This is of the same order of magnitude of the observed delay between the two gamma–ray flares originated from the center of the galaxy Markarian 501 [6].

5.3 Compton Scattering

The conserved quantities π^α can be used to study the scattering processes subject to invariance under the de Sitter group. An important property of the conserved charges is that they are additive and its meaning for a multi-particle process is also well defined. In a generic scattering process involving n particles we can use conservation of the de Sitter energy and momentum to solve the dynamics of the process [53]. Additionally, we can use separately the dispersion relation for each particle in the process once we can treat them as free particles before and after the interaction. The system of equations describing any scattering process is

$$\sum_i^n E_i = E_{total} \quad \text{and} \quad \sum_i^n \boldsymbol{\pi}_i = \boldsymbol{\pi}_{total}, \tag{5.37}$$

with $\pi^\alpha = (E, \boldsymbol{\pi})$. Asymptotically, the electron dispersion relation is

$$E_e^2 = \boldsymbol{\pi}_e^2 + \mu_e^2, \tag{5.38}$$

whereas the photon dispersion relation is given by

$$E_{ph}^2 = \boldsymbol{\pi}_{ph}^2. \tag{5.39}$$

As a simple, but illustrative example, we are going to use the above conservation laws and the dispersion relation to study the scattering of a photon and an electron. To this end, it is necessary to solve the energy and momentum conservation equations, which are given by

$$E_{ph}^{(0)} + E_e^{(0)} = E_{ph} + E_e \tag{5.40}$$

$$\boldsymbol{\pi}_{ph}^{(0)} + \boldsymbol{\pi}_e^{(0)} = \boldsymbol{\pi}_{ph} + \boldsymbol{\pi}_e, \tag{5.41}$$

with the superscript $^{(0)}$ denoting the initial energy and momentum. To begin with, we recall that the electromagnetic field is conformal invariant. Considering that the de Sitter spacetime is conformally flat, the energy of the photon in the de Sitter relativity is the same as in Minkowski spacetime. Namely, it is given by $E_{ph} = \nu$. We then suppose that initially ($t = 0$) the electron is at rest in the origin ($x^i = 0$). In this case, the conservation laws reduce to

$$(\nu^{(0)} - \nu) + \mu_e = \left(\boldsymbol{\pi}_e^2 + \mu_e^2\right)^{1/2} \tag{5.42}$$

$$\boldsymbol{\pi}_{ph}^{(0)} - \boldsymbol{\pi}_{ph} = \boldsymbol{\pi}_e, \tag{5.43}$$

where we have used the dispersion relation (5.38) for the electron in the first equation. Squaring both equations and taking into account that $\boldsymbol{\pi}_{ph}^2 = \nu^2$, we obtain

$$[(\nu^{(0)} - \nu) + \mu_e]^2 = \boldsymbol{\pi}_e^2 + \mu_e^2 \tag{5.44}$$

$$\nu^2 + (\nu^{(0)})^2 - 2\nu\,\nu^{(0)} \cos\theta = \boldsymbol{\pi}_e^2, \tag{5.45}$$

where θ is the angle between the incident and the scattered photons. Adding μ_e^2 to both sides of the second equation, and equating the left hand sides of both equations, we obtain the following relation between the wavelengths of the incident and the scattered photons:

$$\frac{1}{\mu_e}\,(1 - \cos\theta) = \lambda - \lambda^{(0)}. \tag{5.46}$$

This result resembles the standard one of ordinary special relativity,

$$\frac{1}{m_e}\,(1 - \cos\theta) = \lambda - \lambda^{(0)}, \tag{5.47}$$

but with a crucial difference: the Poincaré mass m_e is replaced by the de Sitter mass μ_e, which from Eq. (7.11) with $\mathsf{s} = 1/2$ is found to be [6]

$$\mu_e^2 = m_e^2 + \frac{5\hbar^2}{4c^2l^2} \equiv m_e^2\left(1 + \frac{5\lambda_e^2}{4l^2}\right), \tag{5.48}$$

where $\lambda_e = \hbar/(m_e c)$ is the (reduced) Compton wavelength of the electron. We see from this expression that, for l of the order of λ_e, the de Sitter corrections to the Compton scattering turn out to be of the same order of the electromagnetic effect, and consequently detectable. In terms of Planck length, on the other hand, we get

$$\mu_e^2 = m_e^2\left(1 + \frac{5}{4\alpha_G}\frac{l_P^2}{l^2}\right), \tag{5.49}$$

where $\alpha_G = Gm_e^2/(\hbar c) \equiv m_e^2/m_P^2$ is the gravitational analog of the fine structure constant, with m_P the Planck mass. In the limit $l \to \infty$, the eigenvalue μ_e reduces to the mass m_e, and we recover the result of ordinary special relativity.

However, for small values of l, or equivalently, for large values of Λ, which according

[6] We re-insert now the fundamental constants $\hbar$ and c to match the usual unities.

to the de Sitter special relativity corresponds to high energy-density phenomena, the corrections can be significant. For example, let us consider a high energy photon. Even though the photons in a beam are not necessarily in thermal equilibrium, we can use the thermodynamic expression [47]

$$\varepsilon = \frac{\pi^2}{15} \frac{E^4}{(\hbar c)^3} \tag{5.50}$$

to estimate the photons energy density. Substituting in Eq. (5.9), we obtain

$$\Lambda \simeq \frac{8\pi^3}{15\hbar^2 c^2} \frac{E^4}{E_P^2}, \tag{5.51}$$

where $E_P = \sqrt{c^5\hbar/G}$ is the Planck energy. In the extreme case of a photon with the Planck energy $E = E_P$, the value of Λ in the region occupied by the photon will be

$$\Lambda \simeq 10^{66} \text{ cm}^{-2}, \tag{5.52}$$

which corresponds to a de Sitter length parameter of order of the Planck length $l \simeq 10^{-33}$ cm. In this case, the eigenvalue of the Casimir invariant turns out to be

$$\mu_e^2 \simeq m_P^2, \tag{5.53}$$

and the Compton scattering expression assumes the form

$$\frac{h}{m_P c} (1 - \cos\theta) \simeq \lambda - \lambda^{(0)}. \tag{5.54}$$

As expected, the Compton shift is very small: it will be at most twice the Planck length, a value obtained when the scattering angle is $\theta = \pi$.

Chapter 6

Spin-2 Fields and Helicity

In this chapter, the different representations of Poincaré group are presented. We will discuss the cases for massless particles where the concept of helicity turns to be the great importance for the different representations [54]. Then we will focus in the case of spin 2, where we can have waves with helicity ± 1 or ± 2, and we will give arguments in favour of the representations with helicity ± 1.

6.1 Representations of the Lorentz group

For every field (or particle) of nature there exists a given representation of the Poincaré group [55, 56], the semi-direct product between Lorentz and the translation groups. The eigenvalues of the translational generators define the mass of the field, whereas the eigenvalues of the Lorentz generators define the spin of the field. The commutation relations of the six Lorentz generators, which we denote here by **a** and **b**, can be written in the form [27]

$$\boldsymbol{a} \times \boldsymbol{a} = i\boldsymbol{a} \tag{6.1}$$

$$\boldsymbol{b} \times \boldsymbol{b} = i\boldsymbol{b} \tag{6.2}$$

$$[a_i, b_j] = 0. \tag{6.3}$$

A general spin s representation of the Lorentz group can be constructed as either a field transforming under an irreducible representation, or as a direct sum of irreducible representations, each characterized by an integer or half-integer A and B, with

$$\boldsymbol{a}^2 = A(A+1) \quad \text{and} \quad \boldsymbol{b}^2 = B(B+1). \tag{6.4}$$

These representations are labeled by the numbers (A, B), where $\mathsf{s} = A + B$. The number of components n of the representation (A, B) is

$$n = (2A+1)(2B+1). \tag{6.5}$$

For massless particles, which is the case we will be interested here,

$$\sigma = B - A, \tag{6.6}$$

represents the helicity of the corresponding wave [57].

When $A \neq B$, the irreducible representation can be written as a direct sum of the form $(A, B) \oplus (B, A)$, with the number of independent components given by $2n$. The simplest example is the (Lorentz invariant) spin-0 field ϕ, which is associated with the one-component representation $(0, 0)$. It satisfies the field equation[1]

$$\Box\, \phi = 0, \tag{6.7}$$

with $\Box = \eta^{\mu\nu}\, \partial_\mu \partial_\nu$ the d'Alembertian operator. The most fundamental representations, however, are the spinor representations. The $(1/2, 0)$ representation has spin $1/2$ and describes a (let us say) left-handed Weyl spinor. The $(0, 1/2)$ representation describes a right-handed Weyl spinor. The linear combination

$$(1/2, 0) \oplus (0, 1/2) \tag{6.8}$$

describes a Dirac spinor. The reason why the spinor representations are the most fundamental is that they can be used to construct, by multiplying them together, any other representation of the Lorentz group.

6.2 A Warming Up Example: The Massless Spin-1 Field

In order to get some insight on the theory of representations, as well as on the corresponding wave equations, let us review the well-known case of a massless vector field.

6.2.1 Potential waves

The first way to construct a spin-1 representation is to consider the direct product

$$(1/2, 0) \otimes (0, 1/2) = (1/2, 1/2), \tag{6.9}$$

where $(1/2, 1/2)$ describes a spin-1 field with $n = 4$ components. The electromagnetic vector potential A_μ transforms according to this representation, and in the Lorenz gauge

[1] We use the Greek alphabet $\mu, \nu, \rho, \cdots = 0, 1, 2, 3$ to denote indices related to spacetime, also known as world indices. The first half of the Latin alphabet $a, b, c, \cdots = 0, 1, 2, 3$ will be used to denote algebraic indices related to the tangent spaces, each one a Minkowski spacetime with metric $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$.

$\partial^\mu A_\mu = 0$ satisfies the wave equation

$$\Box A_\mu = 0. \tag{6.10}$$

The solutions to this equation represent waves with helicity $\sigma = 0$.

6.2.2 Field strength waves

A second way to construct a spin-1 representation follows from the direct product

$$(1/2, 0) \otimes (1/2, 0) = (1, 0) \oplus (0, 0). \tag{6.11}$$

The representation $(1, 0)$ can be identified with an antisymmetric, self-dual second-rank tensor. Analogously, the representation $(0, 1)$ can be identified with an antisymmetric, anti self-dual second-rank tensor. The representation that describes a parity invariant 2-form field is, consequently,

$$(1, 0) \oplus (0, 1). \tag{6.12}$$

The electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $n = 6$ components, transforms under this spin-1 representation. It satisfies the dynamical field equation

$$\partial_\nu F^{\mu\nu} = 0, \tag{6.13}$$

which follows from the lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{6.14}$$

In terms of the potential A_μ, and in the Lorenz gauge, the field equation (6.13) coincides with the potential field equation (6.10).

If an electromagnetic wave has helicity $\sigma = \pm 1$, it cannot be described by a solution of the wave equation (6.10) because the electromagnetic potential A_μ lives in the representation $(1/2, 1/2)$, which describes waves with helicity zero. Rather, it must be identified with the solution of the wave equation (6.13). In fact, since $F_{\mu\nu}$ lives in the representation $(0, 1) \oplus (1, 0)$, its solution will represent waves with helicity $\sigma = \pm 1$. Identifying the electric and magnetic components of the electromagnetic field, respectively, by

$$E^i = F^{0i} \quad \text{and} \quad H^i = \tfrac{1}{2}\, \epsilon^{ijk}\, F_{jk}, \tag{6.15}$$

the field equation (6.13) is found to be equivalent to

$$\Box\, \boldsymbol{E} = 0 \quad \text{and} \quad \Box\, \boldsymbol{H} = 0, \tag{6.16}$$

where $\boldsymbol{E} = (E^i)$ and $\boldsymbol{H} = (H^i)$. Electromagnetic waves are solutions to these equations.

As such, they are not potential waves, but field strength waves.

6.3 Massless spin-2 field

6.3.1 Potential waves

The first way to construct a spin-2 representation is to consider the direct product [58]

$$(1/2,1/2)\otimes(1/2,1/2)=[(1,1)\oplus(0,0)]_s\oplus[(0,1)\oplus(1,0)]_a, \tag{6.17}$$

where the subscripts s and a denote, respectively, the symmetric and the anti-symmetric parts of the direct product. Such representation corresponds to the components of a translational-valued 1-form $\varphi_\nu=\varphi^a{}_\nu P_a$, with 16 components. Conceptually, it is equivalent to a linear perturbation of the tetrad,

$$h^a{}_\nu=e^a{}_\nu+\varphi^a{}_\nu, \tag{6.18}$$

with $e^a{}_\mu$ a trivial background tetrad related to the Minkowski spacetime [10]. As such, it satisfies the condition $\partial_\rho e^a{}_\mu=0$. In the so-called harmonic gauge, in which the field satisfies

$$\partial^\nu\psi^a{}_\nu\equiv\partial^\nu(\varphi^a{}_\nu-\tfrac{1}{2}\,e^a{}_\nu\varphi)=0, \tag{6.19}$$

where $\varphi=e_c{}^\rho\varphi^c{}_\rho$, the field equation for $\psi^a{}_\mu$ assumes the form

$$\Box\psi^a{}_\nu=0. \tag{6.20}$$

This theory, as can be easily verified, is invariant under the gauge transformations

$$\psi'^a{}_\nu=\psi^a{}_\nu-\partial_\nu\xi^a, \tag{6.21}$$

with ξ^a an arbitrary parameter. From the sixteen original components of $\psi^a{}_\mu$, the four coordinate conditions (6.19) and the four constraints implied by the gauge transformations (6.21) eliminate eight components. The invariance of the field equation (6.20) under local Lorentz transformations,

$$\psi'^a{}_\nu=\Lambda^a{}_b\,\psi^b{}_\nu, \tag{6.22}$$

eliminates the additional 6 components of $\psi^a{}_\mu$, reducing the number of independent components to only two, as appropriate for a massless field.

In the study of gravitational waves, however, one usually assumes them to be represented by a symmetric second-rank tensor $\varphi_{\mu\nu}=\varphi_{\nu\mu}$, in which case the representation on the right-hand side of (6.17) reduces to the 10 components representation

$$(1,1)\oplus(0,0). \tag{6.23}$$

Conceptually, this field is equivalent to a linear perturbation of the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + \varphi_{\mu\nu}, \tag{6.24}$$

with $\eta_{\mu\nu}$ the background Minkowski metric. In harmonic coordinates, in which the field satisfies the condition

$$\partial^\nu \psi^\mu{}_\nu \equiv \partial^\nu (\varphi^\mu{}_\nu - \tfrac{1}{2}\, \delta^\mu{}_\nu \varphi) = 0, \tag{6.25}$$

the field equation is found to be

$$\Box \psi_{\mu\nu} = 0. \tag{6.26}$$

This theory, as is well known, is invariant under the gauge transformations

$$\psi'_{\mu\nu} = \psi_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{6.27}$$

with ξ_μ an arbitrary parameter. The four coordinate conditions (6.25) and the four constraints implied by the gauge invariance (6.27) reduce, as appropriate, the original ten components of $\psi_{\mu\nu}$ to only two.

In Minkowski spacetime, $\psi_{\mu\nu}$ and $\psi_{a\mu}$ are related by

$$\psi_{\mu\nu} = e^a{}_\mu \psi_{a\nu} + e^a{}_\nu \psi_{a\mu}. \tag{6.28}$$

It is then easy to see that in this case their field equations, given respectively by Eqs. (6.26) and (6.20), are completely equivalent. This means that the potentials $\psi_{\mu\nu}$ and $\psi_{a\nu}$ describe equivalent free theories. When coupled to gravitation, however, they are no longer equivalent. In the last section we will be back to this point.

6.3.2 Field strength waves

Differently from the electromagnetic field, there are two ways of constructing field strength waves for a fundamental spin-2 field. One of them has helicity $\sigma = \pm 1$ and the other has helicity $\sigma = \pm 2$. In what follows we explore in details each one of them.

Helicity $\sigma = \pm 1$ waves

Let us consider the spin-2 representation that comes from the direct product

$$(1/2, 1/2) \otimes [(1,0) \oplus (0,1)] = [(3/2, 1/2) \oplus (1/2, 3/2)] \oplus \ldots, \tag{6.29}$$

where the dots represent additional terms that can be eliminated by the imposition of supplementary conditions. By construction, it describes a vector-valued 2-form with twenty four components. As a matter of fact, it describes the Fierz tensor $\mathcal{F}_a{}^{\mu\nu}$ [59], which is the excitation 2-form of a fundamental spin-2 field. It satisfies the field equation

$$\partial_\nu \mathcal{F}_a{}^{\mu\nu} = 0, \tag{6.30}$$

which can be obtained by linearizing the potential form [60] of Einstein equation in the tetrad formalism.

A much simpler way to approach the Fierz–Pauli formulation of a spin-2 fundamental field is to note that it is similar to the teleparallel equivalent of general relativity. In fact, the field equation (6.30) follows quite naturally from linearizing the gravitational equation of teleparallel gravity [61]. In this context, the Fierz tensor is easily seen to be given by

$$\mathcal{F}_a{}^{\mu\nu} = e_a{}^\rho \, \mathcal{K}^{\mu\nu}{}_\rho + e_a{}^\mu \, e_b{}^\rho \, F^{b\nu}{}_\rho - e_a{}^\nu \, e_b{}^\rho \, F^{b\mu}{}_\rho, \tag{6.31}$$

where

$$\mathcal{K}^{\mu\nu}{}_\rho = \tfrac{1}{2} \left(e_a{}^\nu \, F^{a\mu}{}_\rho + e^a{}_\rho \, F_a{}^{\mu\nu} - e_a{}^\mu \, F^{a\nu}{}_\rho \right) \tag{6.32}$$

is a contortion–type tensor, with

$$F^a{}_{\mu\nu} = \partial_\mu \varphi^a{}_\nu - \partial_\nu \varphi^a{}_\mu \tag{6.33}$$

a torsion–type tensor which represents the spin-2 field strength. Its lagrangian is, accordingly, given by the teleparallel–type lagrangian

$$\mathcal{L} = \tfrac{1}{4} \, \mathcal{F}_a{}^{\mu\nu} \, F^a{}_{\mu\nu}. \tag{6.34}$$

Analogously to the electromagnetic case, in the harmonic gauge (6.19), the field equation (6.30) reduces to the field equation (6.20) for the potential $\psi^a{}_\mu$. Following the same analogy, we can define the gravitoelectric and the gravitomagnetic components of the gravitational field, respectively, by [62]

$$E^{ai} = \mathcal{F}^{a0i} \quad \text{and} \quad H^{ai} = \tfrac{1}{2} \, \epsilon^{ijk} \, \mathcal{F}^a{}_{jk}. \tag{6.35}$$

Equation (6.30) is then found to be equivalent to

$$\Box \, \boldsymbol{E}^a = 0 \quad \text{and} \quad \Box \, \boldsymbol{H}^a = 0, \tag{6.36}$$

where $\boldsymbol{E}^a = (E^{ai})$ and $\boldsymbol{H}^a = (H^{ai})$. Owing to the fact that $\boldsymbol{E}^a$ and $\boldsymbol{H}^a$ are components of $\mathcal{F}^{a\mu\nu}$, which is a field that lives in the representation $(1/2, 3/2) \oplus (3/2, 1/2)$ of the Lorentz group, the solutions to the equations (6.36) represent spin-2 waves with helicity $\sigma = \pm 1$.

Helicity $\sigma = \pm 2$ waves

Another spin-2 representation can be constructed by considering the direct product

$$[(0,1) \oplus (1,0)] \otimes [(0,1) \oplus (1,0)] = [(0,2) \oplus (2,0)] \oplus \ldots, \tag{6.37}$$

where again the dots represent additional terms coming from the direct product. As is well known, this representation describes a fourth rank tensor which is antisymmetric within each pair of indices, and symmetric between the pairs. It represents actually the twenty components of a Riemann–type tensor $\mathcal{R}^{\rho}{}_{\lambda\mu\nu}$ constructed out of the second derivatives of the symmetric field $\varphi_{\mu\nu}$,

$$\mathcal{R}^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} - \partial_\nu \Gamma^{\rho}{}_{\lambda\mu}, \tag{6.38}$$

with

$$\Gamma^{\rho}{}_{\lambda\nu} = \tfrac{1}{2}\, \eta^{\rho\sigma} \left(\partial_\lambda \varphi_{\nu\sigma} + \partial_\nu \varphi_{\lambda\sigma} - \partial_\sigma \varphi_{\lambda\nu} \right). \tag{6.39}$$

Defining the Ricci–type tensor $\mathcal{R}_{\lambda\nu} = \mathcal{R}^{\rho}{}_{\lambda\rho\nu}$, the curvature wave is found to satisfy the Einstein–type equation

$$\mathcal{R}_{\lambda\nu} = 0, \tag{6.40}$$

which in harmonic coordinates reduces to the potential field equation (6.26), that is,

$$\Box \psi_{\mu\nu} = 0. \tag{6.41}$$

As is well known, it follows from the Einstein-Hilbert–type lagrangian

$$\mathcal{L} = -\mathcal{R} \equiv -\eta^{\lambda\nu}\, \mathcal{R}_{\lambda\nu}. \tag{6.42}$$

Due to the fact that $\mathcal{R}^{\rho}{}_{\lambda\mu\nu}$ lives in the representation $(0,2) \oplus (2,0)$ of the Lorentz group, the solution of the equation (6.40) describes spin-2 waves with helicity $\sigma = \pm 2$.

Up to a surface term, the lagrangian (6.42) is equivalent to the teleparallel–type lagrangian (6.34). The presence of this surface term is related to the fact that $\mathcal{R}$ depends on second derivatives of $\psi_{\mu\nu}$. For the same reason, the field equation (6.40) does not involve derivatives of the curvature tensor. The curvature–like tensor $\mathcal{R}^{\rho}{}_{\lambda\mu\nu}$, therefore, is a derived geometrical entity which does not have its own dynamics: its propagation is simply a consequence of the dynamics of the potential $\psi_{\mu\nu}$. This is different from the electromagnetic case, whose field strength $F^{\mu\nu}$ satisfies a dynamical equation. It is also different from the Fierz–Pauli (or of the teleparallel) formulation of gravity, whose excitation 2-form $\mathcal{F}_a{}^{\mu\nu}$, satisfies a dynamical equation.

Chapter 7

Invariant Field Equations

We see from generators (2.57) that the de Sitter spacetime is transitive under a combination of translations and proper conformal transformations [19]. The replacement of Minkowski by de Sitter as the spacetime representing absence of gravitation, therefore, naturally changes the group governing the spacetime kinematics: instead of Poincaré, it turns out to be the de Sitter group. Since the latter includes the proper conformal transformations, the field equations obtained from the first Casimir operator of the de Sitter group are proved to be conformal invariant [63]. In this section we are going to develop the formalism, as well as obtain the conformal invariant field equation for a general massless bosonic field Ψ with helicity σ.

7.1 Conformal Transformations

Under a conformal re-scaling of the metric tensor,

$$\bar{g}_{\mu\nu} = \Omega^2 \, g_{\mu\nu}, \tag{7.1}$$

with $\Omega = \Omega(x)$ the conformal factor, a general field Ψ transforms according to

$$\bar{\Psi} = \Omega^{d\tau} \, \Psi, \tag{7.2}$$

where d is a matrix that multiplies every Fermi field by $3/2$ and every Bose field by 1, and τ is a real number that depends on the spin s of the field [64]. For fermions, it is given by $\tau = 2(\mathsf{s} - 1)$, in which case we obtain

$$\bar{\Psi} = \Omega^{3(\mathsf{s}-1)} \, \Psi. \tag{7.3}$$

For bosons, on the other hand, it is given by $\tau = \mathsf{s} - 1$, which yields

$$\bar{\Psi} = \Omega^{\mathsf{s}-1} \, \Psi. \tag{7.4}$$

The power of Ω is known as the conformal weight w of the field Ψ. A scalar field has conformal weight $w = -1$, a vector field has $w = 0$, and a spin-2 field has $w = 1$. The corresponding conformal transformations of massless fields are obtained by replacing the spin s with the helicity σ of the field.

7.2 Casimir Operator and Field Equations

When acting on a general field Ψ, the first Casimir invariant of the de Sitter group is

$$\hat{\mathcal{C}}_{dS} = -\frac{1}{2l^2}\,\eta^{AC}\,\eta^{BD}\,\hat{J}_{AB}\,\hat{J}_{CD}, \tag{7.5}$$

where

$$\hat{J}_{AB} = \hat{L}_{AB} + \hat{S}_{AB}, \tag{7.6}$$

with $\hat{L}_{AB}$ the orbital generators and $\hat{S}_{AB}$ the matrix spin generators appropriate for the field Ψ. In terms of the stereographic coordinates, it assumes the form

$$\hat{\mathcal{C}}_{dS} = -\frac{1}{2l^2}\left(\eta^{\alpha\beta}\,\eta^{\gamma\delta}\,\hat{J}_{\alpha\gamma}\,\hat{J}_{\beta\delta} - 2\eta^{\alpha\beta}\,\hat{J}_{4\alpha}\hat{J}_{4\beta}\right). \tag{7.7}$$

Substituting the relations

$$\hat{J}_{\alpha\gamma} = \hat{L}_{\alpha\gamma} + \hat{S}_{\alpha\gamma} \qquad \text{and} \qquad \hat{J}_{4\gamma} = l(\hat{\Pi}_\gamma + \hat{\Sigma}_\gamma), \tag{7.8}$$

it becomes

$$\hat{\mathcal{C}}_{dS} = -\frac{1}{2l^2}\left(\hat{L}^2 + \hat{L}\cdot\hat{S} + \hat{S}\cdot\hat{L} + \hat{S}^2\right) + \hat{\Pi}^2 + \hat{\Pi}\cdot\hat{\Sigma} + \hat{\Sigma}\cdot\hat{\Pi} + \hat{\Sigma}^2. \tag{7.9}$$

For particles belonging to representations on the *principal series*, the eigenvalues $\mathcal{C}_{dS}$ of the Casimir operator $\hat{\mathcal{C}}_{dS}$ are given by [32]

$$\mathcal{C}_{dS} = -m^2 + \frac{1}{l^2}\left[\mathsf{s}(\mathsf{s}+1) - 2\right], \tag{7.10}$$

with m the mass and s the spin of the field. In the massless case, instead of the spin s, the eigenvalue is written in terms of the helicity σ of the field:

$$\mathcal{C}_{dS} = \frac{1}{l^2}\left[\sigma(\sigma+1) - 2\right]. \tag{7.11}$$

Using the identity $\hat{\mathcal{C}}_{dS}\Psi_{\mu_1\ldots\mu_\sigma} = \mathcal{C}_{dS}\Psi_{\mu_1\ldots\mu_\sigma}$, the massless field equation is then found to be

$$\hat{\mathcal{C}}_{dS}\Psi_{\mu_1\ldots\mu_\sigma} = -\frac{R}{12}\left[\sigma(\sigma+1) - 2\right]\Psi_{\mu_1\ldots\mu_\sigma}, \tag{7.12}$$

with R the scalar curvature of the de Sitter spacetime, given by Eq (2.53). The kinematic generators $\hat{L}$ and $\hat{\Pi}$, given by Eqs. (2.56) and (2.57), are the same for all fields. The matrix generators $\hat{S}$ and $\hat{\Sigma}$, on the other hand, depend on the spin of the field and are consequently different for fields with different spins.

Now, similarly to the Minkowski case, the Casimir operator of the de Sitter spacetime gives rise to field equations already in the Lorenz gauge. Since the Lorenz gauge is not conformal invariant, the massless version of the field equation (7.12) is not conformal invariant either. It is then necessary to re-introduce the Lorentz gauge into the field equation. This can be done by adding a Lorenz gauge term, such that the massless field equation becomes

$$\hat{\mathcal{C}}_{dS}\Psi_{\mu_1\ldots\mu_\sigma} + a\,\nabla_{\mu_\sigma}\nabla^\rho\Psi_{\mu_1\ldots\mu_{\sigma-1}\rho} = -\frac{R}{12}\left[\sigma(\sigma+1)-2\right]\Psi_{\mu_1\ldots\mu_\sigma}, \tag{7.13}$$

with a given by [65]

$$a = -\frac{2\sigma}{\sigma+1} \tag{7.14}$$

and ∇_ρ the covariant derivative in the Christoffel connection (2.51). This is the conformal invariant field equation in de Sitter spacetime. Considering that it is covariant, if it is true in de Sitter spacetime, it will be true in any pseudo-Riemannian spacetime with non-constant curvature.

7.3 Examples: Scalar and Vector Field Equations

As an illustration, we are going to obtain in this section the well-known conformal invariant equations for the scalar and the vector fields. First, however, it is necessary to obtain the explicit form of the matrix representations $\hat{S}_{\alpha\beta}$ and $\hat{\Sigma}_\alpha$ for the scalar and the vector cases.

7.3.1 Matrix Representation of the Conformal Transformation

The matrix representation of the conformal transformation can be obtained from the vanishing of the Lie derivative along the de Sitter Killing vectors,

$$(\mathcal{L}_X g)_{\alpha\beta} = X^\gamma\partial_\gamma g_{\alpha\beta} + \partial_\alpha X^\gamma g_{\gamma\beta} + \partial_\beta X^\gamma g_{\gamma\alpha} = 0, \tag{7.15}$$

where $g_{\alpha\beta}$ is the de Sitter metric (2.24). In this expression, X^γ is either the Lorentz Killing vector

$$L^\gamma = \epsilon^{\alpha\beta}L^\gamma_{(\alpha\beta)} = \epsilon^{\alpha\beta}\Big(\eta_{\alpha\delta}x^\delta\delta^\gamma_\beta - \eta_{\beta\delta}x^\delta\delta^\gamma_\alpha\Big), \tag{7.16}$$

or the de Sitter "translation" Killing vector

$$\Pi^\gamma = \frac{1}{l}\epsilon^\alpha L^\gamma_{(4\alpha)} = \epsilon^\alpha\Big[\delta^\gamma_\alpha - \frac{1}{4l^2}(2\eta_{\alpha\delta}x^\delta x^\gamma - \sigma^2\delta^\gamma_\alpha)\Big], \tag{7.17}$$

with $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ and ϵ^{α} the ten parameters of the de Sitter group. They satisfy the algebra

$$\left[L_{(\alpha\beta)}, L_{(\gamma\delta)}\right] = \eta_{\beta\gamma} L_{(\alpha\delta)} + \eta_{\alpha\delta} L_{(\beta\gamma)} - \eta_{\beta\delta} L_{(\alpha\gamma)} - \eta_{\alpha\gamma} L_{(\beta\delta)} \tag{7.18}$$

$$\left[\Pi_{(\alpha)}, \Pi_{(\beta)}\right] = \frac{1}{l^2} L_{(\alpha\beta)}, \tag{7.19}$$

where

$$L_{(\alpha\beta)} \equiv L^{\gamma}_{(\alpha\beta)} \partial_\gamma = \eta_{\alpha\delta} x^{\delta} \partial_\beta - \eta_{\beta\delta} x^{\delta} \partial_\alpha \tag{7.20}$$

and

$$\Pi_{(\alpha)} \equiv \frac{1}{l} L^{\gamma}_{(4\alpha)} \partial_\gamma = \partial_\alpha - \frac{1}{4l^2}(2\eta_{\alpha\delta} x^{\delta} x^{\gamma} - \sigma^2 \delta^{\gamma}_{\alpha}) \partial_\gamma. \tag{7.21}$$

In order to obtain the matrix representations appropriate for vector field ϕ_μ, it is necessary to compute the Lie derivative of the field along the direction of the de Sitter Killing vectors. These derivatives are given by

$$\delta_L \phi_\mu \equiv (\mathcal{L}_L \phi)_\mu = \epsilon^{\alpha\beta} L^{\gamma}_{(\alpha\beta)} \partial_\gamma \phi_\mu + \epsilon^{\alpha\beta} \partial_\mu L^{\gamma}_{(\alpha\beta)} \phi_\gamma \tag{7.22}$$

and

$$\delta_\Pi \phi_\mu \equiv (\mathcal{L}_\Pi \phi)_\mu = \epsilon^{\alpha} \Pi^{\gamma}_{(\alpha)} \partial_\gamma \phi_\mu + \epsilon^{\alpha} \partial_\mu \Pi^{\gamma}_{(\alpha)} \phi_\gamma. \tag{7.23}$$

The first term on the right-hand side of these two equations represent the action of the orbital generators. The last term, on the other hand, represent the action of the spin matrix generators. From these terms, therefore, we get the matrix representations

$$(\hat{S}_{\alpha\beta})_\mu{}^\gamma \phi_\gamma \equiv \partial_\mu L^{\gamma}_{(\alpha\beta)} \phi_\gamma = (\eta_{\alpha\mu} \delta_\beta{}^\gamma - \eta_{\beta\mu} \delta_\alpha{}^\gamma) \phi_\gamma \tag{7.24}$$

and

$$(\hat{\Sigma}_\alpha)_\mu{}^\gamma \phi_\gamma \equiv \partial_\mu \Pi^{\gamma}_{(\alpha)} \phi_\gamma = \frac{1}{2l^2}(\eta_{\mu\beta} x^\beta \delta_\alpha{}^\gamma - \eta_{\alpha\mu} x^\gamma - \eta_{\alpha\beta} x^\beta \delta_\mu{}^\gamma) \phi_\gamma. \tag{7.25}$$

Equation (7.24) is the usual spin-1 representation of the Lorentz group. Equation (7.25) is the matrix representation of the proper conformal transformations. In the flat spacetime limit $l \to \infty$ this representation vanishes, a result consistent with the fact that ordinary translations do not have matrix representation.

In the case of a scalar field ϕ, the Lie derivative along the direction of the de Sitter Killing vectors are given by

$$\delta_L \phi \equiv \mathcal{L}_L \phi = \epsilon^{\alpha\beta} L^{\gamma}_{(\alpha\beta)} \partial_\gamma \phi + 0 \tag{7.26}$$

and

$$\delta_\Pi \phi \equiv \mathcal{L}_\Pi \phi = \epsilon^{\alpha} \Pi^{\gamma}_{(\alpha)} \partial_\gamma \phi_\mu + 0. \tag{7.27}$$

From these expressions we see that both the Lorentz and the conformal spin matrix representations for a scalar field vanish:

$$(\hat{S}_{\alpha\beta})_\mu{}^\gamma \phi = 0 \qquad \text{and} \qquad (\hat{\Sigma}_\alpha)_\mu{}^\gamma \phi = 0. \tag{7.28}$$

7.3.2 The Scalar Field

For a scalar field ϕ, for which $\hat{S}\phi = 0$ and $\hat{\Sigma}\phi = 0$, the Casimir operator (7.9) reduces to

$$\hat{\mathcal{C}}_{dS} \equiv -\frac{1}{2l^2}\hat{L}^2 + \hat{\Pi}^2 = \Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \frac{1}{l^2}\,\Omega^{-1}x^\alpha\partial_\alpha. \tag{7.29}$$

As a simple inspection shows, it can be rewritten in the form

$$\hat{\mathcal{C}}_{dS} = g^{\alpha\beta}\nabla_\alpha\nabla_\beta \equiv \Box, \tag{7.30}$$

with ∇_α the covariant derivative in the Christoffel connection (2.51), and $\Box$ the corresponding Laplace-Beltrami operator for a scalar field. For $\sigma = 0$, therefore, the field equation (7.13) for ϕ is found to be

$$\Box\phi - \frac{R}{6}\,\phi = 0, \tag{7.31}$$

where R is the scalar curvature of the de Sitter spacetime. Although obtained in de Sitter spacetime, since it is invariant under general coordinate transformations, it will be true in any pseudo-Riemannian spacetime with non-constant curvature. In this case, as is well known, it represents the conformal invariant equation for a scalar field.

7.3.3 The Vector Field

Using the kinematic generators (2.56) and (2.57), as well as the vector matrix generators (7.24) and (7.25), the Casimir operator (7.9) for this specific case assumes the form

$$(\hat{\mathcal{C}}_{dS})_\mu{}^\gamma = \Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\delta_\mu{}^\gamma + \frac{\Omega^{-1}}{l^2}(x^\gamma\partial_\mu - \eta_{\mu\lambda}\eta^{\gamma\rho}x^\lambda\partial_\rho) + \frac{\Omega^{-1}}{l^2}\delta_\mu{}^\gamma - \frac{1}{2l^4}\eta_{\mu\lambda}x^\lambda x^\gamma. \tag{7.32}$$

As a simple inspection shows, it can be rewritten in the form

$$(\hat{\mathcal{C}}_{dS})_\mu{}^\gamma = \Box\,\delta_\mu{}^\gamma - R_\mu{}^\gamma, \tag{7.33}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the Laplace-Beltrami operator, and $R_\mu{}^\gamma$ is the Ricci curvature tensor (2.53). Therefore, for $\mathsf{s} = 1$, the field equation (7.13) for A_γ is found to be

$$\Box A_\mu - R_\mu{}^\gamma A_\gamma - \nabla_\mu\nabla^\gamma A_\gamma = 0. \tag{7.34}$$

Using the identity

$$R_\mu{}^\gamma A_\gamma = [\nabla^\gamma, \nabla_\mu] A_\gamma, \tag{7.35}$$

it can be rewritten in the form

$$\Box A_\mu - \nabla^\gamma \nabla_\mu A_\gamma = 0. \tag{7.36}$$

This is the gravitationally coupled Maxwell equation. As is well known, it is conformal invariant. In terms of the field strength $F_{\mu\nu}$, it reads

$$\nabla_\nu F^{\mu\nu} \equiv \partial_\nu(\sqrt{-g}\, F^{\mu\nu}) = 0, \tag{7.37}$$

with $g = \det(g_{\mu\nu})$.

7.4 Conformal Invariant Spin-2 Field Equation

The dynamics of a fundamental spin-2 field in Minkowski spacetime is expected to coincide with the dynamics of a linear perturbation of the metric $\zeta_{\mu\nu}$ around flat spacetime [66]:

$$g_{\mu\nu} = \eta_{\mu\nu} + \zeta_{\mu\nu}. \tag{7.38}$$

For this reason, a fundamental spin-2 field is usually assumed to be described by a symmetric, second-rank tensor $\zeta_{\mu\nu} = \zeta_{\nu\mu}$. However, there are some problems with this assumption. First, if a spin-2 field were defined as a perturbation of the metric, its conformal transformation would coincide with the metric transformation (7.1), and consequently it would have a conformal weight $w = 2$. But, according to the general transformation (7.4), a spin-2 field should have conformal weight $w = 1$. A possible solution to this problem is to introduce a scalar field ϕ, and assume that the metric perturbation $\zeta_{\mu\nu}$ and a fundamental spin-2 field $\varphi_{\mu\nu}$ are related by

$$\varphi_{\mu\nu} = \phi\, \zeta_{\mu\nu}. \tag{7.39}$$

This is actually a matter of necessity because, in addition of presenting the correct field dimension, $\varphi_{\mu\nu}$ has conformal weight $w = 1$, as appropriate for a spin-2 field:

$$\bar{\varphi}_{\mu\nu} = \Omega\, \varphi_{\mu\nu}. \tag{7.40}$$

Let us then construct the Casimir operator for the spin-2 field $\varphi_{\mu\nu}$. It is given by

$$(\hat{C}_{ds})^{\mu\nu}_{\alpha\beta} = -\frac{1}{2l^2}\eta^{AB}\eta^{CD}(\hat{J}_{AC})^{\mu\nu}_{\rho\sigma}\,(J_{BD})^{\rho\sigma}_{\alpha\beta}, \tag{7.41}$$

where generators with *four indices* are the spin-2 generators. Like in the spin-1 case, these generators can be decomposed in orbital and spinorial parts,

$$(\hat{J}_{AB})^{\mu\nu}_{\rho\sigma} = (\hat{L}_{AB})^{\mu\nu}_{\rho\sigma} + (\hat{S}_{AB})^{\mu\nu}_{\rho\sigma}. \tag{7.42}$$

The orbital generators are the same for all fields, and is just

$$(\hat{L}_{AB})^{\mu\nu}_{\rho\sigma} = (\hat{L}_{AB})\delta^{\mu}_{\rho}\,\delta^{\nu}_{\sigma}. \tag{7.43}$$

On the other hand, the spinorial generators are obtained by summing two spin-1 representations, one for each index of $\varphi_{\mu\nu}$,

$$(\hat{S}_{AB})^{\mu\nu}_{\rho\sigma} = (\hat{S}_{AB})^{\mu}_{\rho}\,\delta^{\nu}_{\sigma} + (\hat{S}_{AB})^{\nu}_{\sigma}\,\delta^{\mu}_{\rho}, \tag{7.44}$$

where the generators with *two indices* are the spin-1 generators. In terms of these generators, the Casimir operator (7.41) assumes the form

$$\begin{aligned}(\hat{C}_{ds})^{\mu\nu}_{\alpha\beta} = -\frac{1}{2l^2}\eta^{AB}\eta^{CD}\Big[&(L_{AC})(L_{BD})\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} + (L_{AC})(S_{BD})^{\mu}_{\alpha}\,\delta^{\nu}_{\beta} + (L_{AC})(S_{BD})^{\nu}_{\beta}\,\delta^{\mu}_{\alpha}\\ &+2(S_{AC})^{\mu}_{\alpha}(L_{BD})\delta^{\nu}_{\beta} + 2(S_{AC})^{\nu}_{\beta}(L_{BD})\delta^{\mu}_{\alpha} + (S_{AC})^{\nu}_{\gamma}(S_{BD})^{\gamma}_{\beta}\,\delta^{\mu}_{\alpha}\\ &+(S_{AC})^{\mu}_{\gamma}(S_{BD})^{\gamma}_{\alpha}\,\delta^{\nu}_{\beta} + 2(S_{AC})^{\mu}_{\alpha}(S_{BD})^{\nu}_{\beta}\Big].\end{aligned} \tag{7.45}$$

Like in the previous case, all terms in the right-hand side that involves an orbital generator L_{AC} will contribute to the Laplace-Beltrami operator of the field. On the other hand, terms involving two spin generators S_{AC} will contribute with a non-minimal coupling between the field and the spacetime curvature. More specifically, those bearing one Kronecker delta will contribute with terms involving the Ricci tensor. Since the last term does not involve any Kronecker delta, it can only contribute with a term involving the scalar curvature. Substituting then the representations (7.20, 7.21, 7.24, 7.25), after a long but straightforward calculation, Eq. (7.45) can be shown to reduce to

$$(\hat{C}_{ds})^{\alpha\beta}_{\mu\nu} = \Box\,\delta^{\alpha}_{(\mu}\delta^{\beta}_{\nu)} - 2\,R^{\alpha}{}_{(\mu}\delta^{\beta}_{\nu)} - \frac{1}{6}\,R\,\delta^{\alpha}_{(\mu}\,\delta^{\beta}_{\nu)} + \frac{2}{3}\,R_{(\mu\nu)}\,g^{\alpha\beta}, \tag{7.46}$$

with the parentheses indicating symmetrization with a factor 1/2.

As already mentioned, the Casimir approach gives rise to field equations in the Lorenz gauge. Taking into account that, in the spin-2 theory, it is the combination

$$\psi^{\mu}{}_{\nu} = \varphi^{\mu}{}_{\nu} - \tfrac{1}{2}\,\delta^{\mu}{}_{\nu}\varphi^{\rho}{}_{\rho} \tag{7.47}$$

that satisfies the Lorenz gauge (has vanishing covariant derivative), we assume that the Casimir approach yields field equation for $\psi^{\mu}{}_{\nu}$. Furthermore, the field excitation 2-form of a symmetric second-rank potential $\psi^{\mu}{}_{\nu}$ is a curvature-like tensor $\mathcal{R}^{\alpha\beta}{}_{\mu\nu}$, which is nothing but the linearized Riemann tensor written in terms of $\psi^{\mu}{}_{\nu}$. Since $\mathcal{R}^{\alpha\beta}{}_{\mu\nu}$ belongs to the representation $(2,0)\oplus(0,2)$ of the Lorentz group, such field excitation describes waves with helicity $\sigma = \pm 2$. Considering the case of a traceless field, $\psi \equiv \delta^\alpha{}_\beta\, \psi_\alpha{}^\beta = 0$, it is then an easy task to verify that, for the Casimir operator (7.46), the field equation (7.13) with $\sigma = 2$, yields in this case

$$\Box\,\psi_{\mu\nu} - \frac{4}{3}\,\nabla_{(\mu}\nabla^\alpha\psi_{\alpha\nu)} - 2\,R^\alpha{}_{(\mu}\,\psi_{\alpha\nu)} + \frac{1}{6}\,R\,\psi_{\mu\nu} = 0. \tag{7.48}$$

Using the identity

$$[\nabla_{(\mu}, \nabla^\alpha]\psi_{\alpha\nu)} = -R^\alpha{}_{(\mu}\psi_{\alpha\nu)} + R^\alpha{}_{(\mu}{}^\gamma{}_{\nu)}\,\psi_{\alpha\gamma}, \tag{7.49}$$

it can be rewritten in the form

$$\Box\,\psi_{\mu\nu} - \frac{4}{3}\,\nabla^\alpha\nabla_{(\mu}\psi_{\alpha\nu)} - \frac{2}{3}\,R^\alpha{}_{(\mu}\psi_{\alpha\nu)} - \frac{4}{3}\,R^\alpha{}_\mu{}^\gamma{}_\nu\,\psi_{\alpha\gamma} + \frac{1}{6}\,R\,\psi_{\mu\nu} = 0. \tag{7.50}$$

Although obtained in de Sitter spacetime, owing to its covariance, it holds also in any pseudo-Riemannian spacetime with non-constant curvature. It represents, therefore, the conformal invariant field equation for a symmetric second-rank tensor [67, 65].

7.5 Consistency Problems, and a Possible Solution

Although conformal invariant, the field equation (7.50) has consistency problems. In fact, it is not invariant under the transformations

$$\psi_{\mu\nu} \to \psi_{\mu\nu} - \partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu, \tag{7.51}$$

usually called gauge transformations. As a consequence, it is not possible to remove all spurious components of the field, and it turns out to involve more components than necessary to describe a massless field [68, 69]. On the other hand, a symmetric second-rank tensor is not the only way to represent a spin 2 field. In fact, it can also be represented by a 1-form assuming values in the Lie algebra of the translation group [54],

$$\zeta_\nu = \zeta^a{}_\nu\, P_a, \tag{7.52}$$

with $P_a = \partial_a$ the translation generators. This second possibility is related to the existence of the metric and the tetrad representation of gravity: whereas $\zeta_{\mu\nu}$ is conceptually similar to a perturbation of the metric, $\zeta^a{}_\nu$ is conceptually similar to a perturbation of the tetrad,

$$h^a{}_\nu = e^a{}_\nu + \zeta^a{}_\nu, \tag{7.53}$$

with $e^a{}_\nu$ a trivial tetrad representing the Minkowski spacetime. Like before, the variable representing a fundamental translational-valued 1-form is

$$\varphi^a{}_\nu = \phi\, \zeta^a{}_\nu, \tag{7.54}$$

with ϕ a scalar field.

Let us then use the Casimir approach to obtain the conformal invariant equation for

$$\psi^a{}_\nu = \varphi^a{}_\nu - \tfrac{1}{2}\, h^a{}_\nu \varphi, \tag{7.55}$$

with $\varphi = h_c{}^\rho \varphi^c{}_\rho$. We begin by noting that, similarly to the electromagnetic field, $\psi^a{}_\nu$ has conformal weight $w = 0$:

$$\bar{\psi}^a{}_\nu = \psi^a{}_\nu. \tag{7.56}$$

Considering that $\psi_\nu = \psi^a{}_\nu\, P_a$ is ultimately a (translational-valued) vector field, the Casimir operator (7.9) for ψ_ν coincides with that for a vector field:

$$(\hat{\mathcal{C}}_{dS})_\mu{}^\nu = \Omega^{-2}\eta^{\alpha\beta}\partial_\alpha\partial_\beta\delta_\mu{}^\nu + \frac{\Omega^{-1}}{l^2}(x^\nu\partial_\mu - \eta_{\mu\lambda}\eta^{\nu\rho}x^\lambda\partial_\rho) + \frac{\Omega^{-1}}{l^2}\delta_\mu{}^\nu - \frac{1}{2l^4}\eta_{\mu\lambda}x^\lambda x^\nu. \tag{7.57}$$

In terms of the Laplace-Beltrami operator acting on ψ_ν, it assumes the form

$$(\hat{\mathcal{C}}_{dS})_\mu{}^\nu\psi_\nu = \Box\psi_\mu - R_\mu{}^\nu\, \psi_\nu, \tag{7.58}$$

where $R_\mu{}^\nu$ is the Ricci curvature tensor (2.53). Substituting the identity

$$R_\mu{}^\nu\, \psi_\nu = [\nabla^\nu, \nabla_\mu]\psi_\nu, \tag{7.59}$$

it becomes

$$(\hat{\mathcal{C}}_{dS})_\mu{}^\nu\psi^a{}_\nu = \Box\psi^a{}_\mu - \nabla^\nu\nabla_\mu\psi^a{}_\nu + \nabla_\mu\nabla^\nu\psi^a{}_\nu, \tag{7.60}$$

where we have re-introduced the algebraic index of $\psi^a{}_\nu$. Then comes the point: the excitation 2-form of the translational-valued 1-form $\psi^a{}_\mu$ is the Fierz tensor $\mathcal{F}_a{}^{\mu\nu}$ [59], which is also a field assuming values in the Lie algebra of the translation group [61]. Since it belongs to the representation $(1/2, 3/2) \oplus (3/2, 1/2)$ of the Lorentz group, it represents spin 2 waves with helicity $\sigma = \pm 1$. In this case, the field equation (7.13) with $\sigma = 1$ yields

$$\Box\psi^a{}_\mu - \nabla^\nu\nabla_\mu\psi^a{}_\nu = 0. \tag{7.61}$$

Using the transformation (7.56), it is easy to see that this equation is conformal invariant. Furthermore, it is invariant under the gauge transformations

$$\psi^a{}_\nu \to \psi^a{}_\nu - \partial_\nu\varepsilon^a, \tag{7.62}$$

and consequently it is able to eliminate the spurious components of $\psi^a{}_\nu$, remaining with the correct number of components to describe a massless field.

One may wonder why the theory for $\psi_{\mu\nu}$ results inconsistent, whereas the theory for $\psi^a{}_\nu$ turns out to be consistent. The reason is related to their different structure of the indices: whereas spacetime indices has to do with gravitation, the translational algebraic index has to do with inertial effects of the frame only, not with gravitation [70]. As a consequence, the gravitational coupling prescription will be different for each case: whereas the coupling prescription of $\psi_{\mu\nu}$ will include a connection term for each index of the field, the coupling prescription of $\psi^a{}_\nu$ will include a connection term for the spacetime index, but not for the algebraic (or gauge) index. This difference between the two cases is the responsible for rendering the theory, consistent in one case, but inconsistent in the other case.

Chapter 8

Conclusions

8.1 de Sitter Relativity

The de Sitter special relativity emerges when one adds an invariant length parameter l to the kinematics of the Poincaré group [28, 29]. Since the de Sitter group has Lorentz as subgroup, the introduction of such length scale does not imply that Lorentz symmetry is broken. Rather, it is spacetime translations that are violated, leading to new notions of energy and momentum. These new notions are given by a combination of translations and proper conformal transformations. As a consequence, the source of total spacetime curvature turns out to be a combination of energy-momentum and proper conformal currents: whereas the energy-momentum tensor $T_{\mu\nu}$ appears as source of dynamic curvature, the proper conformal current $K_{\mu\nu}$ appears as source of the local de Sitter spacetime, which is necessary to comply with the local kinematics, now governed by the de Sitter group [20].

When applied to the whole universe, de Sitter general relativity is able to predict, from the current matter content of the universe, the value of Λ for the universe [36]. It gives, furthermore, an explanation for the cosmic coincidence problem. When applied to study the propagation of ultra-high energy photons, it gives a good estimate for the recently observed delay in high energy gamma-ray flares coming from the center of the galaxy Markarian 501 [6]. If this delay is a manifestation of the small-scale fluctuations in the texture of spacetime, predicted to exist at very high energies, de Sitter relativity can be seen as a new paradigm to approach quantum gravity. Of course, the experimental evidences of the above delays are still very fragile. Independently of this fact, de Sitter general relativity predicts the existence of such delay. It is important to mention that the energy of these gamma rays are far away from the Planck energy. However, since the de Sitter-induced interference on their propagation are cumulative, it gives rise to measurable effects.

Concerning the Compton scattering [53], owing to the fact that the de Sitter kinematics involves an invariant length parameter, a gravitational contribution is naturally introduced into the expression describing the electromagnetic effect. In this case, however, the effect is not cumulative, and consequently will become relevant only at energies near the Planck scale, where quantum gravity is expected to be in action. In fact, at this energy scale, all masses become negligible, and the Compton scattering expression appears with the Planck mass replacing the electron mass, which makes the Compton shift very small. It is interesting to remark that, due to the fact that the de Sitter eigenvalues (7.11) vanish for the case of the electromagnetic field ($m = 0$ and $\mathsf{s} = 1$), the de Sitter relativity produces no changes in the photon-photon scattering. This is an expected result because electromagnetism is conformal invariant, and the de Sitter spacetime is conformally flat.

A crucial property of de Sitter general relativity is that, since neither energy-momentum nor proper conformal currents is individually conserved, energy-momentum can be transformed into proper conformal current, and vice-versa. Such mechanism yields a natural scenario for an evolving cosmological term. In fact, from a huge primordial Λ, in which most of the universe energy was in the form of dark energy, the cosmological term is then found to evolve according to the matter content of the universe, as given by the relation (4.33). After a decaying process, in order to allow the formation of the cosmological structures we see today, the universe has necessarily passed through a period of non-accelerating expansion, which means that the cosmological term must have assumed a tiny value during this period. Recent observations, however, indicate that the universe is presently entering an accelerated expansion era. This means that the energy-momentum current is transforming back into proper conformal current, which produces an increase in the value of the cosmological term. This mechanism is possible because, according to the de Sitter special relativity, an increasing Λ makes the usual notions of time and space gradually fade away, until they completely disappear at the Planck scale, where only their conformal counterparts exist.

8.2 Spin-2 and Conformal Invariant Equations

Minkowski is a homogeneous space, transitive under spacetime translations, whose kinematics is ruled by the Poincaré group. The first Casimir operator of the Poincaré group yields the d'Alembertian operator in Minkowski spacetime. When equaled to its eigenvalue, it gives rise to the Klein-Gordon equation. The de Sitter spacetime is also a homogeneous space, but transitive under a combination of translations and proper conformal transformations. The simple replacement of Minkowski by de Sitter, therefore, naturally introduces the conformal transformations in the spacetime kinematics. Due to this property, the first Casimir operator of the de Sitter group, when equaled to its eigenvalue, gives rise to conformal invariant equations for any massless field [63]. One has just to use the appropriate representations for each field. Since these equations are covariant, and true in de Sitter, they will also be true in any spacetime with non-constant curvature. We can then say that this procedure constitutes then a new method of obtaining conformal invariant field equations.

Using this approach, we have initially obtained the well-known conformal invariant equations for scalar and vector fields. We have then turned to the spin-2 case. First, we have found the conformal invariant equation for a symmetric second-rank tensor $\psi_{\mu\nu}$, which is given by the field equation (7.50). The field excitation 2-form in this case is a curvature-like tensor $\mathcal{R}^{\alpha\beta}{}_{\mu\nu}$ written in terms of the the potential $\psi_{\mu\nu}$. Since this field excitation belongs to the representation $(2,0)\oplus(0,2)$ of the Lorentz group, it describes waves with helicity $\sigma = \pm 2$. Although conformal invariant, however, the field equation (7.50) has consistency problems. In fact, it is not gauge invariant, and consequently involves more independent components than necessary to describe a massless field [68, 69].

On the other hand, a symmetric second-rank tensor is not the only way to represent a spin-2 field. As a matter of fact, it can also be represented by a 1-form assuming values in the Lie algebra of the translation group, $\psi_\nu = \psi^a{}_\nu P_a$, with P_a the translation generators. The excitation 2-form in this case is the Fierz tensor $\mathcal{F}_a{}^{\mu\nu}$, which is a field that belongs to the representation $(1/2, 3/2) \oplus (3/2, 1/2)$ of the Lorentz group. As such, it represent spin 2 waves with helicity $\sigma = \pm 1$ [54]. Relying on this interpretation of a spin 2 field, we have used the Casimir operator method to obtain the conformal invariant equation for the translational-valued 1-form $\psi^a{}_\nu$, which is given by Eq. (7.61). In addition to be conformal invariant, it is gauge invariant, and consequently fully consistent. These results may be seen as an evidence that a spin-2 field should be interpreted, not as a a symmetric second-rank tensor with helicity $\sigma = \pm 2$, but as a translational-valued vector field with helicity $\sigma = \pm 1$.

On the other hand, when a spin-2 field is considered to be a 1-form assuming values in the Lie algebra of the translation group, and the teleparallel coupling prescription is applied to the field equation (6.30), a sound and consistent gravitationally-coupled spin-2 field theory emerges [70], which is quite similar to the gravitationally-coupled electromagnetic theory. The reasons behind such consistency are twofold. First, since the index "a" of the translational-valued potential $\psi^a{}_\rho$ is not an ordinary vector index, but a gauge index, it is irrelevant for the gravitational coupling prescription. This contrasts with the usual metric approach, which considers both indices of $\psi_{\mu\nu}$ on an equal footing. As a result of this difference, whereas the gravitational coupling coupling of $\psi^a{}_\rho$ keeps the gauge invariance, the gravitational coupling coupling of $\psi_{\mu\nu}$ is found to break the gauge invariance of the theory. Furthermore, owing to the fact that the spin connection of teleparallel gravity represents inertial effects only, not gravitation [71], the corresponding covariant derivative commutes, and no unphysical constraints are imposed on the geometry of spacetime.

Put together, the above facts can be considered a compelling evidence favoring the

Fierz-Pauli approach (or equivalently, the teleparallel approach) to a fundamental spin-2 field, and consequently the interpretation of spin-2 waves as helicity $\sigma = \pm 1$ waves [54]. However, since the concept of helicity can only be defined at the linear level [72], and taking into account that gravitational waves are essentially non-linear [73], it is not clear whether the concepts related to a fundamental spin-2 field can be automatically extended to gravitational waves. Even if the linear approximation to the gravitational waves theory makes physical sense, it can always be corrected by higher-order terms, in which case the very notion of helicity would be lost.

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