Higher-derivative Schwinger model

R. L. P. G. Amaral, L. V. Belvedere, and N. A. Lemos
Instituto de Física, Universidade Federal Fluminense, Outeiro de São João Batista s/n, 24020 Centro, Niterói, Rio de Janeiro, Brazil

C. P. Natividade
Departamento de Matemática, Universidade Estadual Paulista, Campus de Guaratinguetá, 12500 São Paulo, São Paulo, Brazil

(Received 23 October 1992)

Using the operator formalism, we obtain the bosonic representation for the free fermion field satisfying an equation of motion with higher-order derivatives. Then, we consider the operator solution of a generalized Schwinger model with higher-derivative coupling. Since the increasing of the derivative order implies the introduction of an equivalent number of extra fermionic degrees of freedom, the mass acquired by the gauge field is bigger than the one for the standard two-dimensional QED. An analysis of the problem from the functional integration point of view corroborates the findings of canonical quantization, and corrects certain results previously announced in the literature on the basis of Fujikawa's technique.

PACS number(s): 11.10.Ef, 11.15.Tk

I. INTRODUCTION

Variational problems involving functionals that depend on derivatives of order higher than the first appear to have been first discussed by Ostrogradskii [1], who also established the basis for the Hamiltonian treatment of such problems. Although most physical systems are characterized by Lagrangians that contain, at most, first derivatives of the dynamical variables, there is a continuing interest in the study of model field theories defined by higher-derivative Lagrangians.

Early attempts to investigate higher-derivative theories aimed at generalizing or amending certain physical theories in order to get rid of some of their undesirable properties. Along these lines Weyl and Eddington [2] were, to the best of our knowledge, the first to add curvature-squared terms to the Einstein-Hilbert Lagrangian so as to extend the general theory of relativity. Modifications to Maxwell's electromagnetic theory were proposed by Bopp [3] and Podolsky [4] with the goal of avoiding divergences such as the self-energy of a point charge. Next Pais and Uhlenbeck [5] investigated whether the use of higher-order (including infinite-order) equations of motion might lead to the elimination of the divergent quantities that plague quantum field theory. They concluded that, in general, it is not possible to reconcile finiteness, positivity of free field energy, and causality. In other words, ghost states with negative norm and possibly unitarity violation are unavoidable in higher-order theories, and these facts became strong arguments against such theories.

However, in spite of these shortcomings, higher-order theories have never been entirely abandoned because they also possess some good properties, justifying a sort of revival of this subject in recent years. It has been shown [6] that a quantum theory of gravitation constructed by adding terms quadratic in the curvature to the Einstein-Hilbert Lagrangian is asymptotically free and the problem of its renormalizability is attenuated. It must be emphasized that such curvature-squared terms show up naturally as small corrections in the effective action of superstring theories in the limit of zero slope [7]. Higher-derivative terms appear naturally in the superfield formulation of supersymmetric theories [8] and also occur in the action proposed by Polyakov [9] in string theory, which involves the extrinstic curvature of the world sheet. It is further to be remarked that higher-order corrections are very useful as a mechanism for regularizing ultraviolet divergences [10], especially of gauge-invariant supersymmetric theories, since this is the only available regularization method that preserves both gauge invariance and supersymmetry [11].

Originally with functional methods, quantum and electrodynamics in two spacetime dimensions with massless fermions (Schwinger model) was exactly solved by Schwinger [12] as an example of a theory in which gauge invariance does not necessarily require a gauge field with null physical mass. The physical content of the theory as well as the correct interpretation of Schwinger's solution became clearer with the appearance of the operator formulation by Lowenstein and Swieca [13], in which the fermion field is parametrized in terms of boson fields ("bosonization"), a method that had been previously employed by Klaiber [14] to study the Thirring model. The boson representation of fermion fields turned out to be of great utility for establishing several equivalences between two-dimensional quantum field theories [15]. The Schwinger model is an exactly soluble theory which exhibits charge screening, fermion confinement, asymptotic freedom, and a rich vacuum structure. This is why it came to be regarded as a prototype model for confinement of quarks.[16]

Recently, a modified version of the Schwinger model in which fermion and gauge fields are coupled through third-order derivatives was proposed [17] and studied by functional methods. The axial anomaly and the dynamical mass generated for the photon field were calculated by means of Fujikawa's path-integral technique [18], and the results obtained [17] were the same as those for the standard Schwinger model.

The present work is devoted to the study of a generalized Schwinger model with higher derivatives, which includes as particular cases the original model and the one recently proposed by Barcelos-Neto and Natividade [17]. First the free fermion theory is canonically quantized and the exact expression for the n-point Wightman function is derived. Then it is shown that the free theory is amenable to bosonization, and the representation of the fermion field in terms of scalar and pseudoscalar fields is explicitly constructed. Next the generalized Schwinger model is defined by requiring local gauge invariance of the free fermion theory. The resulting Lagrangian density exhibits an additional chiral gauge invariance at the classical level. The quantum model, however, does not display the same symmetry, and the anomalous divergence of the axial-vector current is obtained. It is established that the gauge field acquires a physical mass that becomes larger as the order of the derivative of the fermion field in the Lagrangian increases.

In the limit in which our theory reduces to the Schwinger model the well-known usual results are recovered. However, when it coincides with the theory proposed by Barcelos-Neto and Natividade, our results for the axial anomaly and dynamical generation of mass for the gauge field differ from those found by the later authors. This state of affairs prompted us to undertake a reexamination of the problem by functional methods. An analysis in terms of Fujikawa's path-integral technique was carried out which corroborated our previous findings in the framework of canonical quantization. Finally, the reason for the discrepancy was identified and an error is pointed out in the work of Barcelos-Neto and Natividade.

This paper is organized as follows. In Sec. II the generalized free fermion theory is introduced, and its canonical quantization and bosonization are performed. In Sec. III the generalized Schwinger model is defined by demanding local gauge invariance of the free theory. The operator solution is found, the axial anomaly and the physical mass acquired by the gauge field are derived, and the vacuum structure is discussed. In Sec. IV the model is investigated by Fujikawa's path-integral formalism and the predictions of canonical quantization are confirmed. Section V is dedicated to general comments and conclusions.

Our notation and conventions are as follows:

$$\eta_{\mu\nu}=\mathrm{diag}(1,-1)\ ,\quad \epsilon^{01}=1\ ,\quad \{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}\ ,$$

$$\gamma^\mu=\gamma^0\gamma^{\mu\dagger}\gamma^0\ ;\quad \gamma^\mu\gamma^\nu=\eta^{\mu\nu}1+\varepsilon^{\mu\nu}\gamma_5\ ;$$

$$\gamma_5=\gamma^0\gamma^1\ ,\quad \gamma_5\gamma_\mu=\varepsilon_{\mu\nu}\gamma^\nu\ ;$$

$$x^\pm=x^0\pm x^1\ ,\quad \partial_\pm=\partial_0\pm\partial_1\ .$$

The following explicit representation for the γ matrices is adopted:

$$\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \gamma^1=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},$$

$$\gamma_5=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

II. FREE THEORY: CANONICAL QUANTIZATION AND BOSONIZATION

Since we need the solution of the free theory in order to obtain a full operator solution of the Schwinger model with higher-order derivative couplings, in this section we will consider in detail the free case.

A generalization of the free theory to derivative of order $(2N+1)$ is given by the Lagrangian density

$$\mathcal{L}_0=i\bar{\zeta}[\partial\!\!\!/\partial\!\!\!/^\dagger]^N\partial\!\!\!/\zeta\ ,\quad N=0,1,2,3,\ldots, \tag{2.1}$$

which exhibits global gauge and chiral symmetries. Introducing the two-dimensional spinor field $\zeta=(\zeta_{(1)},\zeta_{(2)})^T$, the Lagrangian density (2.1) can be written as a sum of two decoupled pieces:

$$\mathcal{L}=i\zeta^*_{(1)}\partial_-^{2N+1}\zeta_{(1)}+i\zeta^*_{(2)}\partial_+^{2N+1}\zeta_{(2)}\ . \tag{2.2}$$

For the sake of simplicity and in order to show details of the quantization procedure as well as some features of the theory, we initially consider the case with third-order ($N=1$) derivatives.

Since in the case of free theory the two spinor components decouple, we must treat them independently. The upper spinor component will be considered first.

As the first step to quantize the theory we must obtain the basic Poisson brackets. In the usual procedure for higher-derivative theories [19], one would take ζ and its two first time derivatives as independent variables comprising the configuration space. In the present case, previous experience in dealing with higher-derivative scalar theories [20] suggests that we make a point transformation and take $\zeta_{(1)}$, $\partial_-\zeta_{(1)}$, and $\partial_-^2\zeta_{(1)}$ as our basic variables. As will be seen, this choice of variables brings much simpler expressions for the momenta. The variation of the action around a solution of the equation of motion results in the following expressions for the respective conjugate momenta to the above variables:

$$\Pi_1=i\partial_-^2\zeta^*_{(1)}\ , \tag{2.3a}$$

$$\Pi_2=-i\partial_-\zeta^*_{(1)}\ , \tag{2.3b}$$

$$\Pi_3=i\zeta^*_{(1)}\ . \tag{2.3c}$$

From these canonical variables we obtain the following nonvanishing equal-time anticommutation relations:

$$\begin{aligned}\{\partial_-^2\zeta^*_{(1)}(x),\zeta_{(1)}(y)\}&=-\{\partial_-\zeta^*_{(1)}(x),\partial_-\zeta_{(1)}(y)\}\\ &=\{\zeta^*_{(1)}(x),\partial_-^2\zeta_{(1)}(y)\}\\ &=\delta(x_1-y_1)\ .\end{aligned} \tag{2.4}$$

It is worth remarking that, as is usually done, we could have considered the $\zeta_{(\alpha)}$ and $\zeta^*_{(\alpha)}$ fields as independent variables. In so doing, although the theory presents constraints the resulting Dirac brackets would lead to the same anticommutators as above.

In order to obtain the quantum solutions, let us introduce the Fourier decomposition

$$\zeta_{(1)}(x)=\int dk_+dk_-e^{-i[k_+x^++k_-x^-]}\tilde{\zeta}(k_+,k_-)\ . \quad (2.5)$$

Thus, the equation of motion in momentum space is given by

$$k_-^3\tilde{\zeta}(k_-,k_+)=0\ , \quad (2.6)$$

whose general solution is

$$\tilde{\zeta}(k_+,k_-)=\tilde{\zeta}_1(k_+)\delta(k_-)+\tilde{\zeta}_2(k_+)\frac{d}{dk_-}\delta(k_-)+\tilde{\zeta}_3(k_+)\frac{d^2}{dk_-^2}\delta(k_-)\ . \quad (2.7)$$

In this way, the upper spinor component (2.5) can be written as

$$\zeta_{(1)}(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk_+e^{-ik_+x^+}\left[\frac{1}{m}\tilde{\zeta}_1(k_+)+\left[\frac{x^-}{2}\right]\tilde{\zeta}_2(k_+)+\frac{m}{2}\left[\frac{x^-}{2}\right]^2\tilde{\zeta}_3(k_+)\right]e^{-|k_+|\epsilon}\ , \quad (2.8)$$

where m is a finite arbitrary mass scale introduced for later convenience, and the $\tilde{\zeta}_i$ fields have been redefined. A convergence factor for large momenta has also been introduced. The anticommutation relations now read

$$\{\tilde{\zeta}_1(k_+),\tilde{\zeta}_3^*(k'_+)\}=-\{\tilde{\zeta}_2(k_+),\tilde{\zeta}_2^*(k'_+)\}=\{\tilde{\zeta}_3(k_+),\tilde{\zeta}_1^*(k'_+)\}=\delta(k_+-k'_+)\ . \quad (2.9)$$

To construct the Fock space of the theory the vacuum has to be defined. The annihilation operators will be chosen through

$$\tilde{\zeta}_i(k_+)|0\rangle=\tilde{\zeta}_i^*(-k_+)|0\rangle=0\ , \quad (2.10)$$

for $k_+>0$ and $i=1$, 2, or 3. With such a choice the one-particle states possess positive-definite energies.

The above-defined field (2.8) represents the operator solution for the free theory described by the Lagrangian (2.1), from which the Wightman functions of the theory can be readily obtained. In particular, the two-point function is then given by

$$\langle 0|\zeta_{(1)}(x)\zeta_{(1)}^*(y)|0\rangle=\frac{i}{16\pi}\frac{(y^--x^-)^2}{y^+-x^++i\epsilon}\ , \quad (2.11)$$

while higher-point functions are obtained by means of Wick's theorem just as in the first-order case. Note that the Wightman two-point function for the standard two-dimensional free fermionic theory coincides with the $-\langle 0|\partial_-\zeta\partial_-\zeta^*|0\rangle$ and $\langle 0|(\partial_-^2\zeta)\zeta^*|0\rangle$ correlators.

In order to achieve a clear understanding of this model and aiming at a bosonization scheme, we can further redefine the fields in such a way as to obtain a diagonal anticommutation structure. To this end, we define the new left spinor component $\psi_{(\alpha)}^i$ (α=Lorentz index)

$$\tilde{\psi}_{(1)}^1(k_+)=\frac{1}{\sqrt{2}}[\tilde{\zeta}_1(k_+)+\tilde{\zeta}_3(k_+)]\ , \quad (2.12a)$$

$$\tilde{\psi}_{(1)}^3(k_+)=\frac{1}{\sqrt{2}}[\tilde{\zeta}_1(k_+)-\tilde{\zeta}_3(k_+)]\ , \quad (2.12b)$$

$$\tilde{\psi}_{(1)}^2(k_+)=\tilde{\zeta}_2(k_+)\ . \quad (2.12c)$$

The commutation relations are now diagonalized and we get

$$\{\tilde{\psi}_{(1)}^1(k_+),\tilde{\psi}_{(1)}^{1*}(k'_+)\}=-\{\tilde{\psi}_{(1)}^2(k_+),\tilde{\psi}_{(1)}^{2*}(k'_+)\}=-\{\tilde{\psi}_{(1)}^3(k_+),\tilde{\psi}_{(1)}^{3*}(k'_+)\}=\delta(k_+-k'_+)\ . \quad (2.13)$$

Taking into account the Fourier representation for the usual two-dimensional free Dirac field

$$\psi_{(1)}^j(x^+)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk_+e^{-ik_+x^+}\tilde{\psi}_{(1)}^j(k_+)\ , \quad (2.14)$$

the original $\zeta(x)$ field (2.8) may be written as

$$\zeta_{(1)}(x)=\frac{1}{\sqrt{2}}\left[\frac{1}{m}+\frac{m}{2}\left[\frac{x^-}{2}\right]^2\right]\psi_{(1)}^1(x^+)+\frac{1}{\sqrt{2}}\left[\frac{1}{m}-\frac{m}{2}\left[\frac{x^-}{2}\right]^2\right]\psi_{(1)}^3(x^+)+x^-\psi_{(1)}^2(x^+)\ . \quad (2.15)$$

From the last expression, it is clear that the basic excitations of the theory are three independent free fermionic particles, one of them quantized with positive metric and the other ones with negative metric.

It is to be stressed that, since the mapping (2.15) between the ζ and ψ_j fields involves the time explicitly, the total Hamiltonian, while expressed in terms of the diagonal ψ_j fields, does not have the usual form. Nevertheless, the part of Hamiltonian that evolves the ψ_j fields will be the canonical one.

The expression for the lower spinor component can be obtained from the upper one by simply making the replacement $x^\pm\to x^\mp$.

The conserved currents associated with the global gauge and chiral symmetries are most easily obtained in terms of the variables ζ, $\partial_-\zeta$, and $\partial_-^2\zeta$ as independent fields. In this way, we obtain the light-cone current components

$$J^+(x)=\zeta_{(1)}^*(x)\partial_-^2\zeta_{(1)}(x)-\{\partial_-\zeta_{(1)}^*(x)\}\partial_-\zeta_{(1)}(x)+\{\partial_-^2\zeta_{(1)}^*(x)\}\zeta_{(1)}(x)\ , \quad (2.16a)$$

$$J^-(x)=\zeta_{(2)}^*(x)\partial_+^2\zeta_{(2)}(x)-\{\partial_+\zeta_{(2)}^*(x)\}\partial_+\zeta_{(2)}(x)+\{\partial_+^2\zeta_{(2)}^*(x)\}\zeta_{(2)}(x)\ , \quad (2.16b)$$

which satisfy independent conservation laws

$$\partial_- J^+(x)=\partial_+ J^-(x)=0 \ . \tag{2.17}$$

In terms of the diagonal ψ^j fermion field operators, the current components (2.16) are given by

$$J^+(x)=\psi^{*1}_{(1)}(x)\psi^1_{(1)}(x)-\psi^{*2}_{(1)}(x)\psi^2_{(1)}(x) -\psi^{*3}_{(1)}(x)\psi^3_{(1)}(x) \ , \tag{2.17a}$$

$$J^-(x)=\psi^{*1}_{(2)}(x)\psi^1_{(2)}(x)-\psi^{*2}_{(2)}(x)\psi^2_{(2)}(x) -\psi^{*3}_{(2)}(x)\psi^3_{(2)}(x) \ . \tag{2.17b}$$

In the expressions above, the minus signs together with the fields quantized with negative metric ensure the correct role for the generators of the global gauge and chiral transformations. In terms of the diagonal fields, the vector current can be written as

$$J^\mu=\sum_{j=1}^{3} p_j \bar{\psi}^j\gamma^\mu\psi^j \ , \tag{2.18}$$

where p_j indicates the positive or negative metric quantization prescription for the ψ^j fields ($p_1=-p_2=-p_3=1$).

The generalization to arbitrary $(2N+1)$ order is straightforward. The upper component of the free fermionic field can be written in terms of a set of $(2N+1)$ diagonal fields as

$$\psi_{(1)}(x)=\sum_{j=1}^{2N+1} f_j(x^-)\psi^j_{(1)}(x^+) \tag{2.19a}$$

with

$$f_j(x^-)=\frac{1}{\sqrt{2}m^N}\left[\frac{(mx^-/2)^{j-1}}{(j-1)!}+\frac{(-mx^-/2)^{2N+1-j}}{(2N+1-j)!}\right] \tag{2.19b}$$

and

$$f_{2N+2-j}(x^-)=\frac{1}{\sqrt{2}m^N}\left[\frac{(mx^-/2)^{j-1}}{(j-1)!} -\frac{(-mx^-/2)^{2N+1-j}}{(2N+1-j)!}\right] \tag{2.19c}$$

for $1\le j\le N$, whereas

$$f_{N+1}=\frac{(x^-/2)^N}{N!} \ . \tag{2.19d}$$

For $j\le N$ the field is quantized with a positive metric, for $j\ge(N+2)$ with a negative metric, and the metric sign of the ψ^{N+1} fields is $(-1)^N$.

For N odd (even) the diagonalization is performed introducing $N+1$ (N) fermionic fields quantized with negative (positive) metric and N ($N+1$) fermionic fields quantized with positive (negative) metric.

The generalized two-point function is given by

$$\langle 0|\zeta^*_\alpha(x)\zeta_\alpha(y)|0\rangle=\frac{i2^{-(2N+1)}}{(2N)!\pi}\frac{(x^\mp-y^\mp)^{2N}}{x^\pm-y^\pm+i\epsilon} \ . \tag{2.20}$$

The higher-order derivative nature of the theory is responsible for the violation of clustering. For the upper (lower) spinor component, the two-point function (2.20) violates the cluster decomposition in the x^- (x^+) light-cone coordinate.

With the purpose of introducing the bosonization scheme, let us consider the case $N=1$ previously studied. Since the $\psi^j_{(\alpha)}$ fields are free and canonical, the bosonization scheme to be employed is the standard one (see Halpern, Ref. [15]). In order to ensure the correct anticommutation relations (2.13), Klein factors must be introduced. For the case $N=1$, the bosonized expressions for the free and canonical Dirac field operators ψ^j are given by

$$\psi^1_{(\alpha)}(x)=\left[\frac{\mu}{2\pi}\right]^{1/2}\mathcal{K}_2\mathcal{K}_3{:}\exp\{i\sqrt{\pi}[\gamma^5_{\alpha\alpha}\tilde{\phi}_1(x)+\phi_1(x)]\}{:} \ , \tag{2.21a}$$

$$\psi^2_{(\alpha)}(x)=\left[\frac{\mu}{2\pi}\right]^{1/2}\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3 \times{:}\exp\{i\sqrt{\pi}[\gamma^5_{\alpha\alpha}\tilde{\phi}_2(x)+\phi_2(x)]\}{:} \ , \tag{2.21b}$$

$$\psi^3_{(\alpha)}(x)=\left[\frac{\mu}{2\pi}\right]^{1/2}\mathcal{K}_3{:}\exp\{i\sqrt{\pi}[\gamma^5_{\alpha\alpha}\tilde{\phi}_3(x)+\phi_3(x)]\}{:} \ , \tag{2.21c}$$

where ϕ_j ($\tilde{\phi}_j$) are free and massless scalar (pseudoscalar) fields satisfying $\partial_\mu\phi_j=\varepsilon_{\mu\nu}\partial^\nu\tilde{\phi}_j$; μ is an arbitrary finite mass scale. The Klein factors $\mathcal{K}_j$ are given by

$$\mathcal{K}_j={:}\exp\left[i\sqrt{\pi}\int_{-\infty}^{+\infty}\partial_{z^0}\phi_j(z)dz^1\right]{:} \ . \tag{2.22}$$

In the expression (2.21) we have suppressed the Klein factors which ensure the anticommutation relations between the two spinor components of ψ^j.

Making use of the bosonized expressions given by Eqs. (2.21) and the standard point-splitting limit procedure [21], we obtain, from (2.18),

$$J^\mu=-\left[\frac{1}{\pi}\right]^{1/2}\sum_{j=1}^{3}\partial^\mu\phi_j \ . \tag{2.23}$$

Introducing the U(1) scalar potential Φ via

$$\Phi=\frac{1}{\sqrt{3}}\sum_{j=1}^{3}\phi_j \ , \tag{2.24}$$

we can write the vector current as

$$J^\mu=-\sqrt{3/\pi}\,\partial^\mu\Phi \ . \tag{2.25}$$

We also introduce two independent canonical free scalar fields φ^{i_D} such that the original fields can be written as [22]

$$\phi_j=\frac{1}{\sqrt{3}}\Phi+\sum_{i_D=1}^{2}\lambda^{i_D}_{jj}\varphi^{i_D} \ , \tag{2.26}$$

where the λ^{i_D} are two diagonal matrices of the SU(3). In this way, the U(1) charge dependence can be factorized in (2.21) and we get

$$\psi_{(\alpha)}(x) = :\exp\left\{ i\left[\frac{\pi}{3}\right]^{1/2}[\gamma^5_{\alpha\alpha}\tilde{\Phi}(x)+\Phi(x)]\right\}: \times \sum_{j=1}^{3} f_j(x^{\mp})\Gamma^j_{(\alpha)}(x^{\pm}) \tag{2.27}$$

with

$$\Gamma^j_{(\alpha)}(x^{\pm}) = \left[\frac{\mu}{2\pi}\right]^{1/2} \times :\exp\left[i\sqrt{\pi}\sum_{i_D}\lambda^{i_D}_{jj}(\gamma^5_{\alpha\alpha}\tilde{\varphi}^{i_D}+\varphi^{i_D})\right]: . \tag{2.28}$$

In the last expression we have suppressed the corresponding Klein factors. Note that the Γ^j operators are U(1) neutral. Using the decomposition (2.26), the charges associated with the "component" fermions ψ^j can be written as

$$Q^j = Q + q^j , \tag{2.29}$$

where the U(1) charge reads

$$Q = -\left[\frac{3}{\pi}\right]^{1/2}\int_{-\infty}^{+\infty}\partial_{z^0}\Phi(z)dz^1 , \tag{2.30}$$

and the charges associated with the residual "infrafermions" Γ^j are

$$q^j = -\frac{1}{\sqrt{\pi}}\sum_{i_D=1}^{2}\lambda^{i_D}_{jj}\int_{-\infty}^{+\infty}\partial_{z^0}\varphi^{i_D} . \tag{2.31}$$

The generalization for arbitrary N is straightforward, and can be accomplished by introducing the $2N$ diagonal matrices of the SU$(2N+1)$.

III. LOCAL GAUGE INVARIANCE

In this section we discuss the Schwinger model (SM) with derivatives of order $(2N+1)$ defined by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}(F_{\mu\nu})^2 + \bar{\Psi}\Delta\!\!\!/^{(2N+1)}\Psi , \tag{3.1}$$

where the covariant derivative of order $(2N+1)$ is defined by

$$\Delta\!\!\!/^{(2N+1)} = [D\!\!\!/ D\!\!\!/^{\dagger}]^N D\!\!\!/ , \tag{3.2}$$

the usual covariant derivative being given by

$$D\!\!\!/ = \gamma^{\mu}(\partial_{\mu} - ieA_{\mu}) . \tag{3.3}$$

As in the usual $(N=0)$ SM [13], the electromagnetic interaction is introduced by performing a chiral operator gauge transformation on the free fermionic field operator. Thus, we write the operator solution as

$$\Psi(x) = :e^{i\gamma^5[\lambda\tilde{\Sigma}(x)+\delta\tilde{\eta}(x)]}:\zeta(x) , \tag{3.4}$$

where ζ is the free fermionic field operator (2.15); λ and δ are constants to be determined later on. The gauge field is then identified as being given by

$$A_{\mu} = -\frac{1}{e}\varepsilon_{\mu\nu}\partial_{\nu}(\lambda\tilde{\Sigma}+\delta\tilde{\eta}) . \tag{3.5}$$

From Eq. (2.18), we see that the vector current can be readily computed by the gauge-invariant point-splitting limit prescription [13]

$$\mathcal{J}_{\mu} = \lim_{\epsilon\to 0} Z^{-1}(\epsilon)\sum_{j=1}^{2N+1} p_j[\bar{\psi}^j(x+\epsilon)\gamma_{\mu}\mathcal{G}(x+\epsilon;x)\psi^j(x) - \text{VEV}] , \tag{3.6a}$$

where VEV denotes the vacuum expectation value,

$$\mathcal{G}(x+\epsilon;x) = \exp\left[i\left[-\gamma^5[\lambda\tilde{\Sigma}(x+\epsilon)+\delta\tilde{\eta}(x+\epsilon)] + e\int_x^{x+\epsilon}A_{\mu}(z)dz^{\mu} + \gamma^5[\lambda\tilde{\Sigma}(x)+\delta\tilde{\eta}(x)]\right]\right] , \tag{3.6b}$$

and Z^{-1} is a finite renormalization constant. From (3.5) and (2.21) we find

$$\mathcal{J}^{\mu}(x) = -\frac{2N+1}{\pi}\lambda\varepsilon^{\mu\nu}\partial_{\nu}\tilde{\Sigma}(x) + L^{\mu}(x) , \tag{3.7}$$

with

$$L^{\mu}(x) = -\frac{(2N+1)\delta}{\pi}\partial^{\mu}\eta(x) + \left[\frac{2N+1}{\pi}\right]^{1/2}\partial^{\mu}\Phi(x) . \tag{3.8}$$

Since the model possesses $(2N+1)$ fermionic degrees of freedom, the summation in Eq. (3.6) is responsible for the appearance of the factor $(2N+1)$ in the expression (3.7) for the current.

As in the usual case [13,21], due to the presence of the longitudinal current L^{μ} the Maxwell equations are only satisfied on the physical subspace defined by the subsidiary condition

$$\langle \Xi | L^{\mu}(x) | \Theta\rangle = 0 , \quad |\Xi\rangle, |\Theta\rangle \in \mathcal{H}_{\text{phys}} , \tag{3.9}$$

with $\tilde{\Sigma}$ satisfying the equation of motion

$$\left[\Box + \frac{(2N+1)e^2}{\pi}\right]\tilde{\Sigma}(x) = 0 . \tag{3.10}$$

Condition (3.9) implies that $L^{\mu}(x)$, applied to the Fock vacuum, generates states with zero norm. Hence, η must be a canonical free field, quantized with indefinite metric. This fixes the constant δ to be $\delta^2 = \pi/(2N+1)$. The constant λ is chosen to be $\lambda=\delta$ in order for Ψ to approach the free canonical fermion field at short distances. This also ensures the canonical commutation relation for the vector field A^{μ}.

The anomalous divergence of the axial-vector current is then given by

$$\partial_{\mu}\mathcal{J}^{\mu}_5 = -\frac{(2N+1)e}{2\pi}\varepsilon_{\mu\nu}F^{\mu\nu} . \tag{3.11}$$

With the use of the decomposition (2.27), the fermion field operator (3.4) can be written as

$$\Psi_\alpha(x) = :\exp[i\sqrt{\pi}\gamma^5_{\alpha\alpha}\tilde{\Sigma}(x)]::\exp\left[i\left[\frac{\pi}{2N+1}\right]^{1/2}\{\gamma^5_{\alpha\alpha}[\tilde{\Phi}(x)+\tilde{\eta}(x)]+\Phi(x)\}\right]:\sum_{j=1}^{2N+1} f_j(x^\mp)\Gamma^j_\alpha(x^\pm) . \tag{3.12}$$

The fermionic field operator which commutes with the longitudinal current (3.8), thus belonging to the physical subspace $\mathcal{H}_{\text{phys}}$, is obtained from the field (3.12) by performing the following operator gauge transformation:

$$\Psi(x)\to\hat{\Psi}(x) = :\exp\left[i\left[\frac{\pi}{2N+1}\right]^{1/2}\eta(x)\right]:\Psi(x) , \tag{3.13a}$$

$$A_\mu(x)\to\hat{A}_\mu(x) = A_\mu(x)+\frac{1}{e}\left[\frac{\pi}{2N+1}\right]^{1/2}\partial_\mu\eta(x) . \tag{3.13b}$$

In this "physical" gauge, the fermionic field operator can be factorized as

$$\hat{\Psi}(x) = :\exp\left[i\left[\frac{\pi}{2N+1}\right]^{1/2}\gamma^5\tilde{\Sigma}(x)\right]: \times\sum_{j=1}^{2N+1} f_j(x^\mp)\Gamma^j(x^\pm)\sigma , \tag{3.14}$$

where σ is an operator with a scale dimension of zero given by

$$\sigma = \exp\left[i\left[\frac{\pi}{2N+1}\right]^{1/2}\{\gamma^5[\tilde{\Phi}(x)+\tilde{\eta}(x)] +[\Phi(x)+\eta(x)]\}\right] . \tag{3.15}$$

The operator σ commutes with all the observables of the theory. On the physical subspace defined by (3.9) it acts as a constant operator which merely carries the bare charge and chiral selection rules. As in the case of the standard SM [13,21], an infinite set of vacuum states is generated by repeated application of σ on the Fock vacuum. As is well known, this vacuum degeneracy implies a violation of clustering.

This can be seen by considering the two-point functions of operators carrying bare U(1) and chiral-U(1) selection rules. Defining the operators $\mathcal{O}_\alpha$ via

$$\mathcal{O}_\alpha(x) = \prod_{j=1}^{2N+1} :\exp\left[i\gamma^5_{\alpha\alpha}\left[\frac{\pi}{2N+1}\right]^{1/2}\Sigma(x)\right]:\Gamma^j_\alpha(x)\sigma_\alpha \tag{3.16}$$

we obtain

$$\lim_{|a|\to\infty}\langle 0|\mathcal{O}_1^*(x)\mathcal{O}_2(x)\mathcal{O}_2^*(x+a)\mathcal{O}_1(x+a)|0\rangle = |\langle 0|\mathcal{O}_1^*(x)\mathcal{O}_2(x)|-(2N+1);(2N+1)\rangle|^2\neq 0 \tag{3.17}$$

with

$$|-(2N+1);(2N+1)\rangle = \sigma_1^{*2N+1}\sigma_2^{2N+1}|0\rangle . \tag{3.18}$$

The vacuum state above carries $(2N+1)$ units of free conserved charge and chirality. As in the standard SM, the cluster decomposition is restored with respect to the physical vacuum obtained in the usual way by considering the coherent superposition

$$|\theta_1;\theta_2\rangle = \frac{1}{2\pi}\sum_{n_1 n_2} e^{-in_1\theta_1-in_2\theta_2}|n_1;n_2\rangle . \tag{3.19}$$

In this way we obtain

$$\sigma_\alpha|\theta_1;\theta_2\rangle = e^{i\theta_\alpha}|\theta_1;\theta_2\rangle , \tag{3.20}$$

thus providing an irreducible representation for the observables.

IV. FUNCTIONAL INTEGRAL APPROACH

Our results for the anomalous divergence of the axial current and the dynamical mass generated for the gauge field, obtained in the framework of the operator solution, appear to disagree with those derived previously [17] through the use of Fujikawa's path-integral technique [18]. Therefore, a reexamination of the problem in such terms becomes a necessity.

The method introduced by Fujikawa to deal with gauge theories with fermion fields rests on his observation that although the classical Lagrangian is invariant under a certain gauge transformation, the fermionic measure (suitably defined) in the path integral may not be invariant. If this is the case, the Jacobian of the transformation induces additional terms in the Lagrangian which are responsible for anomalies and dynamical generation of mass.

Let us consider the vacuum functional

$$Z = \mathcal{N}\int[dA_\mu][d\bar{\Psi}][d\Psi]e^{i\int d^2x\mathcal{L}} \tag{4.1}$$

with $\mathcal{N}$ a normalization factor and $\mathcal{L}$ given by Eq. (3.1) with $N=1$. The Lagrangian density $\mathcal{L}$ is invariant under the infinitesimal transformations

$$\Psi(x) = [1+ie\gamma^5\epsilon(x)]\Psi'(x) , \tag{4.2a}$$

$$\bar{\Psi}(x) = \bar{\Psi}'(x)[1+ie\gamma^5\epsilon(x)] . \tag{4.2b}$$

In order to find out how the fermionic measure changes, one first performs a Wick rotation to an Euclidean spacetime by letting $x_0\to -ix_4$ and $A_0\to iA_4$. Then

$$\not{D} = \gamma^0(\partial_0-ieA_0)+\gamma^1(\partial_1-ieA_1) \to\gamma^4 D_4+\gamma^1 D_1\equiv\not{D}_E , \tag{4.3a}$$

$$\not{D}^\dagger=-\gamma^0(\partial_0-ieA_0)+\gamma^1(\partial_1-ieA_1)$$
$$\rightarrow-\gamma^4D_4+\gamma^1D_1\equiv(\not{D}^\dagger)_E\ , \tag{4.3b}$$

where $\gamma^4=i\gamma^0$. Next one assumes that there exists an orthonormal basis $\{\Omega_n(x)\}$ whose elements are eigenfunctions of the Hermitian operator $\not{D}_E(\not{D}^\dagger)_E\not{D}_E$, which is the Euclidean version of the Dirac operator that occurs in the fermionic part of the Lagrangian:

$$\not{D}_E(\not{D}^\dagger)_E\not{D}_E\Omega_n=\lambda_n^3\Omega_n\ . \tag{4.4}$$

By expanding Ψ and $\overline{\Psi}$ in terms of this basis, one has

$$\Psi(x)=\sum_n a_n\Omega_n(x)\ , \tag{4.5a}$$

$$\overline{\Psi}(x)=\sum_n \Omega_n^\dagger(x)b_n\ , \tag{4.5b}$$

where a_n,b_n are generators of an infinite-dimensional Grassmann algebra. The expansions (4.5) possess the property of diagonalizing the fermionic part of the action, which justifies the following definition for the fermionic measure in the path integral [18]:

$$[d\overline{\Psi}][d\Psi]=\prod_n db_n da_n\ . \tag{4.6}$$

According to standard computations [18], the change undergone by the fermionic measure is given by

$$[d\overline{\Psi}][d\Psi]=[d\overline{\Psi}'][d\Psi']\exp\left[-2ie\int d^2x\ \epsilon(x)I(x)\right]\ , \tag{4.7}$$

where, formally,

$$I(x)=\sum_n\Omega_n^\dagger(x)\gamma^5\Omega_n(x)\ . \tag{4.8}$$

This sum, however, is divergent and must be regularized. Following Fujikawa's prescription, let us define I by

$$\begin{aligned}I&=\lim_{M\to\infty}\sum_n\Omega_n^\dagger(x)\gamma^5\Omega_n(x)e^{-\lambda_n^6/M^6}\\&=\lim_{M\to\infty}\lim_{y\to x}\sum_n\Omega_n^\dagger(y)\gamma^5e^{-[\not{D}_E(\not{D}^\dagger)_E\not{D}_E]^2/M^6}\Omega_n(x)\\&=\lim_{M\to\infty}\lim_{y\to x}\mathrm{Tr}[\gamma^5e^{-[\not{D}_E(\not{D}^\dagger)_E\not{D}_E]^2/M^6}\delta^{(2)}(x-y)]\\&=\lim_{M\to\infty}\mathrm{Tr}\,\gamma^5\int\frac{d^2k}{(2\pi)^2}e^{-ikx}e^{-[\not{D}_E(\not{D}^\dagger)_E\not{D}_E]^2/M^6}e^{ikx}\ .\end{aligned} \tag{4.9}$$

Making use of the explicit representation for the gamma matrices one easily finds that

$$[\not{D}_E(\not{D}^\dagger)_E\not{D}_E]^2=\begin{bmatrix}D_+^3D_-^3 & 0\\ 0 & D_-^3D_+^3\end{bmatrix}\equiv\begin{bmatrix}P & 0\\ 0 & Q\end{bmatrix}\ , \tag{4.10}$$

where

$$D_\pm=iD_4\pm D_1\ . \tag{4.11}$$

Notice that the highest power of k occurring in the operators P and Q applied to e^{ikx} is $-k_B^6$. Thus, having taken the trace before integrating, it is convenient to write Eq. (4.9) in the form

$$\begin{aligned}I&=\lim_{M\to\infty}\int\frac{d^2k}{(2\pi)^2}e^{-ikx}(e^{-Q/M^6}-e^{-P/M^6})e^{ikx}\\&=\lim_{M\to\infty}\int\frac{d^2k}{(2\pi)^2}e^{k_E^6/M^6}e^{-ikx}[e^{-(Q+k_E^6)/M^6}-e^{-(P+k_E^6)/M^6}]e^{ikx}\\&\equiv\lim_{M\to\infty}\int\frac{d^2k}{(2\pi)^2}e^{k_E^6/M^6}g(k_E^2)\ ,\end{aligned} \tag{4.12}$$

with

$$k_E^2=-(k_1^2+k_4^2)\ . \tag{4.13}$$

Expansion of the exponentials shows that $g(k_E^2)$ is of the form

$$g(k_E^2)=\alpha\frac{k_E^4}{M^6}+\beta\frac{k_E^8}{M^{12}}+\cdots\ , \tag{4.14}$$

where in each fraction with the denominator M^6, M^{12}, and so on, only the term containing the highest power of k_E^2 is displayed. Insertion of Eq. (4.14) into Eq. (4.12), followed by the change of integration variable $k\rightarrow Mk$, establishes that only the first term of the series (4.14) gives a nonvanishing contribution to I in the limit $M\rightarrow\infty$. Thus

$$\begin{aligned}I&=\int\frac{-idk_1dk_4}{(2\pi)^2}\frac{\alpha k_E^4}{M^6}e^{-(k_1^2+k_4^2)^3/M^6}\\&=-i\alpha\int_0^\infty\frac{2\pi\kappa\, d\kappa}{(2\pi)^2}\kappa^4e^{-\kappa^6}=-\frac{i\alpha}{12\pi}\ .\end{aligned} \tag{4.15}$$

Now, α in Eq. (4.14) is given by the coefficient of k_E^4 in

$$e^{-ikx}(P-Q)e^{ikx}=e^{-ikx}(D_+^3D_-^3-D_1^3D_+^3)e^{ikx}\ . \tag{4.16}$$

Considering only terms proportional to k_E^4, by just counting one finds ($\mathcal{A}=-ieA$)

$$\begin{aligned}&e^{-ikx}(\partial_++\mathcal{A}_+)(\partial_++\mathcal{A}_+)(\partial_++\mathcal{A}_+)(\partial_-+\mathcal{A}_-)\\&\quad\times(\partial_-+\mathcal{A}_-)(\partial_-+\mathcal{A}_-)e^{ikx}=-9iek_E^4\partial_+A_-\ ,\end{aligned} \tag{4.17}$$

where symmetric terms under the exchange $+\leftrightarrow-$ have been omitted. Therefore

$$\alpha=-9ie(\partial_+ A_- - \partial_- A_+) \tag{4.18}$$

whence

$$I=-\frac{3e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu} \text{ (Minkowski)} . \tag{4.19}$$

This last result leads finally to

$$[d\overline{\Psi}][d\Psi]=[d\overline{\Psi}'][d\Psi'] \times \exp\left[\frac{3ie^2}{2\pi}\int d^2x\, \epsilon(x)\varepsilon_{\mu\nu}F^{\mu\nu}(x)\right] . \tag{4.20}$$

The variation of the fermionic part of the action under the transformation (4.2) is

$$S \rightarrow S + ie\int d^2x\, \epsilon(x)\partial_\mu \mathcal{J}_5^\mu(x) , \tag{4.21}$$

where

$$\mathcal{J}_5^\mu = 3\overline{\Psi}\gamma_5\gamma^\mu \not{D}^\dagger \not{D}\Psi + 3\partial_\nu(\overline{\Psi}\gamma_5\not{D}\gamma^{\nu\dagger}\gamma^\mu\Psi) - \partial_\nu\partial_\lambda(\overline{\Psi}\gamma_5\gamma^\mu\gamma^{\nu\dagger}\gamma^\lambda\Psi) . \tag{4.22}$$

The anomalous axial divergence, defined by means of

$$\left.\frac{\delta Z}{\delta\epsilon(x)}\right|_{\epsilon=0}=0 , \tag{4.23}$$

yields

$$\partial_\mu \mathcal{J}_5^\mu = -\frac{3e}{2\pi}\varepsilon_{\mu\nu}F^{\mu\nu} . \tag{4.24}$$

As to dynamical generation of mass for the gauge field, it can be discussed by a standard procedure [23]. Through successive infinitesimal steps one performs a finite chiral gauge transformation that decouples photon and fermion fields. With the repeated use of Eq. (4.20), at the end of the calculation one reaches an action functional for a massless fermion field and a massive gauge field without interaction between them. The mass acquired by the photon turns out to be $m^2=3e^2/\pi$.

The above results for $N=1$ can be easily extended to arbitrary N. The regularized sum is now

$$I_N=\lim_{M\to\infty}\operatorname{Tr}\gamma^5\int\frac{d^2k}{(2\pi)^2}e^{-ikx} \times \exp(-\{[\not{D}_E(\not{D}^\dagger)_E]^N\not{D}_E\}^2/M^{4N+2}) . \tag{4.25}$$

Following the same line of reasoning as before we find that the only nonvanishing contribution to I_N is

$$I_N=-i\alpha_N\int_0^\infty\frac{2\pi\kappa\, d\kappa}{(2\pi)^2}\kappa^{4N}e^{-\kappa^{4N+2}} = -\frac{i\alpha_N}{2\pi(4N+2)} , \tag{4.26}$$

where, by counting,

$$\alpha_N=-\frac{(2N+1)^2e}{4\pi}\varepsilon_{\mu\nu}F^{\mu\nu} , \tag{4.27}$$

leading to

$$[d\overline{\Psi}][d\Psi]=[d\overline{\Psi}'][d\Psi'] \times \exp\left[\frac{(2N+1)ie^2}{(2\pi)}\int d^2x\, \epsilon(x)\varepsilon_{\mu\nu}F_{\mu\nu}(x)\right] . \tag{4.28}$$

The same arguments used previously to find the mass acquired by the photon now conduce to $m^2=(2N+1)e^2/\pi$.

Our results obtained within the framework of the path-integral formalism coincide with those originally found by means of the canonical quantization procedure, but disagree with the findings of Barcelos-Neto and Natividade [17], who also made use of Fujikawa's method. The reason for the discrepancy is as follows. The operator that was used for regularizing the sum I in Ref. [17] was $\mathcal{D}_E^6$, instead of $[\not{D}_E(\not{D}^\dagger)_E\not{D}_E]^2$. However, these two operators are different inasmuch as $(\not{D}^\dagger)_E\neq(\not{D}_E)^\dagger$, that is, the operations of taking the Hermitian conjugate and rotating to Euclidean spacetime do not commute. As a consequence, expansion of Ψ and $\overline{\Psi}$ in a basis of eigenfunctions of $\not{D}_E$ does not diagonalize the fermionic part of the Euclidean action. But such a diagonalization is a fundamental ingredient to justify Fujikawa's definition (4.6) for the fermionic measure. As shown by Fujikawa, his regularization prescription yields the same result whether one chooses, for example, $\exp(-\not{D}_E^2/M^2)$ or $\exp(-\not{D}_E^6/M^6)$ as regularizing functions. Therefore, what the authors of Ref. [17] really did was the computation of the chiral anomaly and of the mass acquired by the photon for the *standard* Schwinger model. Unsurprisingly, the usual results were obtained.

V. CONCLUSION AND OUTLOOK

The main result presented here was the proof that higher-derivative generalizations of fermion models in two spacetime dimensions are susceptible of bosonization. This was shown both for a free theory and for a generalized Schwinger model by explicitly constructing the relevant Mandelstan operators. In the gauged version the mass acquired by the gauge field becomes bigger as the order of the highest derivative occurring in the Lagrangian increases. This can be quite naturally understood as due to the increasing of the number of fermionic degrees of freedom.

Although our results for the generalized Schwinger model were first obtained within the cadre of canonical quantization, their disagreement with previous results reported on the basis of functional integration techniques prompted us to investigate the model in the latter framework as well. Our findings could be reconciled with those stemming from the application of Fujikawa's formalism by choosing as regularization operator of the fermionic Jacobian the one that is present in the fermionic part of the action, a requirement indeed noted before in the literature [24]. It should be noticed that the choice of regularization operator in the Fujikawa scheme we employed was such as to preserve the classical symmetries of the action, but ultimately it was by comparison with the reliable canonical approach that we fixed the scheme.

We also emphasize the amusing property that the bosonization procedure has taken us from a higher-derivative fermion theory to a bosonic multicomponent model of first order. The point to be stressed is that higher-derivative theories are plagued with undesired properties such as nonunitarity and the existence of ghost states. The absence of these difficulties in their bosonic counterparts can be ascribed to the exponential operator mapping from boson to fermion fields. It should be further stressed the resemblance between the scheme presented here and the Abelian bosonization of non-Abelian fermionic models. In each of these models the explicit symmetries that are present are not the same. In the model treated here the boson counterpart exhibits an $O(2N+1)$ symmetry which is not explicit in the original fermion model. Such a similarity leads us to wonder if in our case there is an alternative scheme that preserves the symmetries explicitly, like non-Abelian bosonization.

There are several interesting questions that can be raised as continuation of the present work. For instance, one may inquire about the possible bosonization of higher-derivative generalizations of interacting fermion theories such as the Thirring and Gross-Neveau massive models. It also appears as promising the search for a not at all unlikely discrete realization of the models investigated here in condensed matter physics.

ACKNOWLEDGMENTS

The authors express their thanks to Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, for partial finanical support.

[1] M. Ostrogradskii, Mem. Acad. St. Pétersburg **6**, 385 (1850).

[2] See P. Havas, Gen. Relativ. Gravit. **8**, 631 (1977), and references therein.

[3] F. Bopp, Ann. Phys. (Leipzig) **38**, 345 (1940).

[4] B. Podolsky, Phys. Rev. **62**, 68 (1942); B. Podolsky and C. Kikuchi, *ibid.* **65**, 228 (1944); B. Podolsky and P. Schwed, Rev. Mod. Phys. **20**, 40 (1948).

[5] A. Pais and G. Uhlenbeck, Phys. Rev. **79**, 145 (1950).

[6] D. G. Boulware, in *Quantum Theory of Gravity*, edited by S. M. Christensen (Hilger, Bristol, 1984).

[7] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1984).

[8] P. Fayet and S. Ferrara, Phys. Rep. **32C**, 249 (1977); P. van Nieuwenhuizen, *ibid.* **68**, 264 (1981); J. Barcelos-Neto and N. R. F. Braga, Phys. Rev. D **39**, 494 (1989).

[9] A. A. Polyakov, Nucl. Phys. **B268**, 406 (1986).

[10] A. A. Slavnov, Theor. Math. Phys. **13**, 174 (1972); **33**, 210 (1977); L. D. Faddeev and A. A. Slavnov, *Gauge Fields* (Benjamin/Cummings, Reading, MA, 1980); J. Barcelos-Neto and N. R. F. Braga, Mod. Phys. Lett. A **4**, 2195 (1989).

[11] P. West, Nucl. Phys. **B268**, 113 (1986).

[12] J. Schwinger, Phys. Rev. Lett. **3**, 269 (1959); Phys. Rev. **128**, 2425 (1962).

[13] J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) **68**, 172 (1971).

[14] B. Klaiber, in *Lectures in Theoretical Physics—Boulder Lectures* (Gordon and Breach, New York, 1967).

[15] S. Coleman, Phys. Rev. D **11**, 2088 (1975); Ann. Phys. (N.Y.) **101**, 239 (1976); S. Mandelstan, Phys. Rev. D **11**, 3026 (1975); M. B. Halpern, *ibid.* **12**, 1684 (1975); **13**, 337 (1976); K. D. Rothe and J. A. Swieca, *ibid.* **15**, 1675 (1977); B. Schroer, *ibid.* **15**, 1684 (1977).

[16] A. Casher, J. Kogut, and L. Susskind, Phys. Rev. Lett. **31**, 792 (1973).

[17] J. Barcelos-Neto and C. P. Natividade, Z. Phys. C **49**, 511 (1991).

[18] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); Phys. Rev. D **21**, 2848 (1980); **22**, 1499(E) (1980).

[19] For recent work on this subject see, for example, C. Batlle, J. Gomis, J. M. Pons, and N. Roman-Roy, J. Phys. A **21**, 2963 (1988); V. V. Nesterenko, *ibid.* **22**, 1673 (1989); C. A. P. Galvão and N. A. Lemos, J. Math. Phys. **29**, 1588 (1988); J. Barcelos-Neto and N. R. F. Braga, Acta Phys. Polonica **20**, 205 (1989); J. Barcelos-Neto and C. P. Natividade, Z. Phys. C **51**, 313 (1991).

[20] R. L. P. G. Amaral and E. C. Marino, J. Phys. A **25**, 5183 (1992).

[21] J. A. Swieca, Fortsch. Phys. **25**, 303 (1977).

[22] L. V. Belvedere, J. A. Swieca, K. D. Rothe, and B. Schroer, Nucl. Phys. **B153**, 122 (1979); L. V. Belvedere, *ibid.* **B276**, 197 (1986).

[23] A. Das and C. Hagen, Phys. Rev. D **32**, 2024 (1985); A. Das and V. S. Mathur, *ibid.* **33**, 489 (1986); R. Roskies and F. A. Schaposnik, *ibid.* **23**, 588 (1981); J. Barcelos-Neto and A. Das, *ibid.* **33**, 2262 (1986).

[24] R. E. Gamboa-Saraví, M. A. Muschietti, F. A. Schaposnik, and J. E. Solomin, Ann. Phys. (N.Y.) **157**, 360 (1984).