AIP | Journal of Mathematical Physics

The Levi–Civita space–time

M. F. A. da Silva, L. Herrera, F. M. Paiva, and N. O. Santos

Citation: Journal of Mathematical Physics **36**, 3625 (1995); doi: 10.1063/1.530986
View online: http://dx.doi.org/10.1063/1.530986
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/36/7?ver=pdfcov
Published by the AIP Publishing

The Levi–Civita space–time

M. F. A. da Silva[a)]
Departamento de Física e Química, Universidade Estadual Paulista, Av. Ariberto Pereira da Cunha 333, 12500, Guaratinguetá, SP, Brazil and Departamento de Astrofísica, CNPq-Observatório Nacional, Rua General José Cristino 77, 20921-400 Rio de Janeiro-RJ, Brazil

L. Herrera[b)]
Departamento de Física, Facultad de Ciencias, Universidade Central de Venezuela and Centro de Física, Instituto Venezolano de Investigaciones Científicas, Caracas, Venezuela

F. M. Paiva and N. O. Santos
Departamento de Astrofísica, CNPq-Observatório, Nacional, Rua General José Cristino 77, 20921-400 Rio de Janeiro-RJ, Brazil

(Received 29 August 1994; accepted for publication 31 March 1995)

Two exact solutions of Einstein's field equations corresponding to a cylinder of dust with net zero angular momentum are considered. In one of the cases, the dust distribution is homogeneous, whereas in the other, the angular velocity of dust particles is constant [Nuovo Cimento B **21**, 64 (1974)]. For both solutions the junction conditions to the exterior static vacuum Levi–Civita space–time were studied. From this study we find an upper limit for the energy density per unit length σ of the source equals $\frac{1}{4}$ for both cases. Thus the homogeneous cluster provides another example [Class. Quant. Grav. **8**, 727 (1991); J. Math. Phys. **27**, 152 (1980)] where the limit of σ is $\frac{1}{4}$. It was also found that the cluster of homogeneous dust has a superior limit for its radius, depending on the constant volumetric energy density ρ_0. © *1995 American Institute of Physics.*

I. INTRODUCTION

The Levi–Civita metric[1] is the most general cylindrical static vacuum metric. It will be used as the exterior space–time of static cylindrical sources.

Some nonvacuum exact solutions with cylindrical symmetry may be found in the literature. One very simple solution is the cluster of particles. This source is constituted by a great number of small gravitational particles which move freely under the influence of the field produced by all of them together. The first model of cluster was presented by Einstein[2] in spherical symmetry. Raychaudhuri and Som,[3] based on the Einstein's ideas, obtained the first cylindrical analog case. They considered the source filled with dust with an equal number of particles moving in clockwise and anticlockwise directions and found some special cylindrical solutions. They also deduced a relation between the gravitational mass and the sum of the free particles' masses of their clusters. Another cluster solution was obtained by Teixeira and Som.[4] They supposed a constant angular velocity for the dust particles. From the Teixeira and Som solution, Lathrop and Orsene[5] found an upper limit for the linear mass density of the linear mass density of the source. They used a definition for this quantity given by Vishveshwara and Winicour.[6]

In 1969 Gautreau and Hoffman[7] demonstrated that there is no timelike circular geodesic in the Levi–Civita metric if $\sigma \geqslant \frac{1}{4}$. Based on this, Bonnor and Martins[8] conjectured that the Levi–Civita metric does not represent an infinite line mass if $\sigma \geqslant \frac{1}{4}$. Later, Bonnor and Davidson[9] presented a cylindrical source, filled with perfect fluid, and showed that the matching of this source with the

---

[a)]Author's Internet addresses, respectively: mfas@on.br, lherrera@conicit.ve, fmpaiva@on.br and nos@on.br
[b)]Mailing address: Apartado 80793, Caracas 1080A, Venezuela.

0022-2488/95/36(7)/3625/7/$6.00

Levi–Civita metric permits values of $\sigma > \frac{1}{4}$, but $< \frac{1}{2}$. In another work, Stela and Kramer[10] found a source with $\sigma \lesssim 0.35$ using a numerical interior solution.

Inspired by the Teixeira and Som solution, and imposing the constancy of the energy density of the source instead of the constancy of the angular velocity, we found an exact solution for a homogeneous cylindrical cluster. For both these clusters we studied the junction conditions to the exterior static vacuum Levi–Civita space–time. We found a superior limit for the linear energy density σ of the source equal to $\frac{1}{4}$ for both cases. The limit obtained for the Teixeira and Som cluster is in accordance with the limit found by Lathrop and Orsene.[5]

The paper is organized as follows. In the next section we present the static cylindrical space–time in general. In Sec. III we give the notation and the space–time at the exterior of the boundary of the source. In Sec. IV we describe the interior space–time. This section is subdivided into two subsections, where we present a new cylindrical cluster solution for homogeneous dust, found by us, and the Teixeira and Som[4] cluster solution. The junction conditions of these clusters to the Levi–Civita space–time are also included in this section. In the conclusion we sum up our main results.

II. SPACE–TIME

The space–time is described by the general cylindrically symmetric static metric

$$ds^2 = -f dt^2 + e^{\mu}(dr^2 + dz^2) + l d\varphi^2, \tag{2.1}$$

where f, μ, and l are functions only of r. The ranges of the coordinates t, z, and φ are

$$-\infty < t < \infty, \quad -\infty < z < \infty, \quad 0 \leq \varphi \leq 2\pi, \tag{2.2}$$

and the hypersurface $\varphi = 0$ and $\varphi = 2\pi$ being identified. The coordinates are numbered

$$x^0 = t, \quad x^1 = r, \quad x^2 = z, \quad x^3 = \varphi. \tag{2.3}$$

We shall impose the Einstein's field equations

$$R_{\mu\nu} = k(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T). \tag{2.4}$$

The nonzero components of $R_{\mu\nu}$ for the metric (2.1) are

$$2e^{\mu} D R_0^0 = \left(\frac{l f'}{D} \right)', \tag{2.5}$$

$$2e^{\mu} D R_3^3 = \left(\frac{f l'}{D} \right)', \tag{2.6}$$

$$2R_{11} = -\mu'' + \mu' \frac{D'}{D} - 2 \frac{D''}{D} + \frac{f' l'}{D^2}, \tag{2.7}$$

$$2R_{22} = -\mu'' - \mu' \frac{D'}{D}, \tag{2.8}$$

where the primes stand for differentiation with respect to r and

$$D^2 = f l. \tag{2.9}$$

For the line element (2.1) we found that the circular geodesics equations are

$$\dot{r}=\dot{z}=0, \tag{2.10}$$

$$l'\dot{\varphi}^2-f'\dot{t}^2=0, \tag{2.11}$$

$$\ddot{t}=0, \tag{2.12}$$

$$\ddot{\varphi}=0, \tag{2.13}$$

$$\left(\frac{ds}{dt}\right)^2=l\omega^2-f, \tag{2.14}$$

where the dot stands for differentiation with respect to s, and the angular velocity of the particle $\omega=(dx^3/ds)(dx^0/ds)^{-1}$ is

$$\omega^2=\left(\frac{d\varphi}{dt}\right)^2=\frac{f'}{l'}. \tag{2.15}$$

For a stationary space–time the normal velocity of the particle defined as the change in the displacement normal to $\tau^\mu=\{1,0,0,0\}$ relative to its displacement parallel to τ^μ, where τ^μ is a timelike Killing vector, is[11]

$$W^\mu\overset{\text{def}}{=}\left[\sqrt{-g_{00}}\left(dx^0+\frac{g_{0a}}{g_{00}}\,dx^a\right)\right]^{-1}V^\mu, \tag{2.16}$$

where

$$V^\mu\overset{\text{def}}{=}\left(-\frac{g_{0a}}{g_{00}}\,dx^a,dx^1,dx^2,dx^3\right),$$

Latin indexes range from 1 to 3. So, for the static metric (2.1), where $g_{0a}=0$, the three velocity associated to the particle, defined by $W^2=W^\mu W_\mu$, is

$$W^2=\frac{l}{f}\,\omega^2=\frac{lf'}{fl'}. \tag{2.17}$$

Considering Eqs. (2.15) and (2.17) the geodesic equation (2.14) can be written as

$$\left(\frac{ds}{dt}\right)^2=(W^2-1)f. \tag{2.18}$$

The above equation shows that circular geodesics are timelike, null, or spacelike for, respectively, $W<1$, $W=1$, and $W>1$.

The space–time is divided into two regions: the interior, with $0\leqslant r\leqslant R$, to a cylindrical surface of radius R centered along z; and the exterior, with $R\leqslant r<\infty$. On the boundary surface $r=R$ the first and second fundamental forms have to be continuous.[12] Choosing the same coordinates for the exterior and interior space–times these conditions become

$$[g^{-}_{\mu\nu}-g^{+}_{\mu\nu}]_\Sigma=0, \tag{2.19}$$

$$[g^{-\prime}_{\mu\nu}-g^{+\prime}_{\mu\nu}]_\Sigma=0, \tag{2.20}$$

where the indexes $-$ and $+$ stand for the interior and exterior spacetimes, respectively.

III. EXTERIOR SPACE–TIME

In this section we present the exterior space–time.

The exterior space–time is filled with vacuum, hence Einstein's equations (2.4) reduce to $R_{\mu\nu}=0$. The general solution for (2.5)–(2.8) is the static Levi–Civita metric,

$$f=ar^{4\sigma}, \quad e^{\mu}=\frac{1}{a}\, r^{4\sigma(2\sigma-1)}, \quad l=\frac{1}{a}\, r^{2(1-2\sigma)}, \tag{3.1}$$

where a and σ are constants. The parameter a is associated with the angular defect[13] while the parameter σ can be interpreted as the linear energy density of the source.[13,14]

We shall now study some properties of the circular geodesics in this space–time. A test particle in circular geodesics has angular velocity (2.15) and three velocity (2.17) given by

$$\omega^2=\frac{2\sigma}{1-2\sigma}\, ar^{2(4\sigma-1)}, \tag{3.2}$$

$$W^2=\frac{2\sigma}{1-2\sigma}. \tag{3.3}$$

From Eq. (2.14) we see that

$$\left(\frac{ds}{dt}\right)^2=-\frac{1-4\sigma}{1-2\sigma}\, ar^{4\sigma}. \tag{3.4}$$

Thus, circular geodesics are timelike, null, or spacelike for, respectively, $0\leqslant\sigma<\frac{1}{4}$, $\sigma=\frac{1}{4}$, and $\frac{1}{4}<\sigma\leqslant\frac{1}{2}$. The limit $\sigma=\frac{1}{2}$ implies $W\rightarrow\infty$.

The nonvanishing Cartan scalars[15] for this metric are

$$\begin{gathered}\Psi_2=-(2\sigma-1)\sigma r^{4\sigma-8\sigma^2-2}, \quad \Psi_4=\Psi_0=(4\sigma-1)\Psi_2,\\ \nabla\Psi_{01'}=\nabla\Psi_{50'}=\sqrt{2}(8\sigma^2-4\sigma+1)(4\sigma-1)(2\sigma-1)\sigma r^{6\sigma-12\sigma^2-3}\\ \nabla\Psi_{10'}=\nabla\Psi_{41'}=\sqrt{2}(4\sigma-1)(2\sigma-1)\sigma r^{6\sigma-12\sigma^2-3}\\ \nabla\Psi_{21'}=\nabla\Psi_{30'}=\sqrt{2}(4\sigma^2-2\sigma+1)(2\sigma-1)\sigma r^{6\sigma-12\sigma^2-3}.\end{gathered} \tag{3.5}$$

Note that the metric is locally flat if and only if $\sigma=0$ or $\frac{1}{2}$.

IV. INTERIOR SPACE–TIME

The interior space–time is described by a cylinder filled with a rotationally symmetric cluster of dust with zero net angular momentum. The energy momentum tensor is

$$T^{\mu}_{\nu}=\tfrac{1}{2}\rho(u^{\mu}u_{\nu}+v^{\mu}v_{\nu}), \tag{4.1}$$

where ρ is the energy density and u^{μ} and v^{μ} are the four velocities

$$u^{\mu}=(u^0,0,0,\omega), \quad v^{\mu}=(u^0,0,0,-\omega) \tag{4.2}$$

satisfying

$$u^{\mu}u_{\mu}=v^{\mu}v_{\mu}=-1. \tag{4.3}$$

From Einstein's equations (2.4) and from (2.5)–(2.8) and (4.1) we obtain

$$fl=r^2, \tag{4.4}$$

$$\mu'=-\frac{f'}{f}\left(1-\frac{rf'}{2f}\right), \tag{4.5}$$

$$k\rho re^{\mu}=-(r\mu')', \tag{4.6}$$

$$W^2=\left(\frac{\omega r}{f}\right)^2=\frac{rf'}{2f}=\frac{1}{(1-rf'/2f)}, \tag{4.7}$$

where we have computed in (4.7) the three-velocity (2.17) of a particle in the cluster. Note that, although the particles of the cluster are rotating, this source, with null net angular momentum, generates a static space–time.

A. Cluster of homogeneous dust

Considering a homogeneous distribution of dust inside the cylinder $0\leqslant r\leqslant R$ we have

$$\rho=\rho_0=\text{const.} \tag{4.8}$$

The solution of (4.5) and (4.6) with (4.8) suitable for a matching to the Levi–Civita metric is

$$f=\frac{1}{\sqrt{2}}\,[1-3br^2+\sqrt{(1+br^2)(1-7br^2)}]^{1/2}\exp\left[\frac{7}{2\sqrt{7}}\left(\arcsin\left[-\frac{(3+7br^2)}{4}\right]-\arcsin\left[-\frac{3}{4}\right]\right)\right], \tag{4.9}$$

$$e^{\mu}=(1+br^2)^{-2}, \tag{4.10}$$

where we imposed that the geometry is Euclidean on the rotation axis and $b\overset{\text{def}}{=}k\rho_0/8$. From (4.7) and (4.9) we have for the three velocity of the dust particles

$$W^2=\frac{\sqrt{1+br^2}-\sqrt{1-7br^2}}{\sqrt{1+br^2}+\sqrt{1-7br^2}}. \tag{4.11}$$

From equation (4.9) we can see that there is a restriction on the radius of the orbit of the homogeneous cluster particles, for a given gravitational mass per unit length of the cylinder. More specifically, we should have that

$$r^2\leqslant\frac{8}{7k\rho_0}, \tag{4.12}$$

in order that the metric potential f could be real. Equation (4.11) shows that in this limit the three velocity of the dust particles is equal to the unit. A similar behavior is also found in the van Stockum solution.[16,17]

Considering the matching between the interior and exterior space–times, given by (2.19) and (2.20), we obtain

$$\sigma=\frac{1}{4}\left(1-\sqrt{\frac{1-7bR^2}{1+bR^2}}\right), \tag{4.13}$$

$$a=\frac{b^2R^{4(2\sigma^2-\sigma+1)}}{\sigma^2(2\sigma-1)^2}=(1+bR^2)^2R^{-4bR^2/(1+bR^2)}. \tag{4.14}$$

Equation (4.13) imposes a superior limit on the radius of the source's boundary. This limit is identical to Eq. (4.12). Furthermore, (4.13) implies that $0\leqslant\sigma\leqslant\frac{1}{4}$, with $\rho_0=0$ corresponding to $\sigma=0$. The value $\sigma=\frac{1}{4}$ corresponds to $W=1$ from equation (4.11). Observe that the limits imposed by Eqs. (3.3) and (3.4) are related with the existence of circular geodesics for test particles in the Levi–Civita space–time, while the above limit arises from the junction conditions. Although the junction conditions (4.13) and (4.14) do not impose any restriction on the value $\sigma=\frac{1}{4}$, Eq. (4.11), as we had seen, shows that when $\sigma=\frac{1}{4}$, which means $R^2=8/(7k\rho_0)$, the three velocity of the particles is equal to the unit. So, the particles of the dust travel with the speed of light (counter-rotating photons), and in order to avoid it we must exclude the value $\sigma=\frac{1}{4}$.

B. Cluster of constant rotating dust

We consider now the Teixeira and Som solution where

$$\omega=\omega_0=\text{const}, \tag{4.15}$$

given by

$$f=\tfrac{1}{2}[1+(1+4\omega_0^2r^2)^{1/2}], \tag{4.16}$$

$$e^{\mu}=(1+4\omega_0^2r^2)^{-1/4}, \tag{4.17}$$

$$2\pi\rho=\frac{\omega_0^2}{(1+4\omega_0^2r^2)^{7/4}}, \tag{4.18}$$

$$W^2=\frac{4\omega_0^2r^2}{(1+\sqrt{1+4\omega_0^2r^2})^2}. \tag{4.19}$$

Now considering the matching (2.19) and (2.20) we obtain

$$\sigma=\frac{1}{4}\left(1-\frac{1}{\sqrt{1+4\omega_0^2R^2}}\right), \tag{4.20}$$

$$a=R^{4\sigma(2\sigma-1)}(1+4\omega_0^2R^2)^{1/4}. \tag{4.21}$$

We shall now state some properties of this matching. We can see that if $\sigma=0$, Eqs. (4.20) and (4.21) give $R\omega_0=0$ and $a=1$, producing the Minkowski space–time as expected. Note that the solution is well behaved for $R=0$ since this implies $\sigma=0$ (i.e., finite density energy per unit length of the infinite line mass). On the other hand, even when $R=0$, from Eq. (4.18), the density energy ρ of the source vanishes only if we have $\omega_0=0$.

Equation (4.20) shows that

$$\lim_{R\to\infty}\sigma=\tfrac{1}{4}. \tag{4.22}$$

So, $\frac{1}{4}$ represents a superior limit for σ in order that the interior solution generates an exterior Levi–Civita space–time. Lathrop and Orsene[5] found the same limit for the linear mass density of the Teixeira and Som cluster solution. They used a definition for this parameter given by Vishveshwara and Winicour.[6] This source, nevertheless, does not present a limitation on the radius of its boundary. As in the case of the homogeneous cluster, here the value $\frac{1}{4}$ for the parameter σ implies that the particles of the cluster are travelling with the speed of light, as can be seen from Eq. (4.19). So, we again need to avoid the value $\sigma=\frac{1}{4}$ or otherwise consider the possibility of counter-rotating photons.

V. CONCLUSION

We found the exact solutions of Einstein's field equations for a homogeneous cylinder constituted by an equal number of particles of dust moving in clockwise and anticlockwise directions. The matching of this source with the static vacuum of Levi–Civita is allowed only for a specific range of the linear energy density parameter, that is $0\leq\sigma\leq\frac{1}{4}$. We also found that, for a given volumetric energy density, this source presents an upper limit for its radius. In the literature there is, at least, the van Stockum solution as another cylindrical example in which there is a limit for the radius of the source depending on its volumetric density.[16,17] In this case, the limitation on the radius comes out in order to avoid a change in the signature of the metric.

Using the solution for a cluster constituted by dust particles with constant angular velocity and zero net angular momentum, obtained first by Teixeira and Som,[4] and matching it with the Levi–Civita metric, we found that also in this case the parameter σ should be smaller than or equal to $\frac{1}{4}$. Considering σ as the gravitational mass per unit length, this limit is in agreement with the result of Lathrop and Orsene.[5] While for the Teixeira and Som solution the matching does not present a limitation to the radius of the source, for our homogeneous cluster solution the matching imposes a superior limit for its radius, depending on the volumetric energy density ρ_0.

We note from Eq. (3.4) that circular geodesics in the Levi–Civita space–time become null when σ is equal to $\frac{1}{4}$. Some authors[7,8] use this fact as a restriction for the linear density of the source. Nevertheless, as pointed out in Ref. 3, this result is similar to the Newtonian cylindrical analog case, i.e., for a higher density cylinder, all particles (with speed less than the light) should fall. However, this argument is not without problems, since, as shown by Bonnor and Martins,[8] in the interval $\frac{1}{4}\leq\sigma\leq\frac{1}{2}$ the gravitational field seems to get weaker as σ increases.[18] Note that $\sigma=\frac{1}{2}$ means that the space–time is locally flat, in accordance to the Cartan scalars.

ACKNOWLEDGMENTS

The authors would like to acknowledge Bill Bonnor for useful discussions. M.F.A.S. and F.M.P. also acknowledge financial assistance from CAPES and CNPq, respectively.

[1] T. Levi-Civita, Rend. Acc. Lincei **26**, 307 (1917).
[2] A. Einstein, Ann. Math. **40**, 921 (1939).
[3] A. K. Raychaudhuri and M. M. Som, Proc. Camb. Phil. Soc. **58**, 338 (1962).
[4] A. F. F. Teixeira and M. M. Som, Nuovo Cimento B **21**, 64 (1974).
[5] J. D. Lathrop and M. S. Orsene, J. Math. Phys. **27**, 152 (1980).
[6] C. V. Vishveshwara and J. Winicour, J. Math. Phys. **18**, 1280 (1977).
[7] R. Gautreau and R. B. Hoffman, Nuovo Cimento B **61**, 411 (1969).
[8] W. B. Bonnor and M. A. P. Martins, Class. Quant. Grav. **8**, 727 (1991).
[9] W. B. Bonnor and W. Davidson, Class. Quant. Grav. **9**, 2065 (1992).
[10] J. Stela and D. Kramer, Acta Phys. Pol. B **21**, 843 (1990).
[11] J. L. Anderson, *Principles of Relativity* (New York, 1967), Sec. 10.6(a).
[12] G. Darmois, *Mémorial des Sciences Mathematiques* (Gauthier-Villars, Paris, 1927), Fasc. 25.
[13] M. F. A. da Silva, L. Herrera, F. M. Paiva, and N. O. Santos, (1994), to be published in Gen. Rel. Grav.
[14] L. Marder, Proc. R. Soc. London, Ser. A **244**, 524 (1958).
[15] M. A. H. MacCallum and J. E. F. Skea, "SHEEP: A Computer Algebra System for General Relativity," in *Algebraic Computing in General Relativity: Lecture Notes from the First Brazilian School on Computer Algebra*, Vol. 2, edited by M. J. Rebouças and W. L. Roque (Oxford University, Oxford, 1994).
[16] W. J. van Stockum, Proc. R. Soc. Edin. **57**, 135 (1937).
[17] F. J. Tipler, Phys. Rev. D **9**, 2203 (1974).
[18] W. B. Bonnor, private communication.