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The magnetic field of a current carrying cosmic string

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The magnetostatic field of an infinite rectilinear current placed in the stationary gravitational field of a rotating cosmic string is found. An interesting aspect of this problem is that although the metric is mathematically very simple, its physical meaning is not trivial. It depends only on topological parameters. So, the cosmic string vacuum space–time is locally equivalent to the Minkowski space–time, but not globally. The calculations are so simple that they can easily be shown in the classroom.

I. INTRODUCTION

This work is based on the following question: How could a background field of a stationary cosmic string affect the solutions of the Maxwell's equations? By background gravitational field we mean that we are neglecting the contribution of the magnetic field generated by the current to the gravitational field. In other words, the magnetic field is a test field for the gravitational interactions. Many works have developed these ideas in spherical symmetry. The simplest cases, the electrostatic and magnetostatic solutions, are treated in several examples in the literature. Some examples are: (i) the electrostatic field of a point charge,[1–3] (ii) the electrostatic and magnetostatic dipole fields,[4] (iii) magnetostatic field of a loop current around a black hole[5] and (iv) the magnetostatic field of an infinite rectilinear current distribution.[6]

Why do the cylindrically symmetric systems interest us? Two arguments justify our interest. The first one is that cylindrical symmetry has played an important role in the discussion of the internal consistency of general relativity itself. For example, this is the simplest symmetry that can generate gravitational waves, predicted by the general relativity theory of Einstein for nonspherical sources.[7] Besides, the differences between general relativity and Newtonian theories are more evident in cylindrically symmetric systems than in spherically symmetric systems. This is particularly true concerning the number of parameters characterizing the space–time. In the spherical case both theories have a single parameter, the total mass of the system producing the gravitational field. In the cylindrical case, the Newtonian theory needs only one parameter, the mass per unit length. On the other hand, general relativity theory needs two (four) parameters in the static[8] (stationary[9]) case. In recent papers there has been presented a possible physical and geometric interpretation of these four parameters.[10,11] These works showed that three of these parameters refer only to the topological structure of the space–time, while the fourth is responsible for the curvature of the space–time.

The second argument to motivate the study of cylindrically symmetric systems is based on the astrophysical applications of the study of cosmic strings. The vacuum space–time of a cosmic string is locally flat although it globally differs from the Minkowski space–time. The range of the angular coordinate in it is smaller. This difference is known as an angular deficit. Consequently, the vacuum space–time of a cosmic string presents some characteristics that could be astrophysically observed. For example, a cosmic string could produce the gravitational lens effect. It either duplicates the image of a distant object or makes the image of an extensive object seem to be interrupted by a line of discontinuity. Another effect, very important because of its cosmological consequences, could occur due to the motion of a cosmic string. In this case, in the region behind it there would be a higher density region, in comparison with the other regions. So, it would be able to provide the necessary density perturbations for galaxy formation. These and other astrophysical effects of a cosmic string are very well explained in a paper of Helliwell and Konkowski (1987).[12] Recently, a cylindrical metric was considered in another astrophysical application, out of the cosmic string context.[13] The authors considered the van Stockum metric,[14] which represents a dust distribution with rigid rotation around the symmetry axis, to model galaxies' jets.

Besides these two strong arguments for studying cylindrically symmetric systems, and particularly cosmic strings, we point out the simplicity of the cosmic string metric (with or without rotation). In this work we show, with very simple mathematical calculations, that the presence of a stationary cosmic string can amplify the magnetostatic field generated by an infinite rectilinear current. Indeed, if the students already know tensor analysis and Maxwell's equations in covariant form, this work can be presented in undergraduate courses. Knowledge of some basic topics of general relativity would be advisable. Although the problem presented here is mathematically easy, it is a motivation for introducing the cosmic string space–time, in connection with the Maxwell equations, and for exploring a purely topological effect presented in these space–times.

The paper is organized as follows. In Sec. II we present the vacuum metric of a stationary cosmic string and solve the vacuum Maxwell's equations to the problem. In Sec. III a geometric interpretation of our results is given. Finally, in the conclusion we sum up our main results.

II. FIELD EQUATIONS

The metric which describes the vacuum space–time of a rotating cosmic string is given by[15]

$$ds^2 = -dt^2 - 2b\,dt\,d\phi + dr^2 + dz^2 + \left(\frac{r^2}{a} - b^2 a\right) d\phi^2, \tag{2.1}$$

where a and b are constants, a being associated with the angular defect while b is associated with the angular momentum per unit of mass of the string J by[12]

$$b = -\frac{4}{\sqrt{a}}\,J. \tag{2.2}$$

Here, we adopt geometric units, that is $c = G = 1$. This choice implies that time and mass are measured in units of distance.

Our aim here is to solve Maxwell's equations, for the magnetostatic case, in the rotating cosmic string background metric (2.1).

We assume that the only nonvanishing component of the vector potential A_μ is $A_2 = A_z$ and that it is a function of r only. Then the unique independent component of the electromagnetic field tensor $F_{\mu\nu}$ is

$$F_{21} = -F_{12} = -\frac{\partial A_z}{\partial r}. \tag{2.3}$$

Maxwell's source equations in the vacuum, in generalized coordinates, are

$$\frac{1}{\sqrt{-g}}\,(\sqrt{-g}F^{\mu\nu})_{,\nu} = 0, \tag{2.4}$$

where $g \equiv \det g_{\mu\nu}$ and the comma denotes ordinary derivative. They imply

$$\frac{\partial}{\partial r}\,(\sqrt{-g}g^{11}g^{22}F_{21}) = 0. \tag{2.5}$$

Equation (2.5) shows that

$$\sqrt{-g}g^{11}g^{22}F_{12} = K = \text{const.} \tag{2.6}$$

So, using the metric (2.1), we get

$$F_{12} = \frac{\sqrt{a}}{\sqrt{r^2 + ab^2(1-a)}}\,K. \tag{2.7}$$

In order to interpret (2.7) physically, let us consider $a=1$ and $b=0$ which reduce the metric (2.1) to the Minkowski metric. This corresponds to the pseudo-Euclidean background case, in which $F_{\mu\nu}$ can be interpreted classically. Doing this, we obtain

$$F_{12} = \frac{\partial A_z}{\partial \rho} = \frac{K}{r}, \tag{2.8}$$

which is the only nonvanishing component of the magnetic field generated by a cylindrical electric current, along the z direction, at a distance r from the axis. We can also associate the constant K with the magnetic permeability in the vacuum μ_0 and the constant current I; that is,

$$K = \frac{\mu_0 I}{2\pi}. \tag{2.9}$$

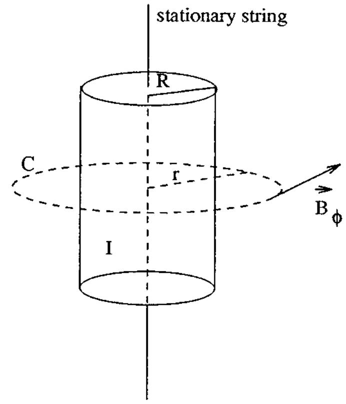

Fig. 1. Magnetic field of a cylindrical electric current distribution. R is the radius of the region which contains the current. Outside this region there is vacuum.

Therefore, Eq. (2.7) represents the generalization of the magnetostatic field of a cylindrical constant electric current that carries a cosmic string. Equation (2.7) also shows that both metric parameters a and b contribute to amplify or diminish the magnetic field B_ϕ, related to that realized, in the same position r, in the Minkowski space–time. The amplification occurs if we consider $a \geqslant 1$ [the restriction upon the value of a is due to its relation with the energy density per unit length of a cosmic string; that is,[11] $\lambda = (1/4)(1 - 1/\sqrt{a})$] and thus $ab^2(1-a) \leqslant 0$. However, the contribution of b to the amplification of B_ϕ only exists if the angular defect exists $(a>1)$. Another intriguing effect is the presence of a lower limit to the radius r in order to preserve a real magnetic field, that is, $r = \sqrt{a(a-1)}b$. This restriction disappears if $b=0$ or $a=1$.

III. GEOMETRIC INTERPRETATION

In Fig. 1 we illustrate the current distribution with the cosmic string along the symmetry axis.

In the globally flat space–time (Minkowski space–time) we have that the magnetic field B_ϕ of an infinite line current is given by

$$B_\phi = \frac{\mu_0 I}{2\pi r}. \tag{3.1}$$

In order to analyze the string vacuum space–time let us simplify the stationary metric (2.1) to the static case, which can be obtained if we put $b = 0$. In this case, the metric (2.1) reduces to

$$ds^2 = -dt^2 + dr^2 + dz^2 + \frac{r^2}{a}\,d\phi^2, \tag{3.2}$$

with $0 \leqslant \phi \leqslant 2\pi$ and the hypersurfaces $\phi = 0$ and $\phi = 2\pi$ being identified.

Observe that this metric differs from the Minkowski metric only by the coordinate transformation

$$\phi' = \frac{1}{\sqrt{a}}\,\phi. \tag{3.3}$$

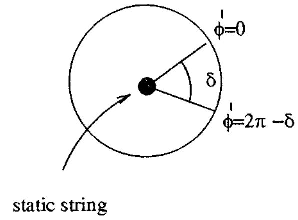

Fig. 2. Geometric representation of the two-surface $dt = dz = 0$ of a static cosmic string. When $\phi = 0$, $\phi' = 0$ but when $\phi = 2\pi$, $\phi' = 2\pi/\sqrt{a}$. So, there is an angular deficit which we call δ.

However, (3.3) implies that the interval of the angular coordinate ϕ' is $[0, 2\pi/\sqrt{a}]$. So, if we consider only the two-dimensional surface defined by $dt = dz = 0$, we have the geometric representation of the space, presented in Fig. 2, where

$$\delta = 2\pi\left(1 - \frac{1}{\sqrt{a}}\right)$$

is called the angular defect, or angular deficit. [Equation (3.3) also shows that the parameter a is unitless. This in connection with Eqs. (2.1) and (2.2) implies that the parameter b is in unit of length.] Observe that $a \geqslant 1$ holds in order that $\delta \geqslant 0$.

Thus, in this space–time, if we calculate the magnetic circulation, the path is not $2\pi r$ but $(2\pi - \delta)r = (2\pi/\sqrt{a})r$. So, we have

$$B_\phi = \frac{\sqrt{a}\mu_0 I}{2\pi r}, \tag{3.4}$$

which is exactly what we have from (2.7), with (2.9), if we put $b=0$.

Another way to give a geometric interpretation to Eq. (2.7), with $b=0$, is to redefine the radius that defines the perimeter around the string, as is shown in Fig. 3. If we substitute $(r^2/a)d\phi^2$ for $R^2 d\phi^2$, we have that ϕ varies from 0 to 2π but the radial distance is changed from r to $r/\sqrt{a}$. In fact, it is not the radial direction which is warped, but the angular direction. That is, meter sticks lying along the ϕ direction either contract or expand, depending upon the value of a ($a >$ or < 1), relative to those sticks measuring radial distances. In this way, the circumference of a circle centered on the z axis is less or greater than $2\pi r$. However, this circle is equivalent to that obtained if we change the radius r by R.

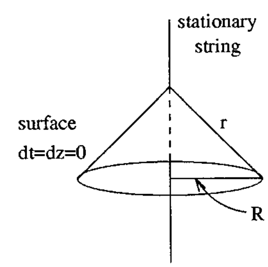

Fig. 3. An alternative geometric interpretation of the two-surface $dt = dz = 0$ to the space–time of a static cosmic string. The effective radius for the calculation of the magnetic circulation is R instead of r, if we consider the correct range of the angular coordinate ϕ, that is $[0,2\pi]$.

Even in the general stationary case, $b \neq 0$, the effect on the magnetic field is analogous to the static case. There the perimeter of a circle centered on the z axis is contracted or expanded depending upon the value of a and b. We can find the equivalent radius which furnishes the same perimeter. In order to calculate the radius that defines the perimeter around the string, in the stationary case, we first find the spatial metric dl^2, that is given by[16]

$$dl^2 = \gamma_{ij} dx^i dx^j, \tag{3.5}$$

where

$$\gamma_{ij} = g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}. \tag{3.6}$$

So, we have $\gamma_{33} = [r^2 + ab^2(1-a)]/a$ or

$$R = \frac{\sqrt{r^2 + ab^2(1-a)}}{\sqrt{a}}.$$

As we can see, the terms involving the parameter b always contribute to reduce the perimeter around the string, as long as we restrict $a > 1$. (In this interpretation we are ignoring the lack of simultaneity between events with the same radial coordinate but different ϕ coordinates since the magnetic field looked for is ϕ independent.)

Thus we conclude that the presence of a stationary cosmic string modifies the magnetostatic field of an infinite rectilinear current. The presence of the topological parameters, a and b, causes an effect on the magnetic field equivalent to diminishing the effective radial distance R of the observer from the current. Note that this effect exists only in the presence of an angular defect.

IV. CONCLUSION

Here, we show that the magnetostatic field of an infinite rectilinear current, placed in the stationary cosmic string space–time, is amplified. This amplification depends on two parameters. One is associated with the angular defect generated by the cosmic string and the other is associated with the angular momentum of the stationary cosmic string.

Although these parameters do not imply a curved space–time, as they are purely topological parameters, the magnetic field is changed by them. It is important to note that the metric (3.2) is locally, but not globally, similar to the Minkowski metric. This global difference is responsible for the effect presented here. So, the local coordinate transformation (3.3) cannot remove this effect.

We believe that this problem can be easily presented in the classroom since the calculations involved here are very simple. On the other hand, some important aspects of cosmic string space–times can be explored, in connection with Maxwell's equations. It would give rise to discussions in the classroom on subjects of present interest in physics.

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[1]W. Israel, ''Event horizons in static electrovac space–times,'' Commun. Math. Phys. **8**, 245–260 (1968).
[2]B. Linet, ''Electrostatic and magnetostatic in the Schwarzschild metric,'' J. Phys. A **9**, 1081–1087 (1976).
[3]B. Leaute and B. Linet, ''Electrostatic in a Reissner–Nordström space–time,'' Phys. Lett. A **58**, 5–6 (1976).
[4]J. A. Souza, ''Field of a dipole in charged black-hole electrostatics,'' Nuovo Cimento B **53**, 81–90 (1979).
[5]J. Petterson, ''Magnetic field of a current loop around a Schwarzschild black-hole,'' Phys. Rev. D **10**, 3166–3170 (1974).
[6]M. F. A. da Silva, O. T. da Silveira Filho, and J. A. Souza, ''Magnetic field of a long wire in a static background field: A classroom exercise in general relativity,'' Am. J. Phys. **57**, 735–737 (1989).
[7]A. Einstein and N. Rosen, ''On gravitational waves,'' J. Franklin Inst. **223**, 43–54 (1937).
[8]T. Levi-Civita, ''ds^2 Einsteiniani in campi Newtoniani. IX-L'analogo del potenziale logarithmico,'' Rend. Acc. Lincei **26**, 101–109 (1919).
[9]T. Lewis, ''Some special solutions of the equations of axially symmetric gravitational fields,'' Proc. R. Soc. London **136**, 176–192 (1932).
[10]M. F. A. da Silva, L. Herrera, F. M. Paiva, and N. O. Santos, ''On the parameters of the Lewis metric for the Lewis class,'' Class. Quant. Grav. **12**, 111–118 (1995a).
[11]M. F. A. da Silva, L. Herrera, F. M. Paiva, and N. O. Santos, ''On the parameters of the Lewis metric for the Weyl class,'' Gen. Rel. Grav. **27**, 859–872 (1995b).
[12]B. Jensen and J. Kučera, ''On a gravitational Aharonov–Bohm effect,'' J. Math. Phys. **34**, 4975–4985 (1993).
[13]T. M. Helliwell and D. A. Konkowski, ''Cosmic strings: Gravitation without local curvature,'' Am. J. Phys. **55**, 401–407 (1987).
[14]R. Opher, N. O. Santos, and A. Wang, ''Geodesic motion and confinement in van Stockum space–time,'' J. Math. Phys. **34**, 1982–1990 (1996).
[15]W. J. van Stockum, ''The gravitational field of a distribution of particles,'' Proc. R. Soc. Edin. **57**, 135–154 (1937).
[16]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975), 4th ed.