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Jacobi ϑ -functions and discrete Fourier transforms

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Properties of the Jacobi ϑ_3 -function and its derivatives under discrete Fourier transforms are investigated, and several interesting results are obtained. The role of modulo N equivalence classes in the theory of ϑ -functions is stressed. An important conjecture is studied. © 2006 American Institute of Physics.

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I. INTRODUCTION

Methods in mathematical physics usually provide an interface between quite different areas of physics, and it is not unusual that such areas advance in parallel, mostly ignoring each other's steps. This is the case with finite dimensional inner product spaces (hereafter mentioned as the "discrete"), with its leading role in quantum mechanics (hence quantum information theory) and in finite signal analysis. References 1–3 provide some links between those theories.

Both quantum mechanics of finite dimensional Hilbert spaces and finite signal analysis rely heavily on the discrete Fourier transform (DFT, sometimes mentioned finite or fractional Fourier transform), and, regarding quantum mechanics, after the seminal work of Weyl on finite dimensional systems,⁴ it was Schwinger who observed and explored the fact that two physical observables whose families of eigenstates are connected *via* DFT share a maximum degree of incompatibility.⁵

Although, at first glance, a finite system might look much simpler than anything defined on a nonenumerable infinite dimensional Hilbert space (hereafter referred to as the "continuum"), there is much more knowledge about the latter than the former. In one phrase, in the continuum we have one, and only one, harmonic oscillator, while in the discrete there are a lot of candidates for that role, each one surely with its virtues, but surely no undisputed champion.

The eigenstates associated to the harmonic oscillator, the Gaussian function and the Hermite polynomials, have a very distinguishable behavior under the action of the (usual) Fourier transform, so widely known that any comment on this regard is completely superfluous. Over such properties rests a huge amount of physical knowledge. On the other hand, however, although the discrete Fourier transform (DFT) is a well known tool, there is nothing on this context which could claim for itself a role analogous to that of the Gaussian function/Fourier transform "duo."

A decisive step in an attempt to "regain," in the discrete, all interpretative power derived from the qualitative behavior of the harmonic oscillator eigenfunctions, lost when one leaves the continuum realm, was given in Ref. 6, where the eigenstates of the DFT are obtained. The purpose of this paper is to further explore this path, showing results which closely parallel those of the continuum. Those results are obtained in a strikingly simple fashion, exploring the technique of breaking infinite sums in modulo N equivalence classes. Pertinent research on the eigenstates of the DFT can also be found in Ref. 7.

A remark must be made about the orthogonality of the DFT's eigenstates. Mehta has conjectured that those states are indeed orthogonal, what seems to be most reasonable. One may be led to believe that, just as in the continuum, the eigenstates of the DFT may be also (nondegenerate)

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eigenstates of some other (unitary or self-adjoint) operator, and thus orthogonal. Further evidence supporting such conjecture is that the continuous limit of the DFT eigenstates recovers, as expected, the Gaussian times the Hermite polynomials. However, as it will be shown, quite surprisingly, the conjecture *does not* hold, giving another fine example of the peculiarities of the finite dimensional context.

The eigenstates of the DFT are seen to be the Jacobi ϑ_3 -function and its derivatives.⁸ Interest in Jacobi ϑ -functions, by their own turn, may come from a variety of directions. First, its mathematical interest goes without saying (see, for example, Ref. 9 and references therein). To cite relatively recent examples in physics, in quantum physics it is deeply related to coherent states associated to both circle¹⁰ and finite lattice topology.¹¹ Its modular properties have proven to be of fundamental importance in superstring theory, as it is shown by standard literature in this field.¹²

The basic notation adopted in this paper and some preliminary results are presented in the next section. Following, orthogonality of the DFT's eigenstates is discussed. Section IV contains the main results, for which a two variable generalization is verified in the subsequent section. Further relations among ϑ_3 -functions are obtained in Sec. VI, which precedes the concluding section.

II. PRELIMINARY RESULTS

In Ref. 6 it is shown that there is a set of functions with the following remarkable property

$$f_n(j) = \frac{i^n}{\sqrt{N}} \sum_{k=0}^{N-1} f_n(k) \exp\left[\frac{2\pi i}{N} kj\right], \quad (1)$$

where N is a natural number. The functions

$$f_n(j) = \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}(\alpha N + j)^2\right] H_n(\epsilon(\alpha N + j)), \quad \epsilon = \sqrt{\frac{2\pi}{N}} \quad (2)$$

are defined making use of the Hermite polynomial H_n . Writing $H_n(x)$ in terms of its generating function, $H_n(x) = \frac{\partial^n}{\partial t^n} \exp[2xt - t^2] \Big|_{t=0}$, it is possible to write this state (to use a quantum mechanical terminology) as⁸

$$f_n(j) = \frac{1}{\sqrt{N}} \frac{\partial^n}{\partial t^n} \vartheta_3\left(\frac{j}{N} - \frac{\epsilon}{\pi} t, \frac{i}{N}\right) \exp[t^2] \Big|_{t=0}, \quad (3)$$

where

$$\vartheta_3(z, \tau) = \sum_{\alpha=-\infty}^{\infty} \exp[i\pi\tau\alpha^2] \exp[2\pi i\alpha z], \quad \text{Im}(\tau) > 0, \quad (4)$$

is the Jacobi ϑ_3 -function, following Vilenkin's notation.¹³ In this notation the basic properties of this even function read as

$$\vartheta_3(z + m + n\tau, \tau) = \exp[-i\pi n^2] \exp[-2\pi i n z] \vartheta_3(z, \tau), \quad (5)$$

$$\vartheta_3(z, i\tau) = \tau^{-1/2} \exp\left[-\frac{\pi z^2}{\tau}\right] \vartheta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right), \quad (6)$$

emphasizing its period 1 and quasiperiod τ . A beautiful consequence of (6) is that this function can be written as a sum of Gaussians,

$$\vartheta_3\left(\frac{z}{L}, \frac{i}{\sigma^2}\right) = \sigma \sum_{\alpha=-\infty}^{\infty} \exp\left[-\pi\left(\frac{\sigma}{L}\right)^2 (\alpha L + z)^2\right], \quad (7)$$

a form in which the width L/σ becomes apparent. Property (6) also provides an easy way to obtain the additional identity (also given by Ref. 6)

$$f_n(j) = \epsilon(-i)^n \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}\alpha^2 + \frac{2\pi i}{N}j\alpha\right] H_n(\epsilon\alpha), \quad (8)$$

which is in fact a generalization of Eq. (7) (if one compares it to Eq. (2)).

III. ORTHOGONALITY OF THE f_n 'S

According to Eq. (1), the functions $\{f_n(j)\}$ are eigenstates of the DFT with associated eigenvalue i^n . Mehta has conjectured that $\{f_n(j)\}_{n=0}^{N-1}$ is an orthogonal set, and thus complete, over a finite set of N points (for odd N . For even N one must replace $f_{N-1}(j)$ by $f_N(j)$). This reasonable conjecture, quite surprisingly indeed, does not hold for arbitrary N (it holds for large N). As, in the following, evidence will be collected *against* the original conjecture, details shall be kept to a level higher than usual.

Let (f_n, f_m) denote the inner product

$$(f_n, f_m) = \sum_{j=0}^{N-1} f_n^*(j) f_m(j) = \epsilon^2 (-i)^{n+m} \sum_{j=0}^{N-1} \sum_{\alpha, \beta=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}(\alpha^2 + \beta^2) + \frac{2\pi i}{N}j(\alpha - \beta)\right] H_n(\epsilon\alpha) H_m(\epsilon\beta).$$

The sum over $\{j\}$ is a realization of the modulo N Kronecker delta,

$$\delta_{\alpha, \beta}^{[N]} = \begin{cases} 1 & \alpha = \beta \pmod{N}, \\ 0 & \alpha \neq \beta \pmod{N}, \end{cases}$$

thus

$$(f_n, f_m) = 2\pi (-i)^{n+m} \sum_{\alpha, \beta=-\infty}^{\infty} \delta_{\alpha, \beta}^{[N]} \exp\left[-\frac{\pi}{N}(\alpha^2 + \beta^2)\right] H_n(\epsilon\alpha) H_m(\epsilon\beta). \quad (9)$$

The well-known identity,

$$\exp\left[-\frac{1}{2}x^2\right] H_k(x) = \frac{i^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy \exp\left[-\frac{1}{2}y^2 + ixy\right] H_k(y),$$

together with the sum over $\{\beta\}$ leads to

$$(f_n, f_m) = \sum_{\alpha, \gamma=-\infty}^{\infty} \int_{-\infty}^{\infty} dy dz \exp\left[-\frac{1}{2}(y^2 + z^2) + iy\epsilon\alpha + iz\epsilon(\alpha + \gamma N)\right] H_n(y) H_m(z),$$

where the infinite sum on $\{\gamma\}$ covers the equivalence class present in $\delta_{\alpha, \beta}^{[N]}$. Now, the sum over $\{\alpha\}$ by its turn is a realization of a modulo 2π Dirac delta, thus, with the integration over $\{z\}$ and convenient changes of variables,

$$(f_n, f_m) = 2\pi \sum_{\gamma, \nu=-\infty}^{\infty} \int_{-\infty}^{\infty} dy \exp\left[-y^2 - \frac{\pi^2 \nu^2}{\epsilon^2} + i(y\epsilon N - \nu\pi N)\gamma\right] H_n\left(y - \frac{\pi\nu}{\epsilon}\right) H_m\left(y + \frac{\pi\nu}{\epsilon}\right),$$

where again an infinite sum is introduced due to the modulo 2π delta.

The above expression is rather elucidative. It is not hard to realize that the infinite sums over $\{\gamma, \nu\}$ are a direct consequence of the equivalence classes brought in by the modulo N Kronecker

delta present in Eq. (9). For large N , the term corresponding to $\gamma=v=0$ becomes increasingly important, and a simple check shows that this term is exactly $\delta_{n,m}$. Thus, as expected, the limit $N \rightarrow \infty$ recovers the usual harmonic oscillator results. For finite (and small) N , however, all terms in the above summation must be taken into account.

Following then, the sum on γ is seen to be a realization of the modulo 2π Dirac delta, $\delta^{2\pi 1}(\gamma \in N - v\pi N)$, and after a change of variables one has

$$(f_n, f_m) = 2\pi\epsilon \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\left(\mu + \frac{N\nu}{2}\right)^2 - \frac{N\pi\nu^2}{2}\right] H_n(\epsilon\mu) H_m\left(\frac{2\pi\nu}{\epsilon} + \epsilon\mu\right).$$

Again, summation over $\{\mu\}$ must be included to account for the 2π periodicity of the Dirac delta. Splitting the sum on ν in two sums, over the odd and even integers and shifting the sum on μ by νN results in

$$(f_n, f_m) = 2\pi\epsilon \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\mu^2 - 2\pi N\nu^2\right] H_n(\epsilon\mu - \epsilon N\nu) H_m(\epsilon\mu + \epsilon N\nu) \\ + 2\pi\epsilon \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\left(\mu + \frac{N}{2}\right)^2 - 2\pi N\left(\nu + 1/2\right)^2\right] H_n(\epsilon\mu - \epsilon N\nu) H_m(\epsilon(\mu + \nu N + N)).$$

Denoting the second term above by $(f_n, f_m)_{\text{odd}}$, if $N=2h+k$, where the binary variable k controls the parity of N , then

$$(f_n, f_m)_{\text{odd}} = 2\pi\epsilon \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\left(\mu + h + k/2\right)^2 - 2\pi N\left(\nu + 1/2\right)^2\right] H_n(\epsilon\mu - \epsilon N\nu) H_m(\epsilon(\mu + \nu N + N))$$

and yet again shifting the sum on μ by $h+k/2$ and the one on ν by $1/2$,

$$(f_n, f_m)_{\text{odd}} = 2\pi\epsilon \sum_{\mu=-\infty}^{\infty} (k) \sum_{\nu=-\infty}^{\infty} (1) \exp\left[-\frac{2\pi}{N}\mu^2 - 2\pi N\nu^2\right] H_n(\epsilon(\mu - N\nu)) H_m(\epsilon(\mu + \nu N)),$$

where now $\sum_{\mu=-\infty}^{\infty} (k)$ denotes a sum over the integers (half-integers) if $k=0$ ($k=1$), so that back to the general expression,

$$(f_n, f_m) = 2\pi\epsilon \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\mu^2 - 2\pi N\nu^2\right] H_n(\epsilon(\mu - N\nu)) H_m(\epsilon(\mu + \nu N)) \\ + 2\pi\epsilon \sum_{\mu=-\infty}^{\infty} (k) \sum_{\nu=-\infty}^{\infty} (1) \exp\left[-\frac{2\pi}{N}\mu^2 - 2\pi N\nu^2\right] H_n(\epsilon(\mu - N\nu)) H_m(\epsilon(\mu + \nu N)).$$

Now, recourse to the Hermite polynomial's generating function gives

$$(f_n, f_m) = 2\pi\epsilon \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial s^m} \left\{ \sum_{\mu, \nu=-\infty}^{\infty} \exp\left[-\frac{2\pi}{N}\mu^2 + 2\mu\epsilon(t+s) - 2\epsilon\nu N(t-s) - t^2 - s^2\right] \right. \\ \left. + \sum_{\mu=-\infty}^{\infty} (k) \sum_{\nu=-\infty}^{\infty} (1) \exp\left[-\frac{2\pi}{N}\mu^2 + 2\mu\epsilon(t+s) - 2\epsilon\nu N(t-s) - t^2 - s^2\right] \right\}_{t=s=0}.$$

The sum on $\{\mu\}$ results in a ϑ_3 -function in the first term, and a ϑ_3 for $k=0$ or a ϑ_2 for $k=1$ in the second. The sum on $\{\nu\}$, by its turn, gives ϑ_3 -function in the first term, and a ϑ_2 in the second, as

$$(f_n, f_m) = 2\pi\epsilon \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial s^m} \left\{ \left[\vartheta_3\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_3\left(\frac{i\epsilon N(t-s)}{\pi}, 2Ni\right) + \vartheta_{3-k}\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_2\left(\frac{i\epsilon N(t-s)}{\pi}, 2Ni\right) \right] \exp[-t^2 - s^2] \right\} \Big|_{t=s=0}.$$

Using the basic properties,

$$\vartheta_3(z, i\tau) = \tau^{-1/2} \exp\left[-\frac{\pi z^2}{\tau}\right] \vartheta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right),$$

$$\theta_2(z, i\tau) = \tau^{-1/2} \exp\left[-\frac{\pi z^2}{\tau}\right] \vartheta_4\left(\frac{z}{i\tau}, \frac{i}{\tau}\right),$$

one gets

$$(f_n, f_m) = \frac{2\pi^{3/2}}{N} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial s^m} \left\{ \left[\vartheta_3\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_3\left(\frac{\epsilon(t-s)}{2\pi}, \frac{i}{2N}\right) + \vartheta_{3-k}\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_4\left(\frac{\epsilon(t-s)}{2\pi}, \frac{i}{2N}\right) \right] \exp[-2ts] \right\} \Big|_{t=s=0}.$$

Finally, compact expressions can be achieved with

$$\theta_3(z, \tau) = \frac{1}{2} \left[\theta_3\left(\frac{z}{2}, \frac{\tau}{4}\right) + \theta_4\left(\frac{z}{2}, \frac{\tau}{4}\right) \right],$$

$$\theta_2(z, \tau) = \frac{1}{2} \left[\theta_3\left(\frac{z}{2}, \frac{\tau}{4}\right) - \theta_4\left(\frac{z}{2}, \frac{\tau}{4}\right) \right],$$

thus for $k=0$

$$(f_n, f_m) = \frac{\pi^{3/2}}{N} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial s^m} \vartheta_3\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_3\left(\frac{\epsilon(t-s)}{\pi}, \frac{2i}{N}\right) \exp[-2ts] \Big|_{t=s=0}$$

and for $k=1$

$$(f_n, f_m) = \frac{\pi^{3/2}}{N} \frac{\partial^n}{\partial t^n} \frac{\partial^m}{\partial s^m} \left\{ \vartheta_3\left(\frac{i\epsilon(t+s)}{\pi}, \frac{2i}{N}\right) \vartheta_3\left(\frac{\epsilon(t-s)}{\pi}, \frac{2i}{N}\right) - 2\vartheta_4\left(\frac{i\epsilon(t+s)}{2\pi}, \frac{i}{2N}\right) \vartheta_4\left(\frac{\epsilon(t-s)}{2\pi}, \frac{i}{2N}\right) \exp[-2ts] \right\} \Big|_{t=s=0}.$$

Again, the limit $N \rightarrow \infty$ easily recovers the usual results, as the i factor inside the ϑ -functions guarantees that, in this limit, only a term proportional to $(\partial^n / \partial t^n)(\partial^m / \partial s^m) \exp[-4ts] \Big|_{t=s=0}$ survives. Anyhow, with the above expressions any term (f_n, f_m) can be calculated as a sum of ϑ -function derivatives evaluated at zero. The particular situation $m=0$, for example, for N even, is quite instructive. In this case

$$(f_n, f_0) = \frac{\pi^{3/2}}{N} \frac{\partial^n}{\partial t^n} \vartheta_3\left(\frac{i\epsilon t}{\pi}, \frac{2i}{N}\right) \vartheta_3\left(\frac{\epsilon t}{\pi}, \frac{2i}{N}\right) \Big|_{t=0},$$

$$(f_n, f_0) = \frac{\pi^{3/2}}{N} \sum_{j=0}^n \binom{n}{j} i^j \frac{\partial^j}{\partial t^j} \vartheta_3\left(\frac{\epsilon t}{\pi}, \frac{2i}{N}\right) \Bigg|_{t=0} \left(\frac{\partial^{n-j}}{\partial t^{n-j}} \vartheta_3\left(\frac{\epsilon t}{\pi}, \frac{2i}{N}\right) \Bigg|_{t=0} \right),$$

and it is immediate to see that all n -odd terms are zero. For $n=2$ (and for all even numbers not multipliers of 4), the symmetry of the binomial term and the multiplicity of the powers of i lead to a pairwise cancellation of all non-zero terms. For $n=4$ (and its multipliers), the situation is different. The simplest case is $n=4$,

$$(f_4, f_0) = \frac{\pi^{3/2}}{N} \sum_{j=0}^4 \binom{4}{j} i^j \frac{\partial^j}{\partial t^j} \vartheta_3\left(\frac{\epsilon t}{\pi}, \frac{2i}{N}\right) \Bigg|_{t=0} \left[\frac{\partial^{4-j}}{\partial t^{4-j}} \vartheta_3\left(\frac{\epsilon t}{\pi}, \frac{2i}{N}\right) \Bigg|_{t=0} \right]$$

$$(f_4, f_0) = \frac{\pi^{3/2}}{N} \left\{ 2 \vartheta_3\left(0, \frac{2i}{N}\right) \vartheta_3'''\left(0, \frac{2i}{N}\right) - 6 \left[\vartheta_3''\left(0, \frac{2i}{N}\right) \right]^2 \right\}.$$

This term (with proper normalization) goes to zero quite fast with increasing N . In fact, for $N=10$ it is already of order of 10^{-6} . On the other hand, it is immaterial to discuss the case $N=4$ (or smaller), as in this situation the distinct eigenvalues of the Fourier operator are enough to guarantee orthogonality of the whole set. Considering all this, it comes down to, literally, one-half a dozen different values of the dimensionality N (the range $[5, 10]$) for which a significant deviation from the “expected” results (that is, orthogonality) can be observed.

IV. DFT AND WIDTH INVERSION

Starting from the own definition of the ϑ_3 -function, Eq. (4), with $\xi \in \mathbf{R}$, a fractional shift of the ϑ_3 function can be calculated,

$$\vartheta_3\left(z + \frac{k}{N}, \frac{i\xi^2}{N}\right) = \sum_{\alpha=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}\xi^2\alpha^2\right] \exp\left[2\pi i\alpha\left(z + \frac{k}{N}\right)\right],$$

where k is an integer. The sum over $\{\alpha\}$ can be broken into modulo N equivalence classes as

$$\vartheta_3\left(z + \frac{k}{N}, \frac{i\xi^2}{N}\right) = \sum_{j=0}^{N-1} \sum_{\beta=-\infty}^{\infty} \exp\left[-\frac{\pi}{N}\xi^2(j + \beta N)^2\right] \exp\left[2\pi i(j + \beta N)\left(z + \frac{k}{N}\right)\right].$$

Conveniently regrouping the terms one gets

$$\vartheta_3\left(z + \frac{k}{N}, \frac{i\xi^2}{N}\right) = \sum_{j=0}^{N-1} \left(\sum_{\beta=-\infty}^{\infty} \exp[-\pi N\xi^2\beta^2] \exp[2\pi i\beta(i\xi^2 j + Nz)] \right)$$

$$\times \exp\left[-\frac{\pi}{N}\xi^2 j^2 + 2\pi i j z + \frac{2\pi i}{N} j k\right],$$

where the term inside the brackets can be identified as ϑ_3 -function,

$$\vartheta_3\left(z + \frac{k}{N}, \frac{i\xi^2}{N}\right) = \sum_{j=0}^{N-1} \vartheta_3(i\xi^2 j + Nz, iN\xi^2) \exp\left[-\frac{\pi}{N}\xi^2 j^2 + 2\pi i j z + \frac{2\pi i}{N} j k\right].$$

Use of property (6) leads to

$$\vartheta_3\left(z + \frac{k}{N}, \frac{i\xi^2}{N}\right) = \frac{1}{\sqrt{N\xi^2}} \sum_{j=0}^{N-1} \vartheta_3\left(\frac{iz}{\xi^2} - \frac{j}{N}, \frac{i}{N\xi^2}\right) \exp\left[-\frac{\pi N}{\xi^2} z^2 + \frac{2\pi i}{N} j k\right] \quad (10)$$

and taking advantage of the Fourier coefficients $\exp\left[\frac{2\pi i}{N} j k\right]$ it is easy to obtain the inverse relation

$$\vartheta_3\left(\frac{iz}{\xi^2} - \frac{k}{N}, \frac{i}{N\xi^2}\right) = \sqrt{\frac{N}{\xi^2}} \sum_{j=0}^{N-1} \vartheta_3\left(z + \frac{j}{N}, \frac{i\xi^2}{N}\right) \exp\left[\frac{\pi N}{\xi^2} z^2 - \frac{2\pi i}{N} jk\right]. \quad (11)$$

Particular cases of these equations are most interesting, and a lot of peculiar relations can be obtained with the different possible choices of z, k and ξ . Two straightforward examples are: First, setting $z=0$ in (10),

$$\vartheta_3\left(\frac{k}{N}, \frac{i\xi^2}{N}\right) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \vartheta_3\left(\frac{j}{N}, \frac{i}{N\xi^2}\right) \exp\left[\frac{2\pi i}{N} jk\right], \quad (12)$$

and, according to Eq. (7), the ϑ_3 -function on the left-hand side has width ξ , while the one under the action of the DFT has width ξ^{-1} . This property is the obvious discrete counterpart of the well-known behavior of the Gaussian function under the usual Fourier transform.

The case $k=0$, by its turn, after some manipulation gives

$$\vartheta_3(Nz, iN\xi^2) = \sqrt{\frac{N}{\xi^2}} \sum_{j=0}^{N-1} \vartheta_3\left(z + \frac{j}{N}, \frac{i\xi^2}{N}\right).$$

A. Application

With the above results it is possible to generalize the result of Ref. 6 in a straightforward way. Introducing

$$f_n(j, \xi) = \sqrt{\frac{N}{\xi}} \frac{\partial^n}{\partial t^n} \vartheta_3\left(\frac{j}{N} - \frac{\epsilon}{\pi} \xi t, \frac{i\xi^2}{N}\right) \exp[t^2] \Big|_{t=0},$$

its DFT can be directly calculated,

$$\bar{f}_n(k, \xi) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left[\frac{2\pi i}{N} jk\right] f_n(j, \xi),$$

$$\bar{f}_n(k, \xi) = \frac{1}{\sqrt{\xi}} \frac{\partial^n}{\partial t^n} \sum_{j=0}^{N-1} \exp\left[\frac{2\pi i}{N} jk\right] \vartheta_3\left(\frac{j}{N} - \frac{\epsilon}{\pi} \xi t, \frac{i\xi^2}{N}\right) \exp[t^2] \Big|_{t=0},$$

and use of Eq. (11) together with change of variables from t to it leads to

$$\bar{f}_n(k, \xi) = \sqrt{N\xi} i^n \frac{\partial^n}{\partial t^n} \vartheta_3\left(\frac{k}{N} - \frac{\epsilon t}{\pi\xi}, \frac{i}{N\xi^2}\right) \exp[t^2] \Big|_{t=0},$$

thus

$$f_n(k, \xi^{-1}) = i^n \sum_{j=0}^{N-1} \exp\left[\frac{2\pi i}{N} jk\right] f_n(j, \xi), \quad (13)$$

which reproduces Eq. (1) for $\xi=1$. From this relation, most identities obtained in Ref. 6 may also be generalized.

V. TWO VARIABLE'S DFT

Yet another generalization of the main result of Ref. 6 regards a two variable DFT, which, for the sake of brevity, here it will be merely verified. Apart from the obvious product solution $f_m(j)f_n(l)$, if one considers the quantity

$$F_{m,n}(j,l) = \sum_{k=0}^{N-1} f_m(k)f_n(k-l)\exp\left[\frac{2\pi i}{N}jk\right],$$

which obeys

$$(F_{m,n}(j,l))^* = F_{m,n}(j,l)\exp\left[\frac{2\pi i}{N}jl\right],$$

use of Eq. (1), and some simple manipulations lead to the nontrivial result

$$|F_{m,n}(j,l)|^2 = \frac{(-i)^{m+n}}{N} \sum_{a,b=0}^{N-1} |F_{m,n}(a,b)|^2 \exp\left[\frac{2\pi i}{N}(ma+nb)\right].$$

As in the one variable case, these states obey

$$\sum_{j,l=0}^{N-1} |F_{m,n}(j,l)|^2 |F_{m',n'}(j,l)|^2 = \delta_{m,m'} \delta_{n,n'}, \quad m+n \neq m'+n' \pmod{4},$$

which imply a multitude of relations involving derivatives of the ϑ_3 -functions (or the Hermite polynomials). Motivated by the preceding section, it should be investigated whether this relation holds for $m+n=m'+n' \pmod{4}$.

VI. FURTHER RELATIONS INVOLVING THE WIDTH

So far it has been seen that to break up the infinite sum present in the definition of the Jacobi ϑ_3 -function leads to interesting properties of this very function. In order to further explore this technique, from Eq. (7) it is straightforward to write

$$\vartheta_3\left(\frac{z}{L}, \frac{i\xi^2}{L}\right) = \frac{\sqrt{L}}{\xi} \sum_{\alpha=-\infty}^{\infty} \exp\left[-L\pi\left(\frac{z}{\xi L} + \frac{\alpha}{\xi}\right)^2\right], \tag{14}$$

with L a positive real number. Choosing ξ integer, it is possible to break the sum over $\{\alpha\}$ into modulo ξ equivalence classes

$$\begin{aligned} \vartheta_3\left(\frac{z}{L}, \frac{i\xi^2}{L}\right) &= \frac{\sqrt{L}}{\xi} \sum_{j=0}^{\xi-1} \sum_{\mu=-\infty}^{\infty} \exp\left[-L\pi\left(\frac{z}{\xi L} + \frac{j+\mu\xi}{\xi}\right)^2\right], \\ \vartheta_3\left(\frac{z}{L}, \frac{i\xi^2}{L}\right) &= \frac{\sqrt{L}}{\xi} \sum_{j=0}^{\xi-1} \sum_{\mu=-\infty}^{\infty} \exp\left[-L\pi\left(\frac{z+jL}{\xi L} + \mu\right)^2\right], \end{aligned}$$

and the infinite sum can be identified as a ϑ_3 ,

$$\vartheta_3\left(\frac{z}{L}, \frac{i\xi^2}{L}\right) = \frac{1}{\xi} \sum_{j=0}^{\xi-1} \vartheta_3\left(\frac{z+jL}{\xi L}, \frac{i}{L}\right). \tag{15}$$

It is quite interesting to set $z=z\xi$ above and observe that

$$\vartheta_3\left(\frac{z\xi}{L}, \frac{i\xi^2}{L}\right) = \frac{1}{\xi} \sum_{j=0}^{\xi-1} \vartheta_3\left(\frac{z}{L} + \frac{j}{\xi}, \frac{i}{L}\right),$$

which, for the particular case $\xi=2$ gives the well-known result

$$\vartheta_3\left(\frac{2z}{L}, \frac{4i}{L}\right) = \frac{1}{2} \left[\vartheta_3\left(\frac{z}{L}, \frac{i}{L}\right) + \vartheta_3\left(\frac{z}{L} + \frac{1}{2}, \frac{i}{L}\right) \right],$$

$$\vartheta_3\left(\frac{2z}{L}, \frac{4i}{L}\right) = \frac{1}{2} \left[\vartheta_3\left(\frac{z}{L}, \frac{i}{L}\right) + \vartheta_4\left(\frac{z}{L}, \frac{i}{L}\right) \right].$$

Similar reasoning would lead to the complementary relation

$$\vartheta_3\left(\frac{z}{L}, \frac{i}{L}\right) = \frac{1}{\xi} \sum_{j=0}^{\xi-1} \vartheta_3\left(\frac{z}{\xi L} + \frac{j}{\xi}, \frac{i}{L\xi^2}\right). \quad (16)$$

And again, the particular case $\xi=2$ gives

$$\vartheta_3\left(\frac{z}{L}, \frac{i}{L}\right) = \frac{1}{2} \left[\vartheta_3\left(\frac{z}{2L}, \frac{i}{4L}\right) + \vartheta_4\left(\frac{z}{2L}, \frac{i}{4L}\right) \right].$$

Equations (15) and (16) can be combined to provide an alternative width inversion relation

$$\vartheta_3\left(\frac{z\xi}{L}, \frac{i\xi^2}{L}\right) = \frac{1}{\xi^2} \sum_{j,j'=0}^{\xi-1} \vartheta_3\left(\frac{z}{\xi L} + \frac{j'}{\xi} + \frac{j}{\xi^2}, \frac{i}{L\xi^2}\right).$$

VII. CONCLUSIONS

The results here presented seem to argue in favor of one basic point: The Jacobi ϑ_3 -function, together with the DFT, plays, in finite dimensional spaces, the same role played by the Gaussian function in conjunction with the usual Fourier transform. Concerning quantum mechanics, Schwinger has already noted that, if the families of eigenstates of two different observables are connected *via* DFT, then those observables share a maximum degree of incompatibility.⁵ In this connection, the width inversion relation obeyed by the $f_n(j, \xi)$ functions strongly suggests that one may be able to construct, for finite dimensional spaces, states which behavior resembles that of the continuous minimum uncertainty states.

However, such a reasoning meets an important hindrance if one considers that the orthogonality of the DFT's eigenstates ultimately fails. It is a fact, however, that with increasing N it becomes, in a numerical sense, true, and in this case the $N \rightarrow \infty$ limit is reached, as witty as it may sound, somewhere near one dozen. This fact may illustrate a true finite dimensional idiosyncrasy, or it might lead one to look for the possibility of finding different sets of DFT's eigenstates, an issue which is a matter of current research.

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