# Consequences of Quadratic Frictional Force on the One Dimensional Bouncing Ball Model

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**Abstract.** Some dynamical properties of the one dimensional Fermi accelerator model, under the presence of frictional force are studied. The frictional force is assumed as being proportional to the square particle's velocity. The problem is described by use of a two dimensional non linear mapping, therefore obtained via the solution of differential equations. We confirm that the model experiences contraction of the phase space area and in special, we characterized the behavior of the particle approaching an attracting fixed point.

**Keywords:** Fermi accelerator model; classical billiard; nonlinear mapping
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## 1. INTRODUCTION

The one dimensional bouncing ball model, sometimes also referred as to the Fermi accelerator model, consists of a classical particle confined between two rigid walls and suffering elastic collisions with them. One wall is assumed to be fixed while the other one is periodically time varying. The fundament of this model back to earlies 1949 when Enrico Fermi [1] proposed a simple model as an attempt to describe a mechanism in which cosmic rays would be accelerated by intense time varying magnetic fields. After that, many other different versions of the model including e.g. external fields, quantum effects, damping forces and many other modifications were considered [2, 3, 4, 5, 6, 7, 8, 9] (see also refs. [10, 11, 12] for recent results of the Fermi accelerator model). Moreover, the model may also be considered as a toy model and many tools developed to study and characterize such system can be extended to more complex models including classical static and time dependent billiard problems.

For the non dissipative case, i.e. the case where dissipative forces are absent in the model, the phase space is of mixed kind in the sense that there are a set of Kolmogorov-Arnold-Moser (KAM) islands surrounded by a chaotic sea (characterized by a positive Lyapunov exponent) which is also confined by a set of invariant spanning curves. The introduction of damping forces however yield profounds consequences on the dynamics of the problem. It is worth stressing that there are different ways of introducing dissipation in the model. One of them is to consider inelastic collisions with the walls [13, 14, 15]. Thus, after the colision, the particle experiences a fractional loss of energy. Some consequences of such a dissipation include: events of crisis [16, 17] yielding the particle to experience the phenomenon of transient [18]; annihilation of fixed points [19]; depending on the strength of the damping coefficient, the particle can also shows the phenomenon of locking [20, 21]. Other kind of dissipation which is eventually present in real experiments is the effect of drag force [22]. Such force is generated by a relative motion of a particle inside a fluid, like a gas. Thus the particle looses energy proportionally to a power law of its velocity. The phase space of the problem no longer present a mixed form with chaos and regularity. Besides, it is possible to observe attracting sinks and a chaotic attractors. Recently, Leonel and McClintock [23, 24] had shown that, for a damping force proportional the velocity and considering a simplified version,[1] the system has a determinant of the Jacobian matrix equals to the unity, thus confirming that parts of the phase space are area preserving. This apparent paradox was explained by using the well known Poincaré recurrence theorem (see [25] for specific details).

[1] The simplified version assumes that both walls are fixed but that, after the collisions with one wall, the particle suffers an exchange of energy and momentum as if the wall were moving. Such a simplification was often used to speed up numerical simulations when computers were far slower. It also makes easier some analytical treatment.

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In this paper, we revisit the one dimensional Fermi accelerator model seeking to understand and describe some of its dynamical properties in the presence of quadratic frictional force. In our approach, we are considering that the drag force is proportional to the square particle's velocity. As it is so usual, the model is described by a two dimensional non linear mapping for the variables velocity of the particle and corresponding time. The dynamical equations that describes all the possible motion of the model were obtained by solution of the Newton's law. The paper is organized as follows, in section 2 we describe the procedures used in the construction of the mapping. Section 3 is devoted to discuss some numerical results while final remarks and conclusions are drawn in section 4.

## 2. THE MODEL AND THE MAPPING

Let us now describe the model and all the details needed to construct the equations of the mapping. The model consists of a classical particle of mass $m$ confined between two rigid walls and suffering elastic collisions with them. The particle also suffers the effect of a frictional force whose magnitude is assumed to be of the type $F = -\eta v^2$, where $\eta$ is the strength of the viscosity and $v$ is the particle's velocity. In our approach on this paper, we will consider only the simplified version (see footnote 1). Thus we assume that either walls are fixed. One of them is at $x = l$ while the other is at the origin $x = 0$. However, when the particle hits the wall located at the origin, it suffers a change of energy and momentum as if the wall were moving according to $x_w(t) = \epsilon' \cos(\omega t)$, where $\epsilon'$ is the amplitude of oscillation and $\omega$ is the frequency of oscillation. This approximation retain the nonlinearity of the problem and avoid the inconveniency of finding numerical solutions from transcendental equations.[2]

The problem is them described in terms of a two dimensional mapping $T(v_n, t_n) = (v_{n+1}, t_{n+1})$. In order to construct the map, we will suppose that in the instant $t = t_n$ the particle is at the position $x = 0$ with velocity $v = v_n > 0$. After solving the Newton's law $-\eta v^2 = m dv/dt$, consider the effects of elastic collisions with both wall and define the dimensionless variables $V_n = v_n/(\omega l)$, $\delta = \eta l$, $\epsilon = \epsilon'/l$ and $\phi_n = \omega t_n$ we find that the mapping is given by

$$\mathrm{T} : \begin{cases} V_{n+1} = |V_n^* - 2\epsilon \sin(\phi_{n+1})| \\ \phi_{n+1} = \phi_n + 2\left[\frac{e^{\delta}-1}{V_n \delta}\right] \mod 2\pi \end{cases} , \tag{1}$$

where the auxiliary term is $V_n^* = V_n/(2e^{\delta} - 1)$. It is important to stress that in the simplified model, non-positive velocities are forbidden because they are equivalent to the particle traveling beyond the wall. In order to avoid such problems, if after the collision the particle has a negative velocity, we inject it back with the same modulus of velocity. This procedure is effected by use of a module function. Note that the velocity of the particle is reversed by the module function only if, after the collision, the particle remains traveling in the negative direction. The module function has no effect on the motion of the particle if it moves in the positive direction after the collision. We stress that this approximation is valid only for small values of $\epsilon$.

We now describe some details of the Jacobian matrix for the mapping (1). It is defined as

$$J = \begin{pmatrix} \frac{\partial \phi_{n+1}}{\partial \phi_n} & \frac{\partial \phi_{n+1}}{\partial V_n} \\ \frac{\partial V_{n+1}}{\partial \phi_n} & \frac{\partial V_{n+1}}{\partial V_n} \end{pmatrix} , \tag{2}$$

with coefficients given by

$$\frac{\partial \phi_{n+1}}{\partial \phi_n} = 1 \ , \ \frac{\partial \phi_{n+1}}{\partial V_n} = -\frac{2e^{\delta} - 2}{V_n^2 \delta} ,$$

$$\frac{\partial V_{n+1}}{\partial \phi_n} = \mathrm{sign}\left[V_n^* - 2\epsilon \sin(\phi_{n+1})\right] \times \left[-2\epsilon \cos(\phi_{n+1}) \frac{\partial \phi_{n+1}}{\partial \phi_n}\right] ,$$

$$\frac{\partial V_{n+1}}{\partial V_n} = \mathrm{sign}\left[V_n^* - 2\epsilon \sin(\phi_{n+1})\right] \times \left[\frac{1}{2e^{\delta} - 1} - 2\epsilon \cos(\phi_{n+1}) \frac{\partial \phi_{n+1}}{\partial V_n}\right] ,$$

where the function $\mathrm{sign}(u) = 1$ if $u > 0$ and $\mathrm{sign}(u) = -1$ if $u < 0$.

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[2] In the complete model, the solutions of the transcendental equations furnish the time that the particle hits the periodically varying wall. Such time is obtained by matching the condition that, at the instant of the colision, the particle's position is the same as that of the time varying wall.

After a careful investigation, we find that the determinant of the Jacobian matrix yields that

$$det\ J = sign[V_n^* - 2\varepsilon\sin(\phi_{n+1})] \times \frac{1}{(2e^{\delta}-1)} \ ,$$

thus confirming that the system now shrinks area in the phase space.

## 3. NUMERICAL RESULTS

Let us now present and discuss some of our numerical results. We begin discussing the behavior for large values of the particle's velocity. Considering the first equation of the mapping (1), we find that

$$\begin{aligned} V_1 &= \frac{V_0}{2e^{\delta}-1} - 2\varepsilon\sin(\phi_1) \ , \\ V_2 &= \frac{V_0}{(2e^{\delta}-1)^2} - \frac{2\varepsilon\sin(\phi_1)}{(2e^{\delta}-1)} - 2\varepsilon\sin(\phi_2) \ , \\ V_3 &= \frac{V_0}{(2e^{\delta}-1)^3} - \frac{2\varepsilon\sin(\phi_1)}{(2e^{\delta}-1)^2} - \frac{2\varepsilon\sin(\phi_2)}{(2e^{\delta}-1)} - 2\varepsilon\sin(\phi_3) \ . \end{aligned}$$

Finally, the general expression can be written as

$$V_n = \frac{V_0}{(2e^{\delta}-1)^n} - 2\varepsilon\sum_{i=1}^{n}\frac{\sin(\phi_i)}{(2e^{\delta}-1)^{n-i}} \ . \tag{3}$$

Before present the behavior of the velocity $V$ as function of the iteration number, let us discuss some approximations of the eq. (3). Firstly, it is important to say that the second term in eq. (3) contributes with an oscillation on the velocity with maximum amplitude of $2\varepsilon$ (recall that $\varepsilon$ is small). However, it is easy to see that on the average, the second term can be disregarded[3]. If we then expand the first term of eq. (3) in powers of $\delta$, we obtain that

$$V_n \cong V_0 \times \left[1 - 2n\delta + (n+2n^2)\delta^2 + \left(-n - 2n^2 - \frac{4}{3}n^3\right)\delta^3 + \left(\frac{13}{12}n + \frac{5}{2}n^2 + 2n^3 + \frac{2}{3}n^4\right)\delta^4 + O(\delta^5)\right] \ . \tag{4}$$

Moreover, if we consider that $n$ is relatively large[4], one might conclude that only the larger terms inside each brackets dominate. Thus, disregarding the lower powers of $n$ inside the brackets, we can rewrite eq. (4) as

$$V_n \cong V_0 \times \left[1 - 2n\delta + 2n^2\delta^2 - \left(\frac{4}{3}n^3\right)\delta^3 + \left(\frac{2}{3}n^4\right)\delta^4 + O(\delta^5)\right] \ . \tag{5}$$

The power series shown in eq. (5) is the own definition of the exponential given by

$$V_n = V_0 e^{-2\delta n} \ . \tag{6}$$

In order to confirm our approach, it is shown in figure 1 the behavior of the velocity $V$ as function of iteration number $n$. The control parameters and initial conditions used in the construction of the figure were $\varepsilon = 10^{-2}$, $\delta = 10^{-4}$ and $V_0 = 3$. After doing an exponential fit of the type $Y = Ae^{Bx}$, we found that $A = 3.02(1)$ and $B = -1.997(1) \times 10^{-4}$. Such a result thus confirms that the particle approaches the fixed point exponentially, as it was expected in eq. (6).

Suppose now that the particle is evolving in time and then suddenly it is captured by an attracting region, like if the particle was moving in a basin of attraction of a sink. The particle will approach it and we will describe such approaching as function of the iteration number. As an attempt to investigate the asymptotic behavior to the attracting sink, we first define a set of different initial conditions and then allow the system to evolve in time. Moreover, we have to stablish a convergency criterion in order to check whether the particle is near enough to the attracting fixed

[3] Such property can be considered since that $\phi_i$ might be assumed as being uniformally distributed in $\phi_i \in [0, 2\pi)$.

[4] As for example, consider that $n > 4$

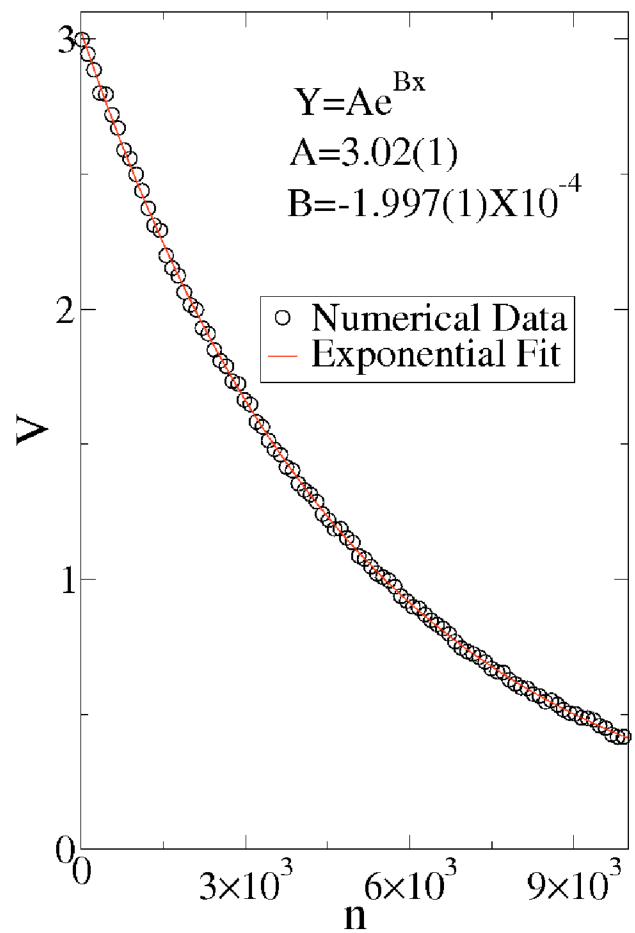


**FIGURE 1.** Behavior of the velocity $V$ plotted against $n$. The control parameters used were $\varepsilon = 10^{-2}$ and $\delta = 10^{-4}$.

point. The criterion lies basically in checking the distance of the particle to the fixed point. We thus define a critical radius $r_c = 10^{-6}$ and evolve each set of initial condition. If the particle is sufficiently close to the attracting sink, say less then $r_c$, we thus save, in an array, the corresponding iteration number spent until match such condition and start a new initial condition. The evolution of an ensemble of different initial conditions allow us to define the average iteration number $\bar{n}_x = 1/M \sum_{i=1}^{M} n_i$. It is shown in figure 2(a) the behavior of the average iteration number $\bar{n}_x$ plotted against the proximity of the attracting sink for a set of different trajectories approaching it. The horizontal axis denotes the distance $r(n) = \sqrt{(V_n - V^*)^2 + (\phi_n - \phi^*)^2}$, where the coordinates of the fixed point are given by $V^*$ and $\phi^*$. The average was made using the 500 different initial conditions in the range $(V_0, \phi_0) = ([0.325, 0.33], \pi)$.

We have fitted each curve of figure 2(a) by the function $\bar{n}_x = A + B\ln(r)$. In figure 2, we assume as fixed the control parameter $\varepsilon = 10^{-2}$ and we obtain that, for $\delta = 10^{-4}$, $A = -4.680(4) \times 10^4$ and $B = -9.9991(4) \times 10^4$. On the other hand, using $\delta = 5 \times 10^{-5}$, we found that $A = -9.379(8) \times 10^4$ and $B = -1.99984(4) \times 10^4$. Finally, for $\delta = 10^{-5}$, we obtain $A = -4.702(3) \times 10^5$ and $B = -9.996(3) \times 10^5$. We can conclude that trajectories approach the attracting sink exponentially as the iteration number evolves.

Let us now describe how the trajectory evolves towards the fixed point as function of the drag coefficient. Thus, it is shown in figure 2(b) the behavior of $\bar{n}_x \times \delta$. In that figure, we have evolved our simulations up to $r < 10^{-6}$. Such a behavior can be described as

$$\bar{n}_x \propto \delta^{\mu} \ . \tag{7}$$

After fitting a power law, as shown in figure 2(b), we obtain that $\mu = -0.9993(7)$. It is worth stressing that, in the

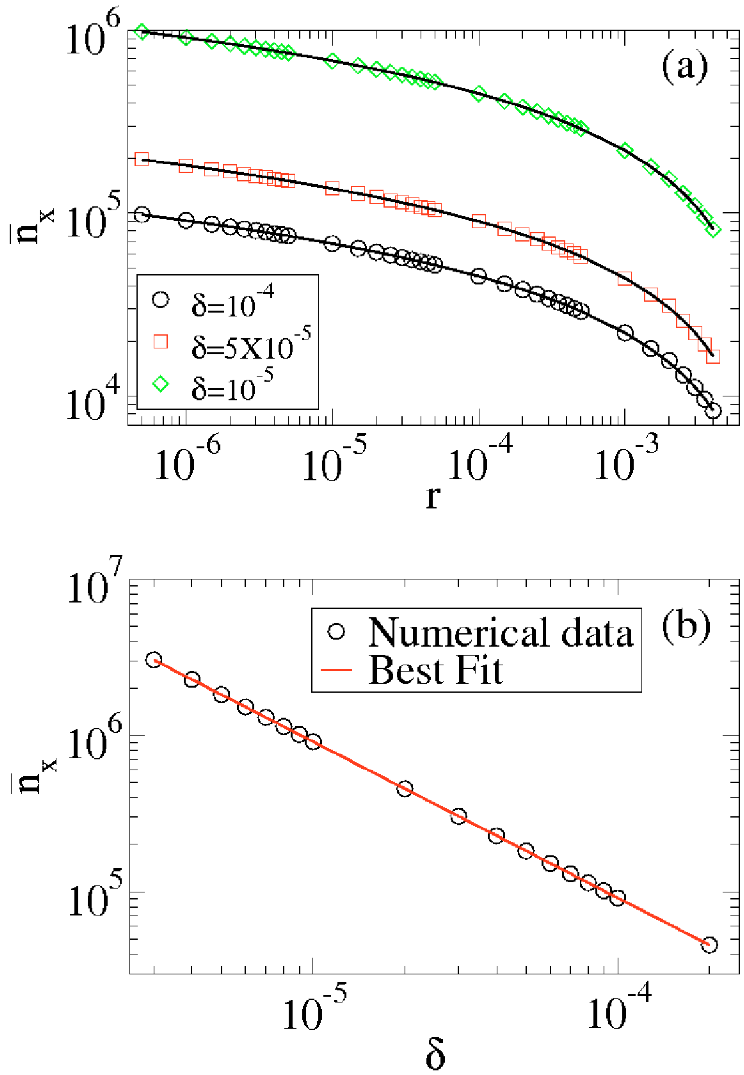


**FIGURE 2.** (a) It is shown the behavior of the average $\bar{n}_x \times r$ while (b) plots the transient $\bar{n}_x \times \delta$. The control parameter used was $\varepsilon = 10^{-2}$.

limit $\delta \to 0$, equation (7) gives us that $\bar{n}_x \to \infty$, thus confirming that no convergency to the fixed point is observed. Note however that the preservation of the phase space measure is recovered for that limit of $\delta$. We can also conclude that, beyond the particle approach exponentially the attracting sink, the speed of the approach depends on the strength of the damping coefficient.

## 4. CONCLUSIONS

As a final remark, we have studied the one dimensional Fermi accelerator model in the presence of frictional force proportional to the square particle's velocity. Our results confirm that the model experiences contraction of the phase space area. We have also characterized the behavior of the particle approaching an attracting sink. It was shown that the particle approaches a fixed point exponentially as the iteration number evolves and with a speed of approach that depends on the strength of the drag force.

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