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SZEGŐ POLYNOMIALS FROM HYPERGEOMETRIC FUNCTIONS

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Abstract. Szegő polynomials with respect to the weight function $\omega(\theta) = e^{\eta\theta}[\sin(\theta/2)]^{2\lambda}$, where $\eta, \lambda \in \mathbb{R}$ and $\lambda > -1/2$ are considered. Many of the basic relations associated with these polynomials are given explicitly. Two sequences of para-orthogonal polynomials with explicit relations are also given.

1. Introduction

With the publications of [20] and [21] in the early 20th century, Szegő introduced *the orthogonal polynomials on the unit circle*. For many interesting results on these polynomials we refer to the classical book [22] of Szegő, the first edition of which was published in 1939. Since then, these polynomials which bear the name of Szegő have been extensively studied by many. We cite, for example, [3], [4], [5], [8], [14], [15], [16] and [19] as some of the very recent contributions. The recent publications of the two excellent volumes [17] and [18] by Simon have given, apart from a thorough perspective of the subject, a new boost to the interest in these polynomials.

Given a positive measure $\mu(z) = \mu(e^{i\theta})$ on the unit circle $\mathcal{C} = \{z = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, the associated sequence of "monic" Szegő polynomials $\{\Phi_n\}$ can be defined by

$$\int_{\mathcal{C}} \bar{z}^j \Phi_n(z) d\mu(z) = \int_0^{2\pi} e^{-ij\theta} \Phi_n(e^{i\theta}) d\mu(e^{i\theta}) = \kappa_n^{-2} \delta_{nj}, \quad 0 \leq j \leq n-1,$$

where $\kappa_n^{-2} = \|\Phi_n\|^2 = \int_{\mathcal{C}} |\Phi_n(z)|^2 d\mu(z)$. Since the integration is along the unit circle, there holds $\int_{\mathcal{C}} \bar{z}^j \Phi_n(z) d\mu(z) = \int_{\mathcal{C}} z^{-j} \Phi_n(z) d\mu(z)$. The orthonormal Szegő polynomials are $\varphi_n(z) = \kappa_n \phi_n(z)$, $n \geq 0$. Results up to (1.2) given below are some of the well known results on Szegő polynomials.

$$\text{(1.1)} \qquad \begin{aligned} \Phi_n^*(z) &= \overline{a}_n\, z\Phi_{n-1}(z) + \Phi_{n-1}^*(z), \\ \Phi_n(z) &= a_n \Phi_n^*(z) + (1 - |a_n|^2) z\Phi_{n-1}(z), \end{aligned} \quad n \geq 1,$$

where $a_n = \Phi_n(0)$ and $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$. The numbers a_n, $n \geq 1$, were traditionally known to mathematicians working in this area as the Szegő or reflection

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coefficients. However, in Simon [17] they have been referred to as the Verblunsky coefficients (in the form of $\alpha_n = -\bar{a}_{n+1}$). The coefficients a_n satisfy

$$|a_n| < 1 \quad \text{and} \quad \prod_{m=1}^{n} (1 - |a_m|^2) = \kappa_n^{-2} = \frac{D_n}{D_{n-1}} \quad \text{for} \quad n \geq 1,$$

where the Toeplitz determinants D_n are such that

$$D_0 = \mu_0 \quad \text{and} \quad D_n = \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix}, \quad n \geq 1.$$

Here, the so-called moments μ_n have been defined by $\mu_n = \int_0^{2\pi} e^{-in\theta} d\mu(e^{i\theta})$ and satisfy $\mu_{-n} = \overline{\mu}_n$, $n \geq 1$.

It is also well known that the Szegő polynomials are completely characterized by the coefficients $\{a_n\}$, as given by the following theorem.

Theorem 1.1. *Given an arbitrary sequence of complex numbers* $\{a_n\}_{n=1}^{\infty}$*, with* $|a_n| < 1$*,* $n \geq 1$*, associated with this sequence there exists a unique measure* μ *on the unit circle such that the polynomials* $\{\Phi_n\}$ *generated by* (1.1) *are the respective Szegő polynomials.*

This theorem, previously known as Favard's theorem for the circle, is referred to as Verblunsky's theorem in Simon [17], where many proofs of this theorem can also be found.

Moreover, the zeros of $\Phi_n(z)$ are in $\mathbb{D} = \{z : |z| < 1\}$ and if

$$\Psi_n(z) = \int_{\mathcal{C}} \frac{z+w}{z-w} [\Phi_n(z) - \Phi_n(w)] d\mu(w), \quad n \geq 1,$$

then

$$\lim_{n\to\infty} \frac{\Psi_n^*(z)}{\Phi_n^*(z)} = F(z) = \int_{\mathcal{C}} \frac{w+z}{w-z} d\mu(w), \tag{1.2}$$

uniformly on compact subsets of $\mathbb{D}$ (see Geronimus [7]). The function $F(z)$ is called the Carathéodory function of the measure μ.

There are very few examples of measures where the associated Szegő polynomials or their basic relations have explicit formulas. Most of the known examples with such desired property come from examples of real orthogonal polynomials on $[-1, 1]$ through the Szegő (or Joukowski) transformation $x = z + z^{-1}$.

The current paper deals with the Szegő polynomials with respect to the measure $d\mu(e^{i\theta}) = \omega(\theta)d\theta$, where $\omega(\theta) = e^{\eta\theta}[\sin(\theta/2)]^{2\lambda}$, defined for $\eta, \lambda \in \mathbb{R}$ and $\lambda > -1/2$. We give these Szegő polynomials and many of their basic relations explicitly. However, polynomials that correspond to the case $\eta = 0$ are well known, as they come from the Gegenbauer or Ultraspherical polynomials.

2. Three term recurrence relations for Szegő polynomials

As already given in [7, p. 91], if $a_n \neq 0$, then the Szegő polynomials satisfy a three term recurrence relation. Results in this section provide some information on Szegő polynomials given by such a three term recurrence relation. These results permit us to derive easily the remaining results in the paper.

Theorem 2.1. *Let $\{\Phi_n\}$ be a sequence of polynomials given by the three term recurrence relation*

$$\Phi_{n+1}(z) = \big(z + \beta_{n+1}\big)\Phi_n(z) - \alpha_{n+1} z \Phi_{n-1}(z), \quad n \geq 1,$$

with $\Phi_0 = 1$ and $\Phi_1(z) = z + \beta_1$. If

$$\beta_{n+1} \neq 0, \quad \alpha_{n+1} \neq 0 \quad \text{and} \quad 0 < \frac{\alpha_{n+1}}{\beta_{n+1}} = 1 - |\Phi_n(0)|^2, \quad \text{for} \quad n \geq 1, \tag{2.1}$$

then there exists a positive measure μ on the unit circle such that $\{\Phi_n\}$ are the associated Szegő polynomials.

Proof. Let $a_n = \Phi_n(0)$, $n \geq 1$. From $\Phi_{n+1}(0) = \beta_{n+1}\Phi_n(0)$, $n \geq 0$, and (2.1) we have

$$\beta_1\beta_2 \cdots \beta_n = a_n, \quad \alpha_{n+1} = \beta_{n+1}(1 - |a_n|^2), \quad n \geq 1, \tag{2.2}$$

and $|a_n| < 1$, $n \geq 1$. Hence, by Theorem 1.1 there exists a unique measure μ associated with the sequence $\{a_n\}$ and that the polynomials generated by the corresponding recurrence system (1.1) are the Szegő polynomials with respect to this measure.

If $\beta_1 \neq 0$, observing that $a_n \neq 0$, from (1.1) we can identify these Szegő polynomials as the polynomials given by the three term recurrence relation (2.1) with coefficients satisfying (2.2).

On the other hand, if $\beta_1 = 0$, then $a_n = 0$ and $\alpha_{n+1} = \beta_{n+1}$, $n \geq 1$, thus giving the sequence of Chebyshev-Szegő polynomials. This sequence of polynomials is treated as the example "free case" in [17]. This completes the proof of the theorem. □

Having established the existence of the measure μ and the orthogonality of the polynomials $\{\Phi_n\}$, the following results in this section can be easily derived from known results in the literature.

The coefficients of the three term recurrence relations satisfy $\beta_1 = -\mu_{-1}/\mu_0$,

$$\alpha_{n+1} = \frac{\int_{\mathcal{C}} z\Phi_n(z)d\mu(z)}{\int_{\mathcal{C}} z\Phi_{n-1}(z)d\mu(z)} \quad \text{and} \quad \frac{\alpha_{n+1}}{\beta_{n+1}} = \frac{\int_{\mathcal{C}} z^{-n}\Phi_n(z)d\mu(z)}{\int_{\mathcal{C}} z^{-(n-1)}\Phi_{n-1}(z)d\mu(z)}, \quad n \geq 1.$$

Hence, $\kappa_0^{-2} = \int_{\mathcal{C}} |\Phi_0(z)|^2 d\mu(z) = \mu_0$,

$$\kappa_n^{-2} = \int_{\mathcal{C}} |\Phi_n(z)|^2 d\mu(z) = \int_{\mathcal{C}} z^{-n}\Phi_n(z)d\mu(z) = \mu_0 \frac{\alpha_2\alpha_3 \cdots \alpha_{n+1}}{\beta_2\beta_3 \cdots \beta_{n+1}}, \quad n \geq 1,$$

and $\displaystyle\int_{\mathcal{C}} z\Phi_n(z)d\mu(z) = \mu_0\beta_1\alpha_2\alpha_3 \cdots \alpha_{n+1}, \quad n \geq 1.$

If one considers the sequence of polynomials $\{Q_n\}$ given by

$$Q_n(z) = -\int_{\mathcal{C}} \frac{\Phi_n(z) - \Phi_n(w)}{z - w} w d\mu(w), \quad n \geq 0, \tag{2.3}$$

then

$$Q_{n+1}(z) = \big(z + \beta_{n+1}\big)Q_n(z) - \alpha_{n+1} z Q_{n-1}(z), \quad n \geq 1,$$

with $Q_0 = 0$, $Q_1(z) = \mu_0\beta_1$ and $\mu_0 = \int_{\mathcal{C}} d\mu(z)$. Note that Q_n is a polynomial of degree $n - 1$ with leading coefficient $\mu_0\beta_1$.

From (2.3) and the theory of continued fractions (see [12, 13]),

$$\begin{aligned} \Phi_n(z)\, L_0(z) - Q_n(z) &= \mu_0 \frac{\alpha_2 \cdots \alpha_{n+1}}{\beta_2 \cdots \beta_{n+1}} z^n + O(z^{n+1}), \\ L_\infty(z) - \frac{Q_n(z)}{\Phi_n(z)} &= \mu_0 \beta_1 \alpha_2 \cdots \alpha_{n+1} \frac{1}{z^{n+1}} + O(\frac{1}{z^{n+2}}), \end{aligned} \tag{2.4}$$

and

$$\frac{Q_n(z)}{\Phi_n(z)} = \frac{\mu_0\beta_1}{|\, z+\beta_1} - \frac{\alpha_2 z}{|\, z+\beta_2 z} - \cdots - \frac{\alpha_n z}{|\, z+\beta_n z},$$

for $n \geq 1$, where $L_0(z) = \sum_{j=0}^{\infty} \mu_j z^j$ and $L_\infty(z) = -\sum_{j=1}^{\infty} \mu_{-j} z^{-j}$ are the series expansions of the function $\int_{\mathcal{C}} (w-z)^{-1} w d\mu(w)$.

Finally, from (2.3) and (1.2),

$$\lim_{n\to\infty} \frac{zQ_n^*(z)}{\Phi_n^*(z)} = \frac{1}{2}\mu_0 - \frac{1}{2}F(z), \tag{2.5}$$

uniformly on compact subsets of $\mathbb{D}$.

3. Szegő polynomials from hypergeometric functions

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$, the hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by the series expansion

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

for $|z| < 1$ and by analytic continuation for other values of $z \in \mathbb{C}$. We refer to the book of Andrews, Askey and Roy [1].

A representation of the Hypergeometric function for $\mathfrak{Re}\, z < 1/2$ is given by the Pfaff transformation

$${}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1(a, c-b; c; z/(z-1)). \tag{3.1}$$

If $\mathfrak{Re}\, c > \mathfrak{Re}\, b > 0$, then for $z \notin [1, \infty)$,

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \tag{3.2}$$

known as Euler's integral for hypergeometric functions. Here Γ represents the gamma function. If a is an integer, then (3.2) holds for all $z \in \mathbb{C}$.

Two "distinct" hypergeometric functions ${}_2F_1(a_1, a_2; a_3; z)$ and ${}_2F_1(\tilde{a}_1, \tilde{a}_2; \tilde{a}_3; z)$ may be called contiguous if $|a_i - \tilde{a}_i| = 0$ or 1. There are interesting relations between contiguous hypergeometric functions, called contiguous relations. Of the many contiguous relations obtained by Gauss, we consider the following three term relations given as (2.5.3) and (2.5.16) in [1]:

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \Big(1 + \frac{a-b+1}{c} z\Big)\, {}_2F_1(a+1, b; c+1; z) \\ &\qquad - \frac{(a+1)(c-b+1)}{c(c+1)} z\, {}_2F_1(a+2, b; c+2; z) \end{aligned} \tag{3.3}$$

and

(3.4)
$$\begin{aligned}(c-a)\,{}_2F_1(a-1,b;c;z) &= \big(c-2a-(b-a)z\big)\ {}_2F_1(a,b;c;z)\\ &\qquad + a(1-z)\ {}_2F_1(a+1,b;c;z).\end{aligned}$$

Let, $2\mathfrak{Re}(b) \neq -1,-2,-3,\ldots$. Then from the contiguous relation (3.4),

$$\begin{aligned}&(b+\bar b+1+n)\,{}_2F_1(-n-1,b+1;b+\bar b+1;1-z)\\ &\quad= \big(\bar b+n+(b+1+n)z\big)\ {}_2F_1(-n,b+1;b+\bar b+1;1-z)\\ &\qquad\qquad - nz\ {}_2F_1(-n+1,b+1;b+\bar b+1;1-z).\end{aligned}$$

Hence, the monic polynomials

$$\Phi_n(b;z) = \frac{(b+\bar b+1)_n}{(b+1)_n}\ {}_2F_1(-n,b+1;b+\bar b+1;1-z), \quad n\geq 0, \tag{3.5}$$

satisfy the three term recurrence relation

$$\Phi_{n+1}(b;z) = (z+\beta_{n+1}^{(b)})\Phi_n(b;z) - \alpha_{n+1}^{(b)} z\Phi_{n-1}(b;z), \qquad n\geq 1,$$

with $\Phi_0(b;z)=1$ and $\Phi_1(b;z) = z+\beta_1^{(b)}$, where

$$\beta_n^{(b)} = \frac{\bar b+n-1}{b+n}, \quad \alpha_{n+1}^{(b)} = \frac{n(b+\bar b+n)}{(b+n)(b+n+1)}, \quad n\geq 1. \tag{3.6}$$

Note that $\beta_{n+1}^{(b)}\neq 0$, $\alpha_{n+1}^{(b)}\neq 0$,

$$\frac{\alpha_{n+1}^{(b)}}{\beta_{n+1}^{(b)}} = \frac{n(b+\bar b+n)}{(b+n)(\bar b+n)} \ \text{ and } \ 1-|\Phi_n(b;0)|^2 = 1-\Big|\frac{(\bar b)_n}{(b+1)_n}\Big|^2 = \frac{n(b+\bar b+n)}{(b+n)(\bar b+n)},$$

for $n\geq 1$. Hence, these coefficients satisfy the conditions of Theorem 2.1, provided that $\mathfrak{Re}(b) > -1/2$.

Together with the polynomials $\{Q_n(b;z)\}$ given by

$$Q_{n+1}(b;z) = (z+\beta_{n+1}^{(b)})Q_n(b;z) - \alpha_{n+1}^{(b)} zQ_{n-1}(b;z), \qquad n\geq 1,$$

with initial polynomials $Q_0(b;z)=0$ and $Q_1(b;z)=\beta_1^{(b)}$, we can now state the following.

Theorem 3.1. *Let $\mathfrak{Re}(b) > -1/2$. Then $\{\Phi_n(b;z)\}$ are the monic Szegő polynomials defined by*

$$\int_{\mathcal{C}} \bar z^j\,\Phi_n(b;z)d\mu(b;z) = (\kappa_n^{(b)})^{-2}\delta_{nj}, \quad 0\leq j\leq n, \tag{3.7}$$

with respect to a positive measure $\mu(b;z)$ on the unit circle. The coefficients $\kappa_n^{(b)} = \|\Phi_n(b;z)\|^{-1}$ and $a_n^{(b)} = \Phi_n(b;0)$ associated with these polynomials satisfy

$$\kappa_n^{(b)} = \sqrt{\frac{|(b+1)_n|^2}{(b+\bar b+1)_n\,n!}} \quad \text{and} \quad a_n^{(b)} = \frac{(\bar b)_n}{(b+1)_n} \quad n\geq 1.$$

Moreover,

$$\left.\begin{aligned}\Phi_n(b;z)\,L_0(b;z) - \frac{Q_n(b;z)}{\Phi_n(b;z)} &= \frac{(b+\bar b+1)_n\,n!}{(b+1)_n(\bar b+1)_n} z^n + O(z^{n+1}),\\ L_\infty(b;z) - \frac{Q_n(b;z)}{\Phi_n(b;z)} &= \frac{\bar b\,(b+\bar b+1)_n\,n!}{(b+1)_n(b+1)_{n+1}}\frac{1}{z^{n+1}} + O(\frac{1}{z^{n+2}}),\end{aligned}\right\} n\geq 1,$$

where $L_0(b;z)=\sum_{j=0}^{\infty}\mu_j^{(b)}z^j$ *and* $L_\infty(b;z)=-\sum_{j=1}^{\infty}\mu_{-j}^{(b)}z^{-j}$, *with*

$$\mu_0^{(b)}=\int_{\mathcal{C}}d\mu(b;z)=1 \quad \textit{and} \quad \overline{\mu_{-j}^{(b)}}=\mu_j^{(b)}=\int_{\mathcal{C}}z^{-j}d\mu(b;z)=\frac{(-b)_j}{(\bar{b}+1)_j}, \quad j\ge 1.$$

Proof. We only need to show $\mu_j^{(b)}=(-b)_j/(\bar{b}+1)_j$, $j\ge 0$, as the remaining results follow from results given in section 2.

If $b=0$, since $\Phi_n(0;z)=z^n$, the results are clearly true. So we also assume $b\neq 0$.

From the contiguous relation (3.3),

$$\frac{{}_2F_1(a+1,-b;\bar{b}+1;z)}{{}_2F_1(a,-b;\bar{b};z)} = \frac{1}{1+\dfrac{a+b+1}{\bar{b}}z-\dfrac{(a+1)(b+\bar{b}+1)}{\bar{b}(\bar{b}+1)}z\,\dfrac{{}_2F_1(a+2,-b;\bar{b}+2;z)}{{}_2F_1(a+1,-b;\bar{b}+1;z)}}.$$

Hence, if we write $R_n(a,b;z)=\dfrac{{}_2F_1(a+n+1,-b;\bar{b}+n+1;z)}{{}_2F_1(a+n,-b;\bar{b}+n;z)}$, $n=0,1,2,\ldots$, then

$$R_0(a,b;z)=\frac{1}{1+g_1z}-\frac{f_2z}{1+g_2z}-\cdots-\frac{f_{n-1}z}{1+g_{n-1}z}-\frac{f_nz}{1+g_nz-f_{n+1}zR_n(a,b;z)},$$

where $g_n=\dfrac{a+b+n}{\bar{b}+n-1}$, $f_{n+1}=\dfrac{(a+n)(b+\bar{b}+n)}{(\bar{b}+n-1)(\bar{b}+n)}$, $n\ge 1$.

If we restrict ourselves to the case in which $a=0$, then

$$\begin{aligned}R_0(0,b;z)&={}_2F_1(1,-b;\bar{b}+1;z)\\ &=\frac{1}{1+g_1z}-\frac{f_2z}{1+g_2z}-\cdots-\frac{f_{n-1}z}{1+g_{n-1}z}-\frac{f_nz}{1+g_nz-f_{n+1}zR_n(0,b;z)},\end{aligned}$$

where $g_n=\dfrac{b+n}{\bar{b}+n-1}$, $f_{n+1}=\dfrac{n(b+\bar{b}+n)}{(\bar{b}+n-1)(\bar{b}+n)}$, $n\ge 1$.

Equivalently, we can also write

$$R_0(0,b;z)=\frac{\beta_1}{z+\beta_1}-\frac{\alpha_2z}{z+\beta_2}-\cdots-\frac{\alpha_nz}{z+\beta_n}-\frac{\alpha_{n+1}z\,R_n(0,b;z)}{\beta_{n+1}},$$

where

$$\beta_n=\frac{1}{g_n}=\beta_n^{(b)} \quad \alpha_{n+1}=\frac{f_{n+1}}{g_ng_{n+1}}=\alpha_{n+1}^{(b)}, \quad n\ge 1. \tag{3.8}$$

Using the theory of continued fractions, we then observe that

$$\begin{aligned}R_0(0,b;z)-\frac{Q_n(b;z)}{\Phi_n(b;z)}&=\frac{\beta_{n+1}^{(b)}Q_n(b;z)-\alpha_{n+1}^{(b)}zR_n(0,b;z)Q_{n-1}(b;z)}{\beta_{n+1}^{(b)}\Phi_n(b;z)-\alpha_{n+1}^{(b)}zR_n(0,b;z)\Phi_{n-1}(b;z)}-\frac{Q_n(b;z)}{\Phi_n(b;z)}\\ &=\frac{\beta_1^{(b)}\alpha_2^{(b)}\cdots\alpha_n^{(b)}\alpha_{n+1}^{(b)}z^nR_n(0,b;z)}{[\beta_{n+1}^{(b)}\Phi_n(b;z)-\alpha_{n+1}^{(b)}zR_n(0,b;z)\Phi_{n-1}(b;z)]\Phi_n(b;z)}\\ &=\frac{(b+\bar{b}+1)_n\,n!}{(\bar{b})_n(\bar{b}+1)_n}z^n+O(z^{n+1}).\end{aligned}$$

Hence $L_0(b;z) = {}_2F_1(1,-b;\bar{b}+1;z)$, and the theorem follows. □

From $\mu_{-j}^{(b)} = \overline{\mu_j^{(b)}}$, we also have

$$L_\infty(b;z) = \frac{\bar{b}}{b+1} z^{-1}\, {}_2F_1(1,-\bar{b}+1;b+2;z^{-1}). \tag{3.9}$$

The following asymptotic results also hold.

Theorem 3.2.

$$\lim_{n\to\infty} \kappa_n^{(b)} = \frac{\sqrt{\Gamma(b+\bar{b}+1)}}{|\Gamma(b+1)|} \quad \textit{and} \quad \lim_{n\to\infty} n^{b-\bar{b}+1} a_n^{(b)} = \frac{\Gamma(b+1)}{\Gamma(b)}.$$

Proof. In the expressions given in Theorem 3.1 for $\kappa_n^{(b)}$ and $a_n^{(b)}$, use $\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^{z-1}}{(z)_n}$. □

Some orthogonal Laurent polynomials generated by the contiguous relations (3.3) and (3.4) are considered in Hendriksen and van Rossum [9], where they obtain orthogonality for these Laurent polynomials in terms of a non-positive-definite moment functional.

The contiguous relation (3.4) also gives rise to the Meixner-Pollaczek polynomials $P_n^{(\lambda,\theta)}(x) = \frac{(2\lambda)_n}{n!} e^{in\theta}\, {}_2F_1(-n,\lambda+ix;2\lambda;1-e^{-2i\theta})$, which are orthogonal on $(-\infty,\infty)$. For more information on these polynomials see for example [10].

4. More on the measure and related functions

Unless stated otherwise, we assume that $\mathfrak{Re}\, b > -1/2$. The following theorem gives the exact expression for the measure $\mu(b;z)$.

Theorem 4.1. *The measure $\mu(b;z)$ can be given by $d\mu(b;e^{i\theta}) = \omega(b;\theta)d\theta$, where*

$$\omega(b;\theta) = \tau^{(b)}\, e^{(\pi-\theta)\mathfrak{Im}\, b}\, [\sin(\theta/2)]^{2\,\mathfrak{Re}\, b}, \quad 0 \le \theta \le 2\pi.$$

The constant $\tau^{(b)} = \dfrac{2^{b+\bar{b}}\, |\Gamma(b+1)|^2}{2\pi\, \Gamma(b+\bar{b}+1)}$ is such that $\mu_0^{(b)} = 1$.

Proof. Since, $\mu_j^{(b)} = \int_0^{2\pi} e^{-ij\theta}\omega(b;\theta)d\theta = \overline{\int_0^{2\pi} e^{ij\theta}\omega(b;\theta)d\theta} = \overline{\mu_{-j}^{(b)}}$, we only have to show that

$$\mu_j^{(b)} = \tau^{(b)} \int_0^{2\pi} e^{-ij\theta} e^{(\pi-\theta)\mathfrak{Im}\, b} [\sin(\theta/2)]^{2\mathfrak{Re}\, b} d\theta = \frac{(-b)_j}{(\bar{b}+1)_j}, \quad j \ge 0.$$

With $2i\sin(\theta/2) = e^{i\theta/2} - e^{-i\theta/2}$,

$$\mu_j^{(b)} = \tilde{\tau}^{(b)} \int_0^{2\pi} e^{-i(j+\bar{b})\theta} [e^{i\theta}-1]^{b+\bar{b}} d\theta, \quad j \ge 0,$$

where $\tilde{\tau}^{(b)} = (2i)^{-2\mathfrak{Re}(b)} e^{\mathfrak{Im}(b)\pi} \tau^{(b)}$. Thus, observing that we can also write this in the form $\mu_j^{(b)} = \tilde{\tau}(b) \int_0^{2\pi} (i)^{-1} e^{-i(j+\bar{b}+1)\theta} [e^{i\theta}-1]^{b+\bar{b}} i e^{i\theta} d\theta$, $j \ge 0$, by integration by parts we establish that

$$\mu_{j+1}^{(b)} = \frac{-b+j}{\bar{b}+1+j}\, \mu_j^{(b)}, \quad j \ge 0.$$

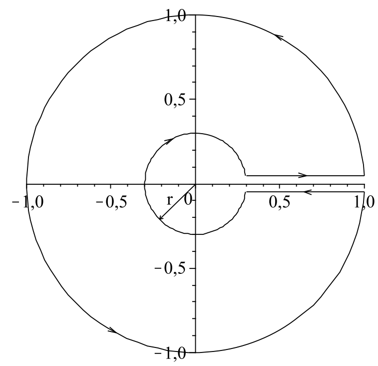

FIGURE 1. Contour Λ

Therefore, the proof will be complete if we can prove $\mu_0^{(b)} = 1$, equivalently, if we can show that $\mu_j^{(b)} = (-b)_j/(\bar{b}+1)_j$ or $\mu_{-j}^{(b)} = (-\bar{b})_j/(b+1)_j$ for some other particular value of j.

With $z = e^{i\theta}$, one can write

$$\mu_{-j}^{(b)} = \frac{i}{2^{b+\bar{b}}}\tau^{(b)} \int_{\mathcal{C}} z^j(-z)^{-\bar{b}-1}(1-z)^{b+\bar{b}}dz, \tag{4.1}$$

where the branch cuts in $(-z)^{-\bar{b}}$ and $(1-z)^{b+\bar{b}}$ are along the positive real axis.

Hence, we choose a j such that $\mathfrak{Re}(j-\bar{b}) > 0$ and evaluate the integral by contour integration using the contour Λ as given in Figure 1. Thus,

$$\mu_{-j}^{(b)} = \frac{i}{2^{b+\bar{b}}}\tau^{(b)} 2i\sin(\bar{b}\pi) \int_0^1 t^{j-\bar{b}-1}(1-t)^{b+\bar{b}}dt.$$

Hence, from the definitions of the gamma function, the beta function and the Euler's reflection formula, we obtain the required result. This completes the proof of the theorem. $\square$

The idea used here to calculate the integral (4.1) is the same as employed in [9], where the authors consider a general set of parameters for the exponents of z and $1-z$, but restricting the values of the parameters to be real.

Theorem 4.2. *If $\mathfrak{Re}\, b > 0$, then for all $z \in \mathbb{C}$ the polynomials $\Phi_n(b;z)$ and their reciprocals $\Phi_n^*(b;z)$ can be given by*

$$\left.\begin{aligned} \Phi_n(b;z) &= \frac{\Gamma(b+\bar{b}+n+1)}{\Gamma(b+n+1)\,\Gamma(b)} \int_0^1 t^b(1-t)^{\bar{b}-1}[1-(1-z)t]^n dt, \\ \Phi_n^*(b;z) &= \frac{\Gamma(b+\bar{b}+n+1)}{\Gamma(\bar{b}+n+1)\,\Gamma(\bar{b})} \int_0^1 t^{b-1}(1-t)^{\bar{b}}[1-(1-z)t]^n dt, \end{aligned}\right\} \quad n \geq 1.$$

Proof. The expression for $\Phi_n(b;z)$ follows immediately from (3.5) with the use of (3.2). To obtain the other, we can directly evaluate $z^n\overline{\Phi_n(b;1/\bar{z})}$ from the above Euler integral for $\Phi_n(b;z)$ or use (3.5) together with (3.1) to get

$$\Phi_n^*(b;z) = \frac{(b+\bar{b}+1)_n}{(\bar{b}+1)_n}\ {}_2F_1(-n,b;b+\bar{b}+1;1-z), \quad n \geq 0,$$

and then use (3.2). $\square$

We can now give an expression for the associated Szegő function

$$D(b;z) = exp\left(\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(\omega(b;\theta))d\theta\right).$$

Theorem 4.3.

$$D(b;z) = \frac{|\Gamma(b+1)|}{\sqrt{\Gamma(b+\bar{b}+1)}}(1-z)^b.$$

Proof. First we assume that $\mathfrak{Re}\, b > 0$ and use the knowledge that $\kappa_n^{(b)}\Phi_n^*(b;z) \to D(b;z)^{-1}$ uniformly on compact subsets of $\mathbb{D}$. With the substitution $u = nt$, we have from Theorem 4.2 that

$$\kappa_n^{(b)}\Phi_n^*(b;z) = \kappa_n^{(b)}\frac{\Gamma(b+\bar{b}+n+1)}{\Gamma(\bar{b}+n+1)\,\Gamma(\bar{b})}\frac{1}{n^b}\int_0^n u^{b-1}\big(1-\frac{u}{n}\big)^{\bar{b}}\big(1-(1-z)\frac{u}{n}\big)^n du.$$

Using

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^{z-1}}{(z)_n} \quad \text{and} \quad e^z = \lim_{n\to\infty}\big(1+\frac{z}{n}\big)^n,$$

together with the help of the Lebesgue's dominated convergence theorem, we then have

$$\lim_{n\to\infty}\kappa_n^{(b)}\Phi_n^*(b;z) = \frac{\sqrt{\Gamma(b+\bar{b}+1)}}{|\Gamma(b+1)|}\frac{1}{\Gamma(b)}\int_0^\infty u^{b-1}e^{-(1-z)u}du$$

for $0 < z < 1$. Hence the theorem follows from the integral representation for a gamma function. The result for remaining values of z follows by analytic continuation. The result can also be analytically extended for $\mathfrak{Re}(b) > -1/2$. □

From (2.5) and (3.9) for the associated Carathéodory function,

$$1 - \lim_{n\to\infty}\frac{2zQ_n^*(b;z)}{\Phi_n^*(b;z)} = F(b;z) = -1 + 2\,{}_2F_1(1,-b+1;\bar{b}+1;z)$$

for z on compact subsets of $\mathbb{D}$.

Using [1, Eq. (2.3.14)], since $\Phi_n(b;z) = \frac{(\bar{b})_n}{(b+1)_n}\,{}_2F_1(-n,b+1;1-n-\bar{b};1-z)$, a generating function for these Szegő polynomials is $G(b;z,t) = (1-t)^{-\bar{b}}(1-tz)^{-b-1}$, and one can verify that

$$G(b;z,t) = \sum_{n=0}^{\infty}\frac{(b+1)_n}{n!}\Phi_n(b;z)\,t^n.$$

This generating function shows that the Szegő polynomials considered here are very similar to orthogonal functions considered by Gasper [6].

5. Para-orthogonal polynomials

As defined by Jones, Njåstad and Thron [11], for any ρ such that $|\rho| = 1$, the polynomial

$$B_n(b,\rho;z) = \frac{\Phi_n(b;z)+\rho\,\Phi_n^*(b;z)}{1+\rho\,\overline{\Phi_n(b;0)}}$$

is a monic para-orthogonal polynomial of degree n. The zeros of these polynomials are simple and lie on the unit circle $\mathcal{C}$. This is different from the Szegő polynomials which have their zeros, not necessarily simple, but which lie within the open unit disk $\mathbb{D}$.

Since $\Phi_n(0;z) = z^n$, we clearly have $B_n(0,\rho;z) = z^n + \rho$, which clearly has n distinct zeros on the unit circle.

Now, assuming $b \neq 0$, let

$$\rho_{n1}^{(b)} = \frac{(\bar{b})_{n+1}}{(b)_{n+1}} \quad \text{and} \quad \rho_{n2}^{(b)} = -\frac{(\bar{b}+1)_n}{(b+1)_n}, \quad n \geq 1.$$

Then the following can be easily established for the sequences of monic para-orthogonal polynomials $\{B_n(b,\rho_{n1}^{(b)};z)\}$ and $\{B_n(b,\rho_{n2}^{(b)};z)\}$.

Theorem 5.1. *With $\mathfrak{Re}\, b > -1/2$ and $b \neq 0$, if $\Phi_n^{(1)}(b;z) = B_n(b,\rho_{n1}^{(b)};z)$, then*

$$\Phi_n^{(1)}(b;z) = \frac{(b+\bar{b})_n}{(b)_n}\ {}_2F_1(-n,b;b+\bar{b};1-z), \quad n \geq 1, \quad \textit{for} \quad \mathfrak{Re}\, b \neq 0,$$

$$\Phi_n^{(1)}(b;z) = \frac{n!}{(b)_n}(z-1)\ {}_2F_1(-n+1,b+1;2;1-z), \quad n \geq 1, \quad \textit{for} \quad \mathfrak{Re}\, b = 0.$$

Moreover,

$$\Phi_{n+1}^{(1)}(b;z) = \big(z + \frac{\bar{b}+n}{b+n}\big)\Phi_n^{(1)}(b;z) - \frac{n(b+\bar{b}+n-1)}{(b+n-1)(b+n)} z\Phi_{n-1}^{(1)}(b;z), \quad n \geq 1,$$

with $\Phi_0^{(1)}(b;z) = 1$ and $\Phi_1^{(1)}(b;z) = z + \bar{b}/b$.

Similarly with $\mathfrak{Re}\, b > -1/2$, if $(z-1)\Phi_{n-1}^{(2)}(b;z) = B_n(b,\rho_{n2}^{(b)};z)$, then

$$\Phi_n^{(2)}(b;z) = \frac{(b+\bar{b}+2)_n}{(b+1)_n}\ {}_2F_1(-n,b+1;b+\bar{b}+2;1-z), \quad n \geq 0,$$

and

$$\Phi_{n+1}^{(2)}(b;z) = \big(z + \frac{\bar{b}+n+1}{b+n+1}\big)\Phi_n^{(2)}(b;z) - \frac{n(b+\bar{b}+n+1)}{(b+n)(b+n+1)} z\Phi_{n-1}^{(2)}(b;z), \quad n \geq 1,$$

with $\Phi_0^{(2)}(b;z) = 1$ and $\Phi_1^{(2)}(b;z) = z + (\bar{b}+1)/(b+1)$.

Again, the three term recurrence relations follow from the contiguous relation (3.4).

6. Concluding remarks

From the properties of the zeros of Szegő polynomials and para-orthogonal polynomials, we can also state the following.

Remark 6.1. If $\mathfrak{Re}\, b > -1/2$, then the zeros of the nth degree hypergeometric polynomial ${}_2F_1(-n,b+1;b+\bar{b}+1;1-z)$ lie within the ring $|b+n|^{-1}|b| \leq |z| < 1$ and, also if $\mathfrak{Re}\, b \neq 0$, then the zeros of the nth degree hypergeometric polynomial ${}_2F_1(-n,b;b+\bar{b};1-z)$ are distinct and lie on the unit circle.

For the lower bound of the zeros of ${}_2F_1(-n,b+1;b+\bar{b}+1;1-z)$ see Corollary 1.7.3 of [17].

As we have already remarked in the introduction, if $b = \lambda$ is real, then the polynomials $\{\Phi_n(\lambda;z)\}$ are the monic Szegő polynomials associated with the weight function $[\sin(\theta/2)]^{2\lambda}$. Hence, these are the polynomials obtained from the Gegenbauer polynomials using the Szegő transformation.

Taking $b = -i\eta$, we can state the following.

Remark 6.2. Let $\eta \in \mathbb{R}$. Then

$$\Phi_n(-i\eta; z) = \frac{n!}{(1-i\eta)_n} \, {}_2F_1(-n, 1-i\eta; 1; 1-z), \quad n \geq 0,$$

are the monic Szegő polynomials associated with the weight function $e^{\eta\theta}$, with the orthogonality given by

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-ij\theta} \Phi_n(-i\eta; e^{i\theta}) \, e^{\eta\theta} d\theta = \frac{e^{\eta\pi}(n!)^2}{|\Gamma(1+n-i\eta)|^2} \delta_{nj}, \quad 0 \leq j \leq n.$$

Moreover, the Verblunsky coefficients $a_n^{(-i\eta)} = (i\eta)_n/(1-i\eta)_n$, $n \geq 1$, are such that $|a_n^{(i\eta)}|^2 = \eta^2/(n^2+\eta^2)$.

Similar orthogonality, however with an inclusion of a mass point at $z = 1$, was recently obtained in Tsujimoto and Zhedanov [24].

Finally, communications with Prof. Richard Askey after the initial submission of this paper has brought to our attention one of his comments regarding [21] on page 304 of [2]. There he mentions the biorthogonality of $\{{}_2F_1(-n, x; x+y-1; 1-z)\}$ and $\{{}_2F_1(-n, y; x+y-1; 1-z)\}$ with respect to the weight function $e^{i(x-y)\theta/2}[\sin\theta/2]^{x+y-2}$, $0 \leq \theta \leq 2\pi$. Surprisingly, it seems no one has done this; the orthogonality results in the current paper can be realized if we take $x = \bar{b}+1$ and $y = b+1$. Professor Askey also informed me of the paper [23] on the asymptotic expansion for these polynomials.

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