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ISW Effect Through Dark Energy Quintessence and ΛCDM Models

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"Todos los años, por el mes de marzo, una familia de gitanos desarrapados plantaba su carpa cerca de la aldea, y con un grande alboroto de pitos y timbales daban a conocer los nuevos inventos."

Gabriel García Márquez
Cien años de soledad

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Dedicated to Angie, mom, dad and sister, who gave me their support through this process, despite the distance.

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Resumo

Observações atuais do satélite Wilkinson Microwave Anisotropy Probe (WMAP) da Radiação Cósmica de Fundo (CMB) e estruturas de grande escala (LSS) têm permitido melhorar os estudos das anisotropias secundárias, especialmente o efeito Sachs-Wolfe Integrado (ISW). Usando a correlação cruzada entre a CMB e mapas da LSS, o sinal do efeito ISW pode ser detectado. Nós podemos usar o efeito ISW junto com o modelo cosmológico padrão (neste caso o Universo esta dominado pela constante cosmológica e a Matéria Escura Fria, ΛCDM) mais algoritmos numéricos para restringir os parâmetros em um modelo cosmológico com energia escura. Para diferentes casos com um único parâmetro livre de um model de Quintessência parametrizado, $w_0 < 0$ e $2,0 < w_a < -2,0$, podemos encontrar bins de largura $[-1,926\,,\,-0,323]$ em w_0 e $[-0,855\,,\,1,190]$. Nestes intervalos, obtemos um sigma de nivel tomando o 68% da amostra que melhor se ajusta ao modelo cosmológico padrão.

Keywords: Dark energy; ΛCDM; Power spectrum; Cosmic Microwave Background; Integrated Sachs-Wolfe effect; Quintessence models.

Areas: Cosmology and Astrophysics

Abstract

Current observations of the Wilkinson Microwave Anisotropy Probe (WMAP) satellite of Cosmic Microwave Background (CMB) and Large Scale Structure (LSS) have allowed to improve studies of the secondary anisotropies, especially the Integrated Sachs-Wolfe effect (ISW). Using the cross-correlation between the CMB and LSS maps, the ISW effect signal can be detected. We can use the ISW effect together with standard cosmological model (in this case the Universe is dominated by the cosmological constant and Cold Dark Matter, ΛCDM) plus numerical algorithms to constrain the parameters in a cosmological model with dark energy. For cases different with a single free parameter of a parameterised Quintessence model, $w_0 < 0$ and $2,0 < w_a < -2,0$, we can find bins of width $[-1,926\,,\,-0,323]$ in w_0 and $[-0,855\,,\,1,190]$ in w_a. In these intervals, we obtain one sigma level by taking the 68% of the sample which best fit the standard cosmological model.

Contents

Introduction

From the measurements of supernovae Ia distances made by Perlmutter and Riess in [1, 2], we changed the vision of the Universe dynamics at the end of the 90's. They discovered that the Universe is expanding in an accelerated way, therefore it is necessary to insert an exotic energy component, which has negative pressure and dominates the Universe dynamics. This component is called dark energy. There are different theories and cosmological models that try to explain this type of energy. Among them we find the cosmological constant and Quintessence models [3, 4]. In the chapter 1 we present the basic notions for a Universe with cosmological constant and a parameterised Quintessence model with three parameters. In addition, we develop the necessary theory to understand the Universe dynamics according to a given cosmological model. This theoretical framework is built using the post-Newtonian description, see [5].

One of the fundamental pieces to understand the modern Cosmology is given by the study of anisotropies of the Cosmic Microwave Background (CMB). This study begins with the discovery of the anisotropies made by the Differential Microwave Radiometer (DMR) experiment on the Cosmic Background Explorer (COBE) satellite in the early 90's, see [6]. Currently we have the results of seven years Wilkinson Microwave Anisotropy Probe (7WMAP) [7]. The acceleration of the Universe seen by supernovae, joint to the analysis of CMB with Large Scale Structure (LSS) requires zero curvature [8, 9], i.e., a locally flat Universe. The anisotropies seen in the CMB is a clear evidence of the big bang theory, and so we can explain the LSS formation that is known today. In the chapter 2 we develop the theory for understanding the evolution of the matter fluctuations in the Universe after the matter radiation equality. Given that the perturbations studied will be consider on the horizon, we shall work in the Newtonian framework.

The main objective is to study the parameters behavior in cosmological models with dark energy (parameterised Quintessence model, it is described by the equation of state (4.1)) and compute their best fit with the standard cosmological model (ΛCDM model, defined in section 4.1) until one sigma level, using the Integrated Sachs-Wolfe (ISW) effect [10].

The main secondary anisotropy at large angle is the so-called Integrated Sachs-Wolfe effect, which is a signature of the decay of the gravitational potential at large scales. This could be either a result of spatial curvature [11, 12] , or presence of a component with negative pressure, the so-called dark energy, in the Universe [13]. Crittenden and Turok [14] proposed to use the cross-correlation between the LSS and the CMB to detect it. They showed that such correlations also include a cosmic magnification signal which may minimize or dampen the ISW effect, especially at high redshift.

Initial CMB-galaxy cross-correlation measurements using the Cosmic Background Explorer (COBE) data failed to find a signal, but this changed dramatically with the arrival of the WMAP data. Weak correlations, at the 2-3σ level, were soon seen between the CMB and numerous probes of large scale structure, such as the NVSS radio galaxy survey and X-ray background [18], and in optical survey like the SDSS [19], while weaker indication been seen using the shallower 2MASS infrared survey [17, 20]. Planck should not dramatically differ from WMAP on the large scale to which the ISW is most sensitive, but its greater resolution and frequency coverage will enable greater control of possible systematic contaminations from the galaxy and extra-galactic point sources. In the chapter 3 we present the main concepts for understanding the properties of matter fluctuations, the CMB angular power spectrum, ISW effect and the cross-correlation between the LSS and the CMB.

Constraining cosmological parameters from measurements of the ISW effect requires developing robust and accurate methods for computing statistical errors in the cross-correlation between maps. We use the likelihood function in this case, see [21]. Numerical algorithms developed in this work are based in the routines given in Numerical Recipes in C [22]. In [17] and [20], Afshordi et al. and Rassat et al. respectively show the formalism to find the statistical errors in the detection of the ISW effect via the cross-correlation. In addition, Rassat et al. determine the value for the linear bias, which is derived directly from the 2 Micron All-Sky Survey (2MASS).

We used two cases with a single free parameter in a parameterised Quintessence model described by equation of state (4.1). We get a value range for the parameter for each case, using the best fit with the standard cosmological model via ISW effect. In addition, we study the behavior of a parameterised Quintessence model with two parameters, in this case, we present the contour curves that enclose the values for the parameters with the best fit in one sigma level. In this way, we accomplish to characterize the parameters in a cosmological model with dark energy using the ISW effect.

Chapter 1

Standard Cosmology notions

In Cosmology we study the nature of large scale and dynamics of the Universe. The objectives are to study the origin of the Universe and the current paradigm given by observations (i.e., Universe is expanding in an accelerated way) which is associated with a type of energy with negative pressure called dark energy. Its nature is still unknown, see [1, 2]. Cosmology as it is known today was prompted at the beginning of the 20^{th} century with the theory of general relativity and the solution of the field equations in a homogeneous and isotropic Universe.

In this chapter the basic concepts will be developed in the framework of the standard Cosmology.

1.1 Universe components

The Universe we see today is composed of photons, baryons (ordinary matter), dark matter, dark energy and neutrinos. The dynamics of the Universe is determined by the amount of each component.

Photons

It is the largest component of the CMB. Today CMB has a temperature of $T_0 = 2,725 \pm 0,002\mathrm{K}$ [28]. Photons are massless bosons satisfying the Bose-Einstein distribution

$$\rho_\gamma = 2\int_0^\infty \frac{\mathrm{p}\,c}{\exp(\frac{\mathrm{p}\,c}{k_B\,T}) - 1}\frac{4\pi\,\mathrm{p}^2}{(2\pi)^3}\mathrm{dp},$$

where p is the momentum, c the speed of light, k_B is the Boltzmann's constant, radiation density ρ_γ due to photons and T the temperature. The factor two multiplying the integral is due to the spin of the photon. The above expression can be accurately calculated obtaining

$$\rho_\gamma = \frac{\pi^2\,k_B^4}{15\,c^3}\,T^4.$$

This is a perfect black body spectrum that contains about 400 photons per cm^3. We can find small anisotropies in the CMB of the order of 10^{-5} around T_0.

Baryons

Technically baryons are composed of three quarks. Protons and neutrons are baryons. In cosmology when we talk about baryons, one refers to everything composed by usual matter including stars, diffuse and ionized gas and heavy elements. The actual amount of this component in the Universe observable is around $\sim 4\%$.

Dark matter

Dark matter has been suggested to fit the shape of the rotation curves in galaxies. A large number of tests have indicated that the suspicion that dark matter really exists is correct and that a large fraction of the matter in the Universe is in fact dark, without pressure and cold, hence the acronym Cold Dark Matter (CDM). It is possible to estimate the amount of dark matter by observing the ratio of dark matter to baryons with some observations, see [29, 30]. These results indicate that the relationship baryon dark matter is about 20%. Anisotropies in the distribution of galaxies are also sensitive to this relationship, current observations indicate that this ratio is $0,15 \pm 0,07$, which is in agreement with studies of clusters. Also the anisotropies in the CMB are extremely sensitive to the density of non-baryonic matter. Recent observations [24] indicate that $\Omega_m h^2 = 0,135 \pm 0,009$ (h is a parameter used in the definition of Hubble's constant now, $H_0 = 100\,h\,\mathrm{km/s\,Mpc^{-1}}$, which vary as $0,5 < h < 1,0$. The current accepted value is $h = 0,72$, see [39]).

Although it is worrying to accept that there is a component of the matter in the Universe whose nature is unknown, there is a strong evidence that such a component does exist, the galaxies rotation curves [25]. It is non-relativistic and spans large distances inside galaxies. The lifetime of dark matter tends to be of the order of the age of the Universe or greater. Its self-interaction can not be too large and the cross section that interacts with baryons should be small.

Dark energy

Dark energy is the component responsible for the Universe acceleration. Dark matter would cause the so-called slowdown to the Universe while dark energy acts in contrary opposition to the intuitive way. The cosmological constant is a special case among many other dark energy models. Dark energy is usually characterized by its equation of state relating the energy density and pressure. Later on there will be a comprehensive description of this component.

Neutrinos

First postulated by Wolfgang Pauli in 1930 to conserve energy and momentum in beta decay the neutrino was long thought to be massless. They come in three different flavors known as neutrino of electron, muon and tau. They interact very weakly with baryons therefore are difficult to be detect. Neutrino's oscillation from one flavors to another implies that they have mass eigenstates which are different from zero.

The photon bath, which was a very hot plasma at early times, was coupled to other particles at the beginning of the Universe. However, as the photons cooled, the came out of equilibrium with other particles. Hence the same must have occurred to neutrinos and we expect there to be a cosmic background of neutrinos left from the early creation of particles, at times close to the beginning of the Universe.

1.2 Cosmological principle

This principle is based on the hypothesis of a homogeneous and isotropic Universe to a large scale $\sim$ 100Mpc. However there are models that are not based on these two concepts, but they are not going to be analyzed here.

At first sight these two concepts seem almost similar and indeed they are closely related. Homogeneity implies that the Universe looks the same regardless the observed position. In Cosmology one often uses density to define uniformity, however other properties which are function of position can also be used. Isotropy implies equal properties along different axes, regardless the direction. One can construct a homogeneous Universe but with anisotropies. This is the case of a Universe with a preferred direction. One can as well build a Universe that is isotropic only in a point but not isotropic in other points and at the same time being inhomogeneous.

The homogeneity and isotropy of the CMB temperature up to an order of 10^{-5}. Experiments different have mapped a large number of galaxies in the Universe which supports the cosmological principle, see Figure 1.4.

1.3 Expanding Universe, Hubble's law

To define the position of objects in space it is necessary to use a suitable coordinate system. Consider coordinates in which galaxies in space do not move, so the physical sensation of the galaxies motion is subject to the expansion of space and no relative movement between them, those kind of coordinates are called comoving. In Figure 1.1 we show an example of this type of coordinates.

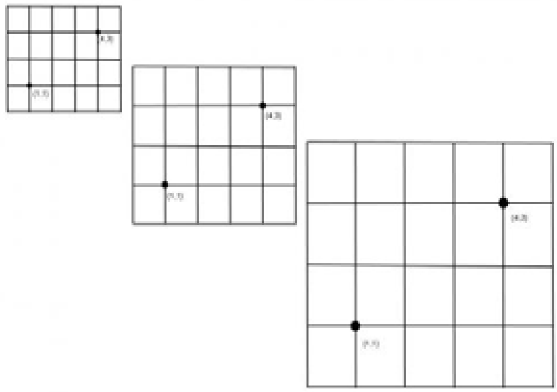


Figure 1.1: Shows the variation in the physical distance between points $(1, 1)$ and $(4, 3)$, however the distance between points in this coordinate system remains constant between different pictures.

When considering two galaxies with position $\mathbf{r}_i$ and $\mathbf{r}_j$ respectively, the distance that separates them is given by $\Delta\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$. We define the position of the galaxies in the comoving coordinate system as $a\,\mathbf{x}_i$ and $a\,\mathbf{x}_j$, where a is the scale factor, which contains information on the space dynamics. The distance between the two galaxies in that coordinate system is

$$\Delta\mathbf{r} = a\,(\mathbf{x}_i - \mathbf{x}_j) = a\,\Delta\mathbf{x}.$$

Since we are assuming the cosmological principle, the scale factor is a function of time only, therefore $a = a(t)$. Here $\Delta\mathbf{x}$ value remains constant for each pair of galaxies as labeled according to their position in the Universe.

By studying the relative velocities of each pair of galaxies, it is necessary to differentiate the above expression with respect to time.

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}(a(t)\,\Delta\mathbf{x}) = \dot{a}\Delta\mathbf{x} = \frac{\dot{a}}{a}\,a\,\Delta\mathbf{x} = H\,\Delta\mathbf{r}, \tag{1.1}$$

where $H = \dot{a}/a$ is the Hubble's constant. It is observed that H generally can be a function of time. It is constant only as a function of space.

These arguments can be announced as *Hubble's law*: the radial velocity between galaxies is proportional to the distance between them. In 1929, Hubble showed this result in an article entitled "*A relation Between distance and radial velocity among extra-galactic nebulae*", at a meeting of the National Academy of Science [26].

1.4 Metric in the Universe

In Cosmology one has to construct definitions of distances and times. Usually the concept of time has not the intuitive meaning thought most situations. Special relativity is a good example.

Considering flat geometry, infinitesimal changes in cartesian space are related by

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

The above expression satisfies the cosmological principle in space. The goal is to find a metric that accomplishes this principle applied to a Universe with a more general geometry. To this end, first we analyze the metric induced by a 3-sphere $S_3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = R^2\}$. The 3-sphere can be parametrized as follows:

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta, \\ w &= \sqrt{R^2 - r^2}, \end{aligned}$$

differentiating

$$\begin{aligned} \mathrm{d}x &= \sin\theta \cos\phi \, \mathrm{d}r + r \cos\theta \cos\phi \, \mathrm{d}\theta - r \sin\theta \sin\phi \, \mathrm{d}\phi, \\ \mathrm{d}y &= \sin\theta \sin\phi \mathrm{d}r + r \cos\theta \sin\phi \mathrm{d}\theta + r \sin\theta \cos\phi \, \mathrm{d}\phi, \\ \mathrm{d}z &= \cos\theta \, \mathrm{d}r - r \sin\theta \, \mathrm{d}\theta, \\ \mathrm{d}w &= -\frac{r \, \mathrm{d}r}{\sqrt{R^2 - r^2}}, \end{aligned}$$

therefore the metric induced by those parameters is

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \, \mathrm{d}\Omega^2,$$

where $k = 1/R^2$ is related to the curvature and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2$, solid angle differential. Although this metric was built for a closed geometry $k > 0$ space, it can be generalized for an open geometry (basically change $R = iR$) $k < 0$ or flat $k = 0$ without losing the assumption of homogeneity and isotropy in space.

Performing a transformation of the r variable to χ we get a compact expression depending on the space geometry

$$r = S_k(\chi) = \begin{cases} |k|^{-1/2} \sin(|k|^{1/2} \chi) & si \quad k > 0; \\ \chi & si \quad k = 0; \\ |k|^{-1/2} \sinh(|k|^{1/2} \chi) & si \quad k < 0. \end{cases}$$

In this case the metric can be written as

$$\mathrm{d}l^2 = \mathrm{d}\chi^2 + S_k(\chi)^2\,\mathrm{d}\Omega^2.$$

Now it is necessary to contextualize these results for the space-time. The metric used in special relativity is the Minkowski metric, which is defined as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}l^2.$$

To extend this concept to the metric that describes the dynamics of the Universe, one needs to introduce a closed hypersurface H_s with an imaginary dimension defined as $\{H_s = (x, y, z, w, v) : x^2 + y^2 + z^2 + w^2 - v^2 = R_0^2\}$.

The metric can be written as

$$\mathrm{d}s^2 = \mathrm{d}v^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 - \mathrm{d}w^2.$$

Now considering the following parameterization

$$\begin{aligned} v &= R_0 \sinh\alpha, \\ w &= R_0 \cosh\alpha\, \cos\beta, \\ z &= R_0 \cosh\alpha\, \sin\beta\, \cos\theta, \\ x &= R_0 \cosh\alpha\, \sin\beta\, \sin\theta\, \cos\phi, \\ y &= R_0 \cosh\alpha\, \sin\beta\, \sin\theta\, \sin\phi, \end{aligned}$$

we obtain

$$\mathrm{d}s^2 = R_0^2\,\mathrm{d}\alpha^2 - R_0^2 \cosh^2\alpha(\mathrm{d}\beta^2 + \sin^2\beta\,\mathrm{d}\Omega^2).$$

Identifying $R_0\alpha = ct$, $a(t) = |k|R_0\cosh(ct/R_0)$ and $\beta = |k|^{1/2}\chi$, the expression above becomes

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)^2(\mathrm{d}\chi^2 + |k|^{-1}\sin^2(|k|^{1/2}\chi)\mathrm{d}\Omega^2).$$

It shows that the function that accompanies the solid angle is $S_k(\chi)$ for a open geometry space. With similar arguments we can generalize this metric to a Universe with a more general geometry, which satisfies the cosmological principle. This is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric and it is defined as

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)^2(\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2). \tag{1.2}$$

This metric actually describes an expanding Universe.

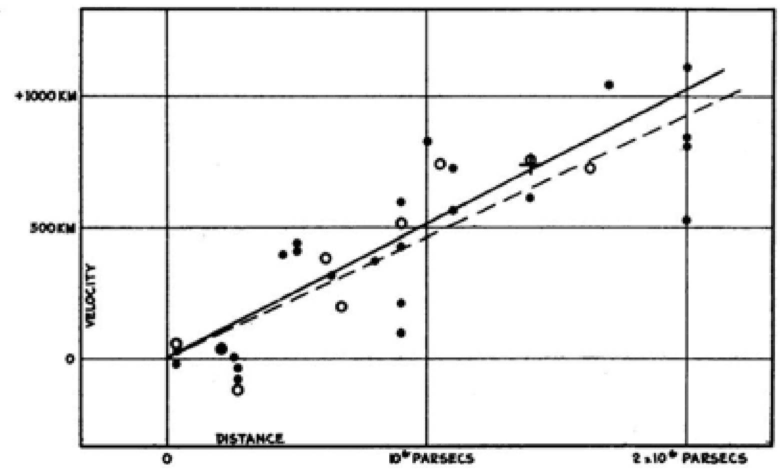


Figure 1.2: Relation between velocity-distance of Hubble. The two lines use a different correction for the movement of the sun. *Hubble 1929, The Realm of the Nebulae, Yale University Press* [26].

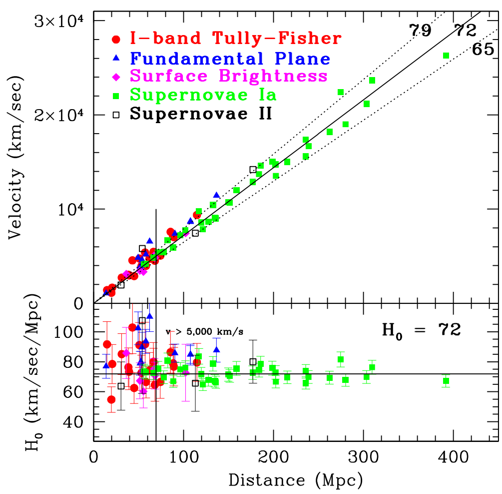
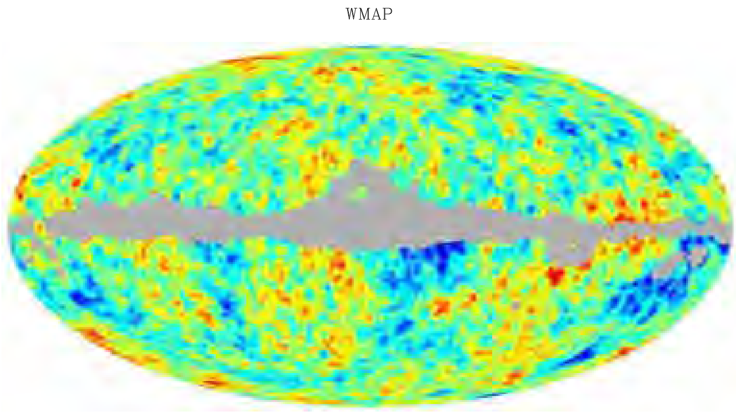


Figure 1.3: Velocity-distance for galaxies with 400 Mpc calibrated by Cepheid distance scale. An adjustment to the slope yields a value of $H_0 = 72\, km/s\, \mathrm{Mpc}^{-1}$, see [39].



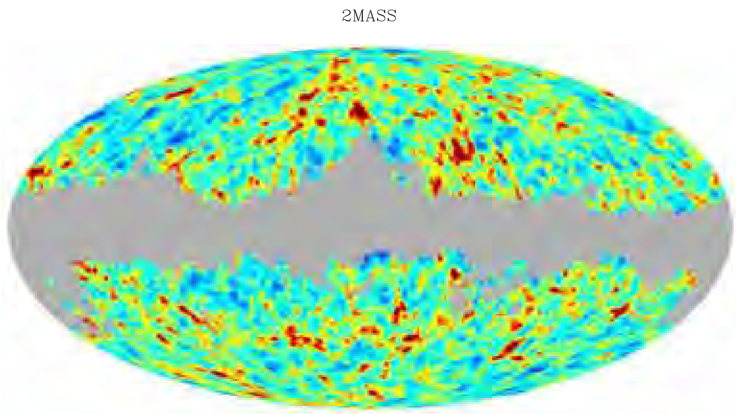


Figure 1.4: *Top*: The 3rd year WMAP (WMAP3) Internal Linear Combination (ILC) map, which are maps of temperature anisotropies in the CMB. *Bottom*: The 2 Micron All-Sky Survey (2MASS) 827,947 overdensities field of galaxies. Red shading represents CMB hot spots and overdensities in the 2MASS and blue shading correspond to cold spots CMB and low densities in the 2MASS. The gray mask corresponds to K_p2 for the CMB and the 2MASS map to galaxies with the galactic extinction $A_k > 0,05$. The release of the data is available on the link http://lambda.gsfc.nasa.gov/ (WMAP3, see [8]) and http://www.ipac.caltech.edu/2mass/ (2MASS, see [31])

1.5 Dynamics of the Universe from a classical perspective

The study of the dynamics of the Universe is based on finding the equation of motion for the scale factor given in the FLRW metric. If the scale factor is zero we would be in the case of a static Universe, otherwise, we would be in the case of a dynamic Universe.

Usually to find the equation of motion for the scale factor we have to use general relativity, but in this section we present a Newtonian description to determine the dynamics of the Universe. Milne and McCrea proposed to use this description in [5].

1.5.1 Friedmann's equation

To develop the theoretical frame that leads to the equation of motion of the scale factor it is necessary to define the system. For this case we consider a galaxy mass test m which is immersed in a gravitational field due to a uniform mass M. Using the Newton's second law

$$m\ddot{r} = -G\frac{m\,M}{r^2}, \quad \text{where} \quad M = \frac{4}{3}\pi r^3 \rho.$$

Now replacing the coordinate r by the comoving coordinate distance x using the relation $r = a(t)x$, we obtain

$$\ddot{a} = -\frac{4}{3}\pi G\rho a,$$

if $\rho = \frac{\rho_0}{a^3}$, then

$$\ddot{a} = -\frac{4}{3}\pi_0 G\rho a^{-2},$$

multiplying $\dot{a}$ in both side, in the above equation

$$\dot{a}\ddot{a} = -\frac{4}{3}\pi_0 G\rho a^{-2}\dot{a},$$

and using $\frac{1}{2}\frac{\mathrm{d}(\dot{a}^2)}{\mathrm{d}t} = \dot{a}\ddot{a}$, $-\frac{\mathrm{d}(a^{-1})}{\mathrm{d}t} = a^{-2}\dot{a}$, we have

$$\frac{\mathrm{d}(\dot{a}^2)}{\mathrm{d}t} = \frac{8}{3}\pi G\rho_0\frac{\mathrm{d}(a^{-1})}{\mathrm{d}t},$$

integrating

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho_0 a^{-3} - \frac{k\,c^2}{a^2},$$

where k is an integration constant, if we define $H = \frac{\dot{a}}{a}$, we can rewrite the above equation as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\,G}{3}\rho - \frac{k\,c^2}{a^2}. \tag{1.3}$$

The above expression is called Friedmann's equation. Comparing with the results obtained using general relativity, see [21, 23], the constant k is exactly the same that we have defined in the FLRW metric.

1.5.2 Fluid Equation

Friedmann's equation is not enough to find $a(t)$, therefore we need to find an equation for the evolution of ρ as a function of time. Considering the first law of thermodynamics

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V, \tag{1.4}$$

where p is the pressure, and assuming that the energy is equal to $E = \varepsilon_{total} V$, where ε_{total} is the total energy density of the fluid in the relativistic sense, then

$$\mathrm{d}E = V\mathrm{d}\varepsilon_{total} + \varepsilon_{total}\mathrm{d}V,$$

as $V \propto a^3$, so $\mathrm{d}V = \frac{3V}{a}\mathrm{d}a$, since we restrict the study only to adiabatic processes $\mathrm{d}S = 0$, we obtain

$$V\mathrm{d}\varepsilon_{total} + \varepsilon_{total}\left(\frac{3V}{a}\mathrm{d}a\right) = -p\left(\frac{3V}{a}\mathrm{d}a\right),$$

dividing by $\mathrm{d}t$, and simplifying

$$\frac{\mathrm{d}\varepsilon_{total}}{\mathrm{d}t} + 3\frac{\dot{a}}{a}(\varepsilon_{total} + p) = 0.$$

This result can be expressed in terms of the inertial mass density associated with the total energy density $\varepsilon_{total} = \rho c^2$, therefore the above equation is rewritten as

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{1.5}$$

The above is called fluid equation. This equation can be solved by specifying the type of component. Defining a general property for each component of the Universe, which is proportional to the ratio of the pressure with the energy density of the component. This amount $w_i = \frac{p_i}{c^2\rho_i}$ can be replaced in (1.5) and we get

$$\dot{\rho}_i + 3\,H\rho_i(1 + w_i) = 0. \tag{1.6}$$

Solving equation (1.6) by using the initial condition $\rho_i(a_0) = \rho_i(1) = \rho_i^0$, where the scale factor today a_0 is considered equal to one, we obtain the following expression for the density of each component ρ_i

$$\rho_i = \rho_i^0 \exp\left(-3\int_1^a \frac{\mathrm{d}a'}{a'}(1 + w(a'))\right). \tag{1.7}$$

Note that non-relativistic matter has no pressure so that for this component $w_m = 0$. Due to the cosmological constant, dark energy has a constant energy density, i.e., $w = -1$ (we cover this topic in more detail later on). Radiation has a pressure that varies as one third of the density, it is a result of thermodynamics, here $w_r = \frac{1}{3}$. We also observe that the curvature acts as a density with $w_k = -\frac{1}{3}$.

1.5.3 Acceleration equation

Friedmann's equation and the fluid equation determine the evolution of the scale factor as a function of time. Now we combine both expressions to find an equation for the acceleration of the Universe eliminating the constant curvature. Differentiating equation (1.3) with respect to time we have

$$2\,\dot{a}\,\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}\,a^2 + 2\rho\,a\,\dot{a}).$$

Solving equation (1.5) for $\dot{\rho}$ and substituting in the above equation we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right). \tag{1.8}$$

This expression is called acceleration equation. If we write the pressure for each component of the Universe as $p_i = c^2 w_i \rho_i$, the above equation can be rewritten as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i \rho_i(1 + 3w_i). \tag{1.9}$$

According to the above expression, for an accelerated expanding Universe, we must have a component with negative pressure, which was unthinkable. By the end of the last century it was considered that the Universe was expanding, but not accelerated. However, distance measures of type Ia supernovae performed by Perlmutter et al. and Riess et al., see [1, 2], show that the Universe is expanding in an accelerated way, therefore we need to include a component with $w < -\frac{1}{3}$ in its equation of state. The natural candidate for this type of energy is the cosmological constant, although there exist other Quintessence models that also may explain this phenomenon.

1.5.4 Generalizing the Friedmann's equation

Knowing that the Universe is expanding in an accelerated way, it is need to add a component with negative pressure. The simplest case to describe this type of exotic energy would be to consider a constant density, observing (1.7) we have $w = -1$. The total density of the Universe is written as $\rho = \rho_{other} + \rho_\Lambda$, where ρ_Λ is the energy density associated to this exotic component, usually called vacuum energy. The density ρ_{other} is associated to the other components of the Universe. The Friedmann's equation in this case is written as

$$H^2 = \frac{8\pi G}{3}\rho_{other} + \frac{8\pi G}{3}\rho_\Lambda - \frac{k\,c^2}{a^2}.$$

Defining $\Lambda \equiv 8\pi G\rho_\Lambda$, where Λ is called the cosmological constant, we obtain

$$H^2 = \frac{8\pi G}{3}\rho_{other} - \frac{k\,c^2}{a^2} + \frac{\Lambda}{3}. \tag{1.10}$$

Einstein was the first to propose the above result, considering in the equations of motion a constant term Λ conserving the Bianchi identity. His aim was to obtain an expression to describe a static Universe. However, after the results obtained by Hubble his claim was rejected.

Currently the cosmological constant is one of the most important models for understanding the dynamics of today's Universe [3].

1.6 Components and cosmological parameters

In this section we find the evolution of ρ_i for different components of the Universe. We define different cosmological parameters that were used in the development of this work.

1.6.1 Evolution of density as a function of the scale factor

- **Radiation:** Radiation has a pressure that varies as one third of the energy density. Its equation of state is $p_r = \frac{c^2}{3}\rho_r$ and its energy density is

$$\rho_r = \frac{\rho_r^0}{a^4}.$$

- **Non-relativistic matter:** The energy density of such particles is equal to its rest mass energy, which remains constant in time. Since that density is inversely proportional to the volume, the density of this component is

$$\rho_m = \frac{\rho_m^0}{a^3}.$$

  This is satisfied for $w = 0$ in the equation of state.

- **Curvature:** Defining $\rho_k^0 = -\frac{3k\,c^2}{8\pi G}$ the curvature term in the Friedmann's equation is rewritten as

$$\rho_k = \frac{\rho_k^0}{a^2}.$$

  This expression is known as curvature density.

- **Cosmological constant:** Recalling the definition of cosmological constant Λ in the above section, we can observed that

$$\rho_\Lambda = \frac{\Lambda}{8\pi G},$$

This expression is known as vacuum energy density or cosmological constant energy density. This energy density describes an accelerated expanding Universe, since $w_\Lambda = -1$ for the equation of state.

1.6.2 Parameters

Definition of parameters used in Cosmology.

- Hubble's constant: $H = \frac{\dot{a}}{a}$.
- Critical density: $\rho_c = \frac{3H^2}{8\pi G}$.
- Density parameter: $\Omega_i = \frac{\rho_i}{\rho_c}$.
- Deceleration parameter: $q = -\frac{\ddot{a}}{a\,H^2}$.

We identify the current value of the cosmological parameters with index zero. In addition we will use the evolution of the energy density of each component in the Universe, the Friedmann's equation is written as

$$H^2(a) = H_0^2\left(\frac{\Omega_m^0}{a^3} + \frac{\Omega_r^0}{a^4} + \frac{\Omega_k^0}{a^2} + \Omega_\Lambda^0\right). \tag{1.11}$$

The above expression can be rewritten as

$$H^2 = \frac{8\pi G}{3}\rho, \quad \text{including} \quad \rho_k \quad \text{and} \quad \rho_\Lambda,$$

where $\rho = \sum_i \rho_i$. Using the definition of the critical density we have

$$\sum_i \Omega_i(a) = 1.$$

If w in the equation of state is constant for all the Universe components, we have

$$\rho_i = \frac{\rho_i^0}{a^{3(1+w_i)}},$$

therefore the evolution of the i^{th} density parameter is given by

$$\Omega_i(a) = \frac{H_0^2}{H^2(a)}\frac{\Omega_i^0}{a^{3(1+w_i)}}.$$

Using the equation (1.11) and rearranging the terms, we can write the above expression as

$$\Omega_i(a) = \frac{\Omega_i^0}{\sum_j \Omega_j^0 \, a^{3(w_i - w_j)}}. \tag{1.12}$$

To finish, using the equation (1.9), the deceleration parameter can be written as

$$q(a) = \frac{1}{2} \sum_i \frac{\Omega_i^0 (1 + 3w_i)}{\sum_j \Omega_j^0 \, a^{3(w_i - w_j)}}. \tag{1.13}$$

Note that the evolution of the cosmological parameters depends on the extent of the density parameters of each component and the Hubble constant today.

1.7 Cosmological models

So far we have obtained the equations to describe the dynamics of the Universe. To find the solution and hence the evolution of the scale factor $a(t)$ it is necessary to consider the initial conditions of the cosmological parameters. In this section we explore some specific cases which can obtain the analytical expression of the evolution of the scale factor with time. Also we will describe some important models to explain the current dynamics of the Universe.

1.7.1 Milne Universe

This Universe is open, it is a model in which there is very little matter or radiation, where the value of $\Omega_k^0 \sim 1$. The Friedmann's equation can be written as

$$H^2 = \frac{H_0^2}{a^2}.$$

Therefore $\dot{a} = H_0$. The evolution with the scale factor is linear in time.

$$a(t) = H_0 \, t \quad \text{so} \quad H = \frac{1}{t}.$$

1.7.2 Einstein-de Sitter Universe

This is a Universe close to the critical density, hence flat, since that its content is mostly of matter, so $\Omega_m^0 \sim 1$. The Friedmann's equation can be written as

$$H^2 = \frac{H_0^2}{a^3}.$$

Therefore $\dot{a}^2 = H_0^2/a$. The evolution of the scale factor is given by

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \text{so} \quad H = \frac{2}{3t},$$

where $t_0 = \frac{2}{3H_0}$.

1.7.3 Universe dominated by radiation

In this case $\Omega_r^0 \sim 1$ therefore the value of the other omegas is close to zero. In the standard cosmological model, initially the Universe was dominated by radiation. The Friedmann's equation can be written as

$$H^2 = \frac{H_0^2}{a^4}.$$

Therefore $\dot{a} = H_0/a$. The evolution of the scale factor is given by

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad \text{so} \quad H = \frac{1}{2t},$$

where $t_0 = \frac{1}{2H_0}$.

1.7.4 de Sitter Universe

This Universe is dominated by a cosmological constant, $\Omega_\Lambda^0 \sim 1$. The Friedmann's equation can be written as $H = H_0$, then $\dot{a} = H_0\, a$. The evolution of the scale factor is given by

$$a(t) = \mathrm{e}^{H_0(t-t_0)}.$$

This is the only case where H is constant.

1.7.5 Quintessence

We can explain the Universe current observations postulating a kind of exotic energy called dark energy. The Quintessence is a model to describe this type of energy, where the parameter $w(a)$ of the equation of state $p = c^2 w\rho$ is not always constant. It can evolve with the scale factor a. The cosmological constant model is contained in the Quintessence models with $w = -1$.

Once Quintessence models attempt to explain an accelerated expanding Universe, therefore $w < -1/3$, see equation (1.8).

In this work we consider the following parameterised Quintessence model where w is parameterised as

$$w(a) = w_0 + (a - a_0)w_a,$$

where w_a, w_0, a_0, are the parameters. This parameterization is called Chevallier-Polarski-Linder (CPL) [33]. For configuration $w_a = 0$, $w_0 = -1$, we return to the model of a Universe with cosmological constant. For Quintessence models with scalar field, see the appendix A and [4, 34].

The standard cosmological model accepted today is the ΛCDM model. It describes a Universe dominated by vaccum energy (i.e., more than two third of the

Universe is composed for this kind of energy) and cold dark matter, hence the acronym.

1.8 Distances and times

The notion of distance and time is different when we are looking at an expanding Universe. These magnitudes depend particularly on the cosmological model itself. For example in an Einstein de-Sitter Universe the Hubble factor is related to its age via $H = 2/(3t)$, hence the today's Universe age is related to the Hubble's constant via $t_0 = 2/(3H_0)$. So in this section we define different concepts to explain as it is possible to measure distances and times in an expanding Universe.

1.8.1 The comoving distance

It is the distance along the line of sight between a distant emitter and us. If we integrate the metric in a flat Universe we obtain the following value for the comoving distance

$$\chi = \int_t^{t_0} \frac{c\,\mathrm{d}t}{a} = \int_a^1 \frac{c\,\mathrm{d}a'}{a'\dot{a}'}. \tag{1.14}$$

1.8.2 Proper distance

The proper distance is the physical distance measured by us, it is the product between the scale factor and the comoving distance, in a flat Universe

$$d_{prop}^{\mathrm{flat}} = a \int_t^{t_0} \frac{c\,\mathrm{d}t}{a} = a \int_a^1 \frac{c\,\mathrm{d}a'}{a'\dot{a}'}.$$

1.8.3 Angular distance

A classic way to determine distance in astronomy is to measure the angle θ subtended by an object of known physical size l. If the angle subtended is small, the distance to that object is given by

$$d_A = \frac{l}{\theta}.$$

To compute the angular distance in an expanding Universe, we first note that the comoving size of the object is l/a. So the angle subtended is $\theta = (l/a)/\chi(a)$. Comparing with the above expression, we see that the angular distance in a flat Universe is

$$d_A^{\mathrm{flat}}(a) = a\chi(a). \tag{1.15}$$

For a Universe with a general geometry

$$d_A(a) = a\,S_k(\chi(a)).$$

1.8.4 Luminosity distance

Another way of inferring distances in astronomy is to measure the flux from an object of known luminosity. Recall that the observed flux F at a distance d from a source of known luminosity L is

$$F = \frac{L}{4\pi d^2}. \tag{1.16}$$

To generalize this result to an expanding Universe, we work on the comoving grid with the source centered at the origin. The flux we observe is

$$F = \frac{L(\chi)}{4\pi \chi^2(a)},$$

where $L(\chi)$ is the luminosity through a comoving spherical shell with radius $\chi(a)$. Also we have considered that all photons are emitted with the same energy. Then $L(\chi)$ is this energy multiplied by the number of photons passing through a spherical shell per unit time. In a fixed time interval, the number of photons crossing a shell will be smaller today than at emission, smaller by a factor of a. Similary, the energy of the photons will be smaller today than at emission, due to the expansion. Therefore, the energy per unit time passing through a comoving shell at a distance $\chi(a)$ from the source will be a factor of a^2 smaller than the luminosity at the source. The flux we observe therefore will be

$$F = \frac{La^2}{a\pi \chi^2(a)}.$$

Comparing the above expression whit (1.16) we define the luminosity distance in a flat Universe as

$$d_L^{\rm flat}(a) \equiv \frac{\chi(a)}{a}.$$

For a Universe with a general geometry

$$d_L(a) = \frac{S_k(\chi(a))}{a}.$$

The relation between the luminosity distance and the angular distance is

$$\frac{d_L}{d_A} = \frac{1}{a^2}.$$

1.8.5 Times

- **Cosmic time.** Cosmic time t is defined as the time measured by a fundamental observer who reads time on a standard clock.

$$t = \int_0^t \mathrm{d}t' = \int_0^a \frac{\mathrm{d}a'}{\dot{a}'}. \tag{1.17}$$

- **Conformal time.** To simplify the definition of comoving distance we define a time scale different to the cosmic time. These time intervals are given by

$$\mathrm{d}t_{conf} = \mathrm{d}\eta = \frac{\mathrm{d}t}{a}.$$

At any epoch, the conformal time has value

$$\eta = \int_0^a \frac{\mathrm{d}t}{a} = \int_0^a \frac{\mathrm{d}a'}{a'\dot{a}'}. \tag{1.18}$$

The comoving distance as conformal time function is

$$\chi(\eta) = c(\eta_0 - \eta).$$

1.8.6 Age of the Universe

An important quantity to measure in cosmology is the age of the Universe. This measure depends on the cosmological model. If we consider Friedmann's equation (1.11) we can derive a general expression for the age of the Universe. Then

$$\left(\frac{\dot{a}}{a}\right) = \left(\frac{\mathrm{d}a}{\mathrm{d}t}\frac{1}{a}\right) = H_0\sqrt{\frac{\Omega_m^0}{a^3} + \frac{\Omega_r^0}{a^4} + \frac{\Omega_k^0}{a^2} + \Omega_\Lambda^0} = H_0 E(a),$$

integrating it yields

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{\mathrm{d}a}{a\,E(a)}.$$

1.9 Horizons

- **Particle horizon.** At any epoch t, the particle horizon is defined to be the maximum distance over which causal communication could have taken place by that epoch. In other words, this is the distance a light signal could have travelled from the origin of the Big Bang at $t = 0$ by the epoch t.

The definition of the particle horizon $r_H(t)$ at the cosmic epoch t is

$$r_H(t) = a\int_0^t \frac{c\,\mathrm{d}t}{a} = a\int_0^a \frac{c\,\mathrm{d}a'}{a'\dot{a}'}.$$

- **Event horizon.** The event horizon is the greatest distance an object can have at a particular cosmic epoch, if it is ever to be observable by an observer who observes the Universe at a cosmic time t_1.

Consider a light ray emitted at time t_1 which arrives at the observer at time t. Then, the comoving distance traversed by the light ray is

$$\chi_{trav}(t) = \int_{t_1}^{t} \frac{c\mathrm{d}t}{a}.$$

The question is whether or not this integral converges as $t \to \infty$ to open models or as $t \to t_{max}$ for collapsing closed models. The definition of the event horizon is therefore

$$r_E = a\int_{t_1}^{t_{max}} \frac{c\,\mathrm{d}t}{a(t)} = a\int_{a_1}^{a_{max}} \frac{c\,\mathrm{d}a}{a\dot{a}}.$$

- **The Hubble sphere.** The Hubble sphere is the distance where all points of the Universe are expanding away from us at exactly the speed of light, in other words it is defined as the limit of all points which we can be in casual contact now.

  Remember the Hubble's law is given by expression (1.1), so if the speed is c, then

  $$c = H\,d_h \quad \text{so} \quad d_h = \frac{c}{H} \quad \text{or} \quad \chi_h = \frac{c}{a\,H}, \quad \text{for any redshift,}$$

  where d_h is the Hubble's radius and χ_h the comoving Hubble's radius, now the Hubble's radius is $d_h = \frac{c}{H_0}$. The Hubble sphere changes as the Universe changes its expansion history and it is defined at a given time.

1.10 Redshift

If we considered the travel of a photon with a specific frequency that was emitted in r_0 and was observed in r_1 the comoving distance is given by

$$\chi = \int_{t_0}^{t_1} \frac{c\mathrm{d}t}{a(t)}.$$

Moreover, if we consider another photon beginning its travel at r_0 but emitted Δt later, such that the frequency of the two photons is the same (the phase is the same $+\pi$), then as the comoving distance to r_1 is the same as in the previous photon we obtain

$$\int_{t_0+\Delta t_0}^{t_1+\Delta t_1} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{t_1} \frac{\mathrm{d}t}{a(t)}.$$

Rearranging the above expression we obtain

$$\int_{t_0}^{t_0+\Delta t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_1}^{t_1+\Delta t_1} \frac{\mathrm{d}t}{a(t)},$$

so if we assume that the scale factor is invariant during these small interval of time we can factor it out of the integral and obtain

$$\frac{\Delta t_0}{a(t_0)} = \frac{\Delta t_1}{a(t_1)},$$

in terms of the light wavelength, we have

$$\frac{\lambda_1}{\lambda_0} = \frac{a(t_1)}{a(t_0)},$$

recalling that the redshift z is defined as

$$1 + z = \frac{\lambda_1}{\lambda_0} \quad \text{so} \quad 1 + z = \frac{a(t_1)}{a(t_0)}.$$

If the photon is observed today, the scale factor $a(t_1) = 1$. The photon is emitted at any time t, so the above equation can be written as

$$1 + z = \frac{1}{a(t)}. \tag{1.19}$$

Usually the negative redshift is called blueshift. On the other hand, we can note that the cosmic time, the scale factor and the redshift are related, see (1.17) and (1.19), then we can write the cosmic time in terms of the scale factor or redshift.

Chapter 2

Linear theory of the structure formation

The inhomogeneity present in the Universe today as galaxies, clusters and superclusters, induces us to think that in the primordial Universe existed small inhomogeneity which evolved until today to big structures. The inflation theory provides a successful mechanism for generating this perturbation, see [21, 13]. In this chapter we will study the matter fluctuations evolution responsible of structures formation in the linear regime. Apart from this, we analyze the growth of fluctuations in the context of different cosmological models.

2.1 Growth of fluctuations in an expanding Universe

The phenomenon of growth of fluctuations we are going to study here is encoded in the case where gravity and non-relativistic matter dominate our Universe. In this case all perturbations are on the horizon H^{-1} so that Newtonian theory can be used for describing them.

Initially we consider the fluid dynamics equations in a gravitational field, i.e., the continuity equation, the equation of motion or Navier-Stokes and the Poisson equation.

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, &(2.1)\\
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} &= -\frac{\nabla p}{\rho} - \nabla \phi, &(2.2)\\
\nabla^2 \phi &= 4\pi G \rho, &(2.3)
\end{aligned}
$$

where ρ is the fluid density, p is the pressure, $\mathbf{v}$ is the velocity distribution and ϕ is the gravitational potential which depends on the density ρ. All of these quantities dependent on the position in a fixed point in space, i.e. this is an Eulerian description of the system. For this analysis it is better to work with a Lagrangian description of the system. For this purpose we perform the following transformation

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)\,. \tag{2.4}$$

Using the above expression in the fluid dynamics equation in a gravitational field, we obtain

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\rho\nabla\cdot\mathbf{v}, && (2.5)\\ \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\frac{\nabla p}{\rho} - \nabla\phi, && (2.6)\\ \nabla^2\phi &= 4\pi G\rho. && (2.7) \end{aligned}$$

Now we apply these equations in a dynamic Universe. We assume that the Universe satisfies the cosmological principle, so it is convenient to use a comoving frame (i.e. the flow is flowing with the expansion of the Universe). In such a frame the velocity will follow the Hubble's law $\mathbf{v} = H\,\mathbf{r}$. Substituting this into equation (2.5) we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H\rho,$$

which is simply the fluid equation for the case of a Universe dominated by the matter.

Initially we derive the solutions for the velocity $\mathbf{v}_0$, density ρ_0, pressure p_0 and gravitational potential ϕ_0 in an unperturbed medium which satisfies equations (2.5), (2.6) and (2.7). Notice that we will use the subscripts 0 to refer to the properties of the unperturbed medium only in this calculation

$$\begin{aligned} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} &= -\rho_0\nabla\cdot\mathbf{v}_0, && (2.8)\\ \frac{\mathrm{d}\mathbf{v}_0}{\mathrm{d}t} &= -\frac{\nabla p_0}{\rho_0} - \nabla\phi_0, && (2.9)\\ \nabla^2\phi_0 &= 4\pi G\rho_0. && (2.10) \end{aligned}$$

Now we write the equations including first-order perturbations

$$\mathbf{v} = \mathbf{v}_0 + \Delta\mathbf{v}, \quad \rho = \rho_0 + \Delta\rho, \quad p = p_0 + \Delta p, \quad \phi = \phi_0 + \Delta\phi.$$

For this purpose, initially we expand up to the first order equation (2.5)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\rho_0 + \Delta\rho) &= -(\rho_0 + \Delta\rho)\nabla\cdot(\mathbf{v}_0 + \Delta\mathbf{v}),\\ &= -\rho_0\nabla\cdot\mathbf{v}_0 - \rho_0\nabla\cdot\Delta\mathbf{v} - \Delta\rho\nabla\cdot\mathbf{v}_0 - \Delta\rho\nabla\cdot\Delta\mathbf{v}, \quad (2.11) \end{aligned}$$

given that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\rho}{\rho_0}\right) = \frac{\mathrm{d}\,(\Delta\rho)}{\mathrm{d}t}\frac{1}{\rho_0} - \frac{1}{\rho_0^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}t}\Delta\rho,$$

replacing $\frac{\mathrm{d}(\Delta\rho)}{\mathrm{d}t}$ of equation (2.11) and $\frac{\mathrm{d}\rho_0}{\mathrm{d}t}$ of equation (2.8) in to the above expression, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\rho}{\rho_0}\right) = \frac{1}{\rho_0}\left(-\Delta\rho\nabla\cdot\mathbf{v}_0 - \rho_0\nabla\cdot\Delta\mathbf{v} - \Delta\rho\nabla\cdot\Delta\mathbf{v}\right) - \frac{1}{\rho_0^2}\Delta\rho\left(-\rho_0\nabla\cdot\mathbf{v}_0\right),$$

simplifying the above expression and defining the overdensity $\delta \equiv \frac{\Delta\rho}{\rho_0}$, we obtain

$$\dot{\delta} = -\nabla\cdot\Delta\mathbf{v}. \tag{2.12}$$

Next, we expand $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ to first order using (2.4)

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial\mathbf{v}_0}{\partial t} + \frac{\partial\Delta\mathbf{v}}{\partial t} + \left(\mathbf{v}_0\cdot\nabla\right)\Delta\mathbf{v} + \left(\mathbf{v}_0\cdot\nabla\right)\mathbf{v}_0 + \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v}_0 + \left(\Delta\mathbf{v}\cdot\nabla\right)\Delta\mathbf{v}.$$

If we ignore second order quantities, this can be rewritten as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\partial\mathbf{v}_0}{\partial t} + \frac{\mathrm{d}\Delta\mathbf{v}}{\mathrm{d}t} + \left(\mathbf{v}_0\cdot\nabla\right)\mathbf{v}_0 + \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v}_0,$$

expanding to first order the equation (2.6), replacing $\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ in the above expression and using the equation (2.2) in the unperturbed medium we obtain

$$\frac{\mathrm{d}\Delta\mathbf{v}}{\mathrm{d}t} + \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v}_0 = -\frac{\nabla\left(p_0+\Delta p\right)}{\rho_0\left(1+\delta\right)} - \nabla\left(\Delta\phi\right) + \frac{\nabla p_0}{\rho_0},$$

Assuming that the initial state satisfies the cosmological principle so that $\nabla p_0 = 0$ and $\nabla\rho_0 = 0$, then

$$\frac{\mathrm{d}\Delta\mathbf{v}}{\mathrm{d}t} + \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v}_0 = -\frac{\nabla\left(\Delta p\right)}{\rho_0\left(1+\delta\right)} - \nabla\left(\Delta\phi\right),$$

if we multiply $(1-\delta)$ to $\frac{\nabla(\Delta p)}{\rho_0(1+\delta)}$ and ignoring second order quantities, we have

$$\frac{\mathrm{d}\Delta\mathbf{v}}{\mathrm{d}t} + \left(\Delta\mathbf{v}\cdot\nabla\right)\mathbf{v}_0 = -\frac{\nabla\left(\Delta p\right)}{\rho_0} - \nabla\left(\Delta\phi\right). \tag{2.13}$$

Finally, subtracting (2.10) of (2.7) we find

$$\nabla^2\left(\Delta\phi\right) = 4\pi G\rho_0\delta \tag{2.14}$$

Equations (2.13), (2.12) and (2.14) are the key differential equations in the present analysis.

Suppose that now we have a particle moving in the comoving frame. We can express the velocity as a general Hubble flow plus a peculiar velocity in a comoving frame. Hence we can write:

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}},$$

if we define the peculiar velocity in the comoving frame as $\mathbf{u} \equiv \dot{\mathbf{x}}$, the above expression is written as

$$\mathbf{v} = H\mathbf{r} + a\mathbf{u},$$

so we can compare the above equation with the expansion to first order of $\mathbf{v}$, then

$$\mathbf{v}_0 = H\mathbf{r}, \quad \Delta\mathbf{v} = a\mathbf{u}, \tag{2.15}$$

replacing the above expression in the equation (2.13) we obtain

$$\frac{\mathrm{d}\,(a\mathbf{u})}{\mathrm{d}t} + (a\mathbf{u}\cdot\nabla)\,(H\mathbf{r}) = -\frac{\nabla\,(\Delta p)}{\rho_0} - \nabla\,(\Delta\phi) \tag{2.16}$$

it is convenient to write down all the derivatives in the comoving coordinates and work in the comoving frame, so that $\nabla \to \frac{1}{a}\nabla_c$, then

$$(a\,\mathbf{u}\cdot\nabla)\,H\,\mathbf{r} = (\mathbf{u}\cdot\nabla_c)\,H\,a\,\mathbf{x} = a\,H\,\mathbf{u} = \dot{a}\mathbf{u},$$

using this result in equation (2.16) we have

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + 2\left(\frac{\dot{a}}{a}\right)\mathbf{u} = -\frac{1}{a^2}\frac{1}{\rho_0}\nabla_c\,(\Delta p) - \frac{1}{a^2}\nabla_c\,(\Delta\phi)\,.$$

Now, let us consider an adiabatic perturbation in which both perturbations in pressure and density are related to the adiabatic sound speed c_s^2 by $c_s^2 = \frac{\partial p}{\partial \rho}$. Hence if we take the gradient of the above expression as we introduce the sound speed we obtain:

$$\nabla_c\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + 2\left(\frac{\dot{a}}{a}\right)\nabla_c\cdot\mathbf{u} = -\frac{1}{a^2}\nabla_c\cdot\left(\frac{c_s^2}{\rho_0}\nabla_c\,(\Delta p)\right) - \frac{1}{a^2}\nabla_c^2\,(\Delta\phi)\,,$$

given that $\nabla\rho_0 = 0$ and using (2.14) we rewrite the previous equation as

$$\nabla_c\cdot\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + 2\left(\frac{\dot{a}}{a}\right)\nabla_c\cdot\mathbf{u} = -\frac{c_s^2}{a^2}\nabla_c^2\delta - 4\pi G\rho_0\delta.$$

Using (2.12) and (2.15) in the above expression, we find the equation for the overdensity

$$\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} = \frac{c_s^2}{a^2}\nabla_c^2\delta + 4\pi G\rho_0\delta, \tag{2.17}$$

we can translate this equation into k space by seeking a solution of the type $\delta = D(t)\exp\left(i\mathbf{k}_c\cdot\mathbf{x}\right)$, where $\mathbf{k}_c$ is the comoving wave-number and relates to the real wave-number via $\mathbf{k}_c = a\,\mathbf{k}$ and $D(t)$ is the linear growth factor. The equation become then

$$\ddot{D} + 2\left(\frac{\dot{a}}{a}\right)\dot{D} = \left(4\pi G\rho_0 - k^2c_s^2\right)D. \tag{2.18}$$

The previous equation describes the growth of perturbations.

In addition to the linear growth factor $D(t)$, we can define the growth function $f(t)$ as

$$f(t) \equiv \frac{\mathrm{d} \ln D}{\mathrm{d} \ln a}. \tag{2.19}$$

Lahav in [35] proposes that the growth function at $z = 0$ in the range $-5 \leq \lambda_0 \leq 5$, $0,003 \leq \Omega_m^0 \leq 2$ can be approximated as

$$f(z=0) \approx (\Omega_m^0)^{0,6} + \frac{1}{70}\lambda_0 \left(1 + \frac{1}{2}\Omega_m^0\right),$$

where λ_0 is related to the cosmological constant $\lambda_0 = \Lambda/(3H_0^2)$. Since the dynamics at $z = 0$ depends mainly on the present matter density, it suggest that f depends on $\Omega_m(z)$ at any epoch. Indeed, $f(z)$ is well approximated for any z, λ_0 and Ω_m^0 by

$$f(z) \approx (\Omega_m)^{0,6}. \tag{2.20}$$

For a flat Universe an even better approximation is provided by analogy with $f(z = 0)$, $f(z) = (\Omega_m)^0, 6 + \frac{1}{70}\left[1 - \frac{1}{2}\Omega_m(1 + \Omega_m)\right]$. The approximation for the growth function depends of the cosmological model that we are employing. In this work, we compare the approximation 2.20 with the exact definition of f, see Figure 2.1.

2.2 Static Universe, Jeans' Instability

Initially we study the case of a static medium, so that $\dot{a} = 0$. Then, for the wave solution of the form $\delta \propto \exp i\left(\mathbf{k}_c \cdot \mathbf{x} - \omega t\right)$, the dispersion relation is

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho.$$

This relation was derived by Jeans in 1902 [27]. The dispersion relation describes oscillations or instabilities depending on the sign of the right-hand side of the equation.

- If $c_s^2 k^2 > 4\pi G\rho$, the right-hand side is positive and the perturbations oscillate. This means that any perturbation in such a medium would produce a wave solution. In other words the pressure is sufficient to provide support to collapsing regions. The limit is given by the so called Jeans' wavelength

$$\lambda_J = \frac{2\pi}{k_J} = c_s \left(\frac{\pi}{G\rho}\right)^{1/2}. \tag{2.21}$$

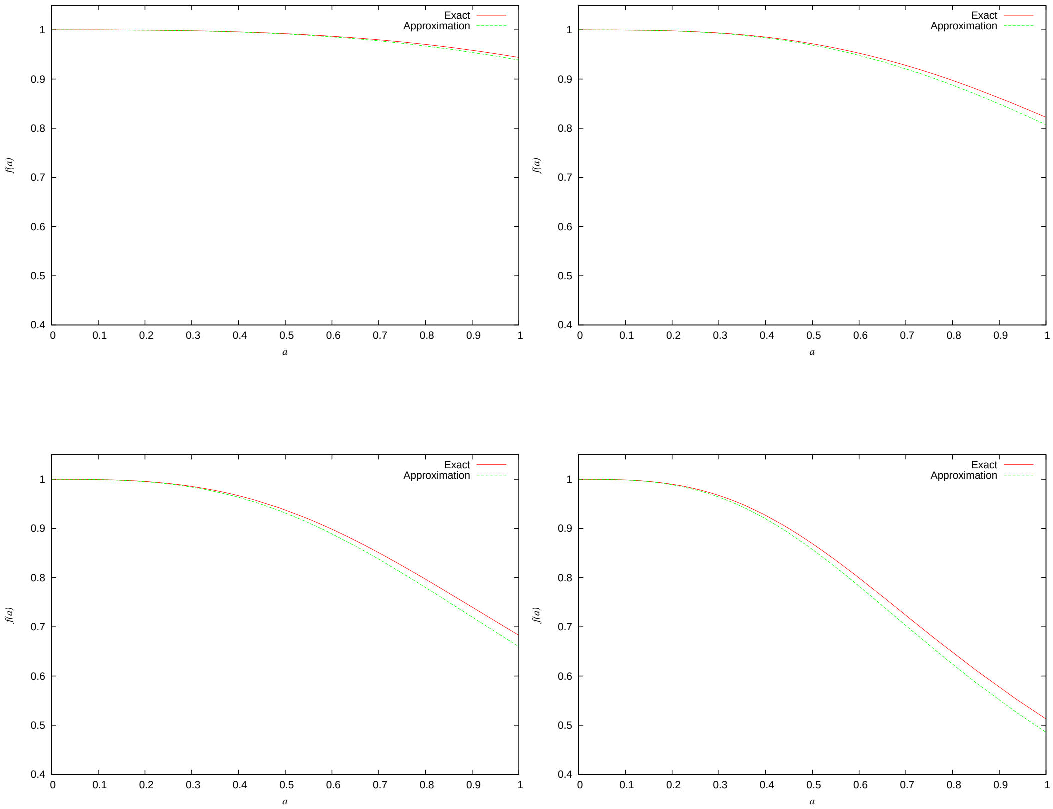


Figure 2.1: Comparative graphics of the growth function f in a flat Universe with cosmological constant for values different $\Omega_\Lambda^0 = 1 - \Omega_m^0$. *Left top*: $\Omega_\Lambda^0 = 0,1$, $\Omega_m^0 = 0,9$; *right top*: $\Omega_\Lambda^0 = 0,3$, $\Omega_m^0 = 0,7$; *left bottom*: $\Omega_\Lambda^0 = 0,5$, $\Omega_m^0 = 0,5$; *right bottom*: $\Omega_\Lambda^0 = 0,7$, $\Omega_m^0 = 0,3$.

- If $c_s^2 k^2 < 4\pi G\rho$ the right-hand side is negative, leading to an imaginary part of ω, corresponding to unstable modes, where the pressure cannot support the collapsing or expanding regions hence we have growing and decaying modes. Therefore in this case we can define

$$\Gamma = \pm i\omega = \pm \left[4\pi G\rho \left(1 - \frac{\lambda_J^2}{\lambda^2} \right) \right]^{1/2},$$

so we can write the solution as

$$\delta = \delta_0 \exp(\Gamma t - i(\mathbf{k} \cdot \mathbf{r})).$$

The positive and negative solutions correspond to exponentially growing and exponentially decaying modes, respectively. For wavelengths much greater than the Jeans' wavelength, $\lambda \gg \lambda_J$, the growth rate Γ becomes $(4\pi G\rho)^{1/2}$.

In this case, the characteristic growth time for the instability is

$$\tau = \Gamma^{-1} = (4\pi G\rho)^{-1/2}.$$

This is called Jeans' instability, τ is interpreted as timescale for collapse.

2.3 Dynamic Universe, expanding medium

In the section 2.2 we studied the perturbations in a static medium, we observed that in this case the perturbations can oscillate or correspond to instability modes. In this section we study the behavior of linear growth factor in different cosmological models. Using the definition of the Jeans' wavelength the equation (2.18) can be written as

$$\ddot{D} + 2\left(\frac{\dot{a}}{a}\right)\dot{D} = 4\pi G\rho\, D\left[1 - \frac{\lambda_J^2}{\lambda^2}\right].$$

If we rewrite the above expression in the long wavelength limit $\lambda \gg \lambda_J$, we have the equation that will be analyzed in this section

$$\ddot{D} + 2\left(\frac{\dot{a}}{a}\right)\dot{D} = 4\pi G\rho\, D. \tag{2.22}$$

We consider in this work that the linear growth factor is normalized today $D(a_0) = 1$ and in a high redshift $D(a \to 0) = 0$.

- **The Einstein-de Sitter Universe**

  In this model $\Omega_m^0 = 1$, the density parameter in the other components is zero. The evolution of scale factor is $a(t) = \left(\frac{t}{t_0}\right)^{2/3}$, then $H = \frac{2}{3t}$, so that the equation for linear growth factor is given by:

  $$\ddot{D} + \frac{4}{3t}\dot{D} - \frac{2}{3t^2}D = 0,$$

  its general solution is

  $$D(t) = A\,t^{-1} + B\,t^{2/3},$$

  where A and B are constant.

  The first term of the solution is a decaying mode and the second term of the solution is a growing mode. Now we have structures as galaxies, clusters and superclusters, in a Universe dominated for the matter, we only considered the growing mode in the solution, then $D \propto t^{2/3} \propto a$.

- **The Milne Universe**

  In this model $\Omega_k^0 = 1$. The evolution of scale factor is $a(t) = H_0\, t$, then $H = \frac{1}{t}$, therefore the equation is given by:

  $$\ddot{D} + \frac{2}{t}\dot{D} = 0,$$

  its general solution is

  $$D(t) = A\, t^{-1} + B,$$

  where A and B are constant. We find two solutions but one is decaying and the second is static. In an empty Universe or nearly empty Universe, perturbations tend not to grow.

- **The de Sitter Universe**

  In this case we only have a cosmological constant, $\Omega_\Lambda^0 = 1$. The evolution of scale factor is $a(t) = \mathrm{e}^{H_0(t-t_0)}$, then $H = H_0$, therefore the equation is given by:

  $$\ddot{D} + 2H_0\dot{D} = 0,$$

  its general solution is

  $$D(t) = A\mathrm{e}^{-2\, H_0 t} + B,$$

  where A and B are constant. In a Universe dominated by a cosmological constant we only have a decaying mode and a constant mode.

- **Universe dominated by radiation**

  In this model $\Omega_r^0 = 1$. The evolution of scale factor is $a(t) = \left(\frac{t}{t_0}\right)^{1/2}$, then $H = \frac{1}{2t}$, therefore the equation is given by:

  $$\ddot{D} + \frac{1}{t}\dot{D} = 0,$$

  its general solution is

  $$D(t) = A + B\ln(t),$$

  where A and B are constant. The first solution is constant and the second solution is logarithmically growing with t.

  To generalize this result to other systems, we have to do a relativistic study of this model, in this work we are not performed this development, however for more details see [23] .

In Figure 2.2 we can observe the graphs for the linear growth factor and the growth function of the above cosmological models.

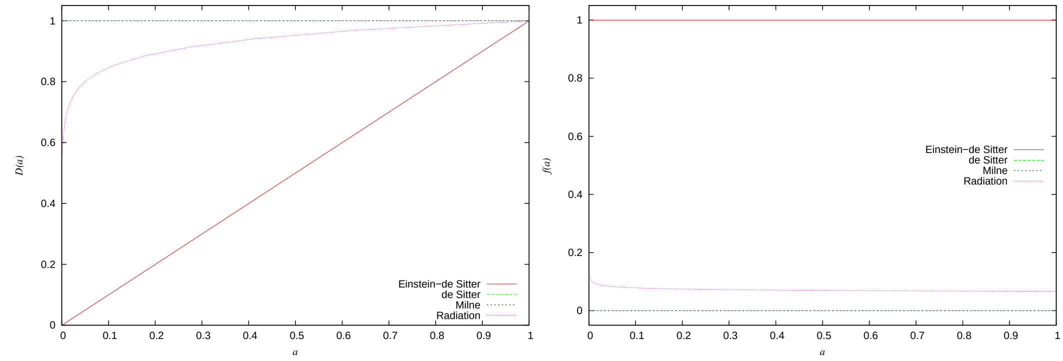


Figure 2.2: Graphs of the linear growth factor $D(a)$ (on the left-hand side) and growth function $f(a)$ (on the right-hand side) in a Einstein-de Sitter Universe, a de Sitter Universe, a Milne Universe and a Universe dominated by radiation, respectively.

2.3.1 Solution for a general model

In this work we need to find a solution for the linear growth factor and the growth function for a general model. In order to find a solution to this problem we have to use numerical methods, since there is not an analytic solution. To that purpose, we will rewrite the equation (2.22). Initially we make the following substitution $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{a}\,\frac{\mathrm{d}}{\mathrm{d}a}$, then $\dot{D} = \dot{a}\,\frac{\mathrm{d}D}{\mathrm{d}a}$, $\ddot{D} = \dot{a}^2\,\frac{\mathrm{d}^2D}{\mathrm{d}a^2} + \frac{\mathrm{d}D}{\mathrm{d}a}\,\ddot{a}$, therefore

$$\dot{a}^2\frac{\mathrm{d}^2D}{\mathrm{d}a^2} + \left(\ddot{a} + 2\left(\frac{\dot{a}^2}{a}\right)\right)\frac{\mathrm{d}D}{\mathrm{d}a} = 4\pi G\rho_0 D.$$

Now we define $y = \ln a$ and $\frac{\mathrm{d}D}{\mathrm{d}y} \equiv D'$ we obtain $\frac{\mathrm{d}D}{\mathrm{d}a} = \frac{1}{a}\,D'$, $\frac{\mathrm{d}^2D}{\mathrm{d}a^2} = \frac{1}{a^2}\,(D'' - D')$, in addition, using the definition of the acceleration parameter, $q = -\frac{\ddot{a}}{H^2\,a}$, we have

$$D'' + (1-q)\,D' = \frac{4\pi G\rho}{H^2}D,$$

once the perturbations are of matter type $\rho_0 = \frac{3H_0^2}{8\pi G}\frac{\Omega_m^0}{a^3}$, then the above expression can be rewritten as

$$D''(y) + (1-q(y))\,D'(y) = \frac{3}{2}\frac{H_0^2\Omega_m^0}{a^3H(y)^2}D(y). \tag{2.23}$$

The above expression allows us to study the linear growth factor and the growth function for any cosmological model. In Figure 2.3 we show two numerical experiments allowing to study the behavior of $D(a)$ and $f(a)$ by varying the density parameter of matter Ω_m^0 in a cosmological model with cosmological constant and the state equation w_0 in a cosmological model with $\Omega_m^0 = 0,3$ and $\Omega_\Lambda^0 = 0,7$. In both models we consider a flat Universe without radiation.

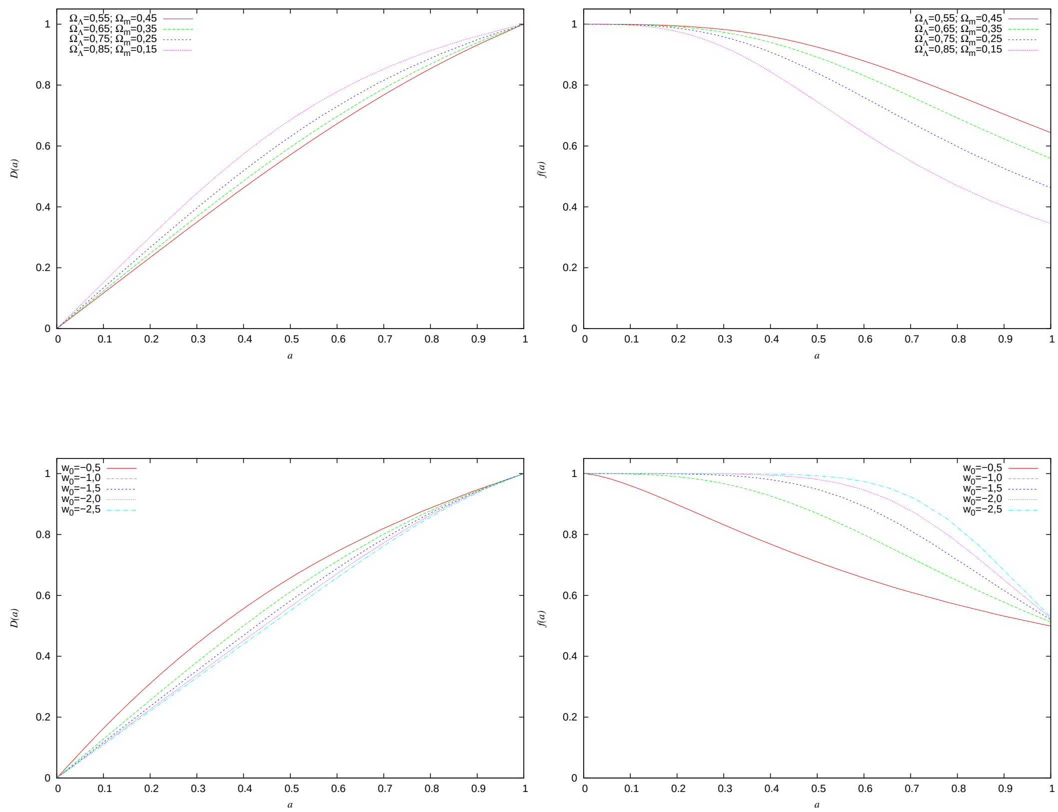


Figure 2.3: Numerical solution of the equation (2.23). *Top*: linear growth factor and growth function for different values of Ω_m^0 and Ω_Λ^0, given that the Universe is considered flat. These parameters satisfy the following relationship $\Omega_m^0 + \Omega_\Lambda^0 = 1$. *Bottom*: linear growth factor and growth function for different values of w_0 in a parameterised Quintessence model $w(a) = w_0 + (a - a_0)w_a$, where $w_a = 0$ and $a_0 = 0$.

Chapter 3

General theoretical framework

One of the key pieces to understand modern Cosmology is the study of the anisotropies in the CMB. First discovered by the DMR experiment on the COBE satellite in the early 90 [6]. Observations of CMB anisotropies have advanced to get to the data of WMAP satellite (Wilkinson Microwave Anisotropy Probe). In [7] we find a description of the data on the seventh year of WMAP (7WMAP).

While most of the CMB fluctuations seen by WMAP and other experiments were generated at $z \sim 1000$, in low redshift ($z < 20$) there are more fluctuations [32], which are called secondary anisotropies. In this chapter the goal is to understand the Integrated Sachs-Wolfe effect (ISW) which is manifested in the secondary anisotropy.

3.1 The two-point correlation function

To study the statistical distribution of galaxies on a large scale it is necessary to define the correlation function of two-points, which describes the excess probability of finding a galaxy at distance r from a galaxy selected at random over that expected in an uniform, random distribution. The correlation function of two-points $\xi(r)$ describes the number of galaxies in a volume $\mathrm{d}V$ at distance r from any galaxy

$$\mathrm{d}N(r) = N_0[1 + \xi(r)]\mathrm{d}V,$$

where N_0 is an amplitude related to the observations. Due to the statistical isotropy, the function $\xi(r)$ does not depend on the direction of $\mathbf{r}$. In addition, this function can be written in terms of the probability of finding pairs of galaxies separated by a ditance r

$$\mathrm{d}N_{pair} = N_0^2[1 + \xi(r)]\mathrm{d}V_1\,\mathrm{d}V_2.$$

The above expression can be written using the matter density

$$\mathrm{d}N_{pair}(r) = \rho(x)\mathrm{d}V_1\,\rho(x + r)\mathrm{d}V_2,$$

since $\rho = \rho_0[1+\delta(r)]$, we have

$$\mathrm{d}N_{pair}(r) = \rho_0^2[1+\delta(x)][1+\delta(x+r)]\mathrm{d}V_1\mathrm{d}V_2.$$

Taking the average over a large number of volume elements and considering that fluctuations are Gaussian (the mean of δ is zero), we obtain

$$\mathrm{d}N_{pair}(r) = \rho_0^2[1+\langle\delta(x)\delta(x+r)\rangle]\mathrm{d}V_1\mathrm{d}V_2.$$

This calculation shows explicitly the relationship between fluctuations $\delta(r)$ at different scales of r and the correlation function of two-points for galaxies

$$\xi(r) = \langle\delta(x)\delta(x+r)\rangle. \tag{3.1}$$

3.2 Power spectrum

The most natural way to study the distribution of fluctuations is in Fourier space. The Fourier transform of $\delta(\mathbf{r})$ is defined as

$$\begin{aligned}\delta(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3k,\\ \delta_{\mathbf{k}} &= \int \delta(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}\mathrm{d}^3r.\end{aligned}$$

Using the Parseval's theorem that relates the integral of the squares of $\delta(\mathbf{r})$ and its Fourier transform $\delta_{\mathbf{k}}$ we obtain

$$\int \delta^2(\mathbf{r})\mathrm{d}^3x = \frac{1}{(2\pi)^3}\int |\delta_{\mathbf{k}}|^2\mathrm{d}k.$$

The amount on the left-hand side of the above equation is the average of the square of the amplitude of the fluctuations, and $|\delta_{\mathbf{k}}|^2$ is the *power spectrum* of fluctuations, which is often written as $P(k)$. The dimensionless power spectrum is defined as $\Delta^2(k) = \frac{k^3}{2\pi^2}P(k)$. Since the correlation function of two points is spherically symmetric, we have that $\mathrm{d}^3k = 4\pi k^2\mathrm{d}k$, then

$$\langle\delta^2\rangle = \frac{1}{2\pi^2}\int P(k)k^2\mathrm{d}k.$$

The goal now is to relate the power spectrum and the correlation function of the two-points. To accomplish this, it is better to expand $\delta(\mathbf{x})$ in Fourier series

$$\delta(\mathbf{x}) = \sum_{\mathbf{k}} \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

Taking the average of the product of $\delta(\mathbf{x})$ and $\delta(\mathbf{x}+\mathbf{r})$, we obtain

$$\xi(\mathbf{r}) = \langle \sum_{\mathbf{k}} \sum_{\mathbf{k'}} \delta_{\mathbf{k}} \delta^*_{\mathbf{k'}} e^{-i(\mathbf{k}-\mathbf{k'})\cdot\mathbf{x}} e^{i\mathbf{k'}\cdot\mathbf{r}} \rangle,$$

the cross-terms in this sum vanish except for $\mathbf{k} = \mathbf{k'}$. Therefore,

$$\xi(\mathbf{r}) = \sum_{\mathbf{k}} |\delta_{\mathbf{k}}|^2 \, e^{i\mathbf{k}\cdot\mathbf{r}}.$$

Now converting the sum in to an integral, the expression above becomes

$$\xi(\mathbf{r}) = \int |\delta_{\mathbf{k}}|^2 \, e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 k.$$

Noting that the function $\xi(r)$ is a real function, therefore it is interesting only the integral of the real part of $e^{i\mathbf{k}\cdot\mathbf{r}}$, i. e., the integral over $\cos(\mathbf{k}\cdot\mathbf{r}) = \cos(k\,r\cos\theta)$, considering the spherical symmetry we obtain

$$\xi(r) = \frac{1}{2\pi^2} \int P(k) \frac{\sin kr}{kr} k^2 \mathrm{d}k = \int \Delta^2(k) \frac{\sin kr}{kr} \mathrm{d}(\ln k).$$

Performing a similar process to the previous results, we obtain the following expression for the power spectrum

$$P(k) = \int_0^\infty \xi(r) \frac{\sin kr}{kr} 4\pi r^2 \mathrm{d}r, \tag{3.2}$$

the dimensionless power spectrum is given by

$$\Delta^2(k) = \frac{2}{\pi} k^3 \int_0^\infty \xi(r) \frac{\sin kr}{kr} r^2 \mathrm{d}r. \tag{3.3}$$

3.2.1 Initial power spectrum of matter

Initial fluctuations are considered to be of a Gaussian nature, these fluctuations are caused by an inflationary phase in the early Universe. However, it is possible to introduce non-Gaussianities into the density field. For example, the nonlinear collapse of matter would introduce non-Gaussianities on small spatial scales but the linear power spectrum will remain the only tool needed to describe the linear behavior of the initial Gaussian field.

In a scenario where inflation puts initial fluctuations in this density field, many models predict an initial fluctuation spectrum which is nearly scale independent, among them there is the cosmological constant model. In other words, this means that the spectral index of scalar fluctuations n_s is close to one, according to recent data from the WMAP observations $n_s \approx 0,95$ [7]. It is convenient to use the following parameterization for the initial power spectrum

$$P_i(k) = A_s\, k^{n_s}.$$

The constant A_s represents the strength of the initial fluctuations that have been created in this early stage of the Universe and may be related to the height of the fluctuations seen in the analysis of the CMB or in LSS. The index i indicate the early Universe. This constant is related to variance of the density field within a sphere of radius $8\,h^{-1}\,\mathrm{Mpc}$, σ_8. The value $8\,h^{-1}\,\mathrm{Mpc}$ has been chosen since that is the scale on which the two-point correlation function for galaxies has roughly unit amplitude $\xi \approx 1$. These fluctuations are considered adiabatic, since they are a result of parts compressing or decompressing of an exactly homogeneous Universe.

It is usual to write the power spectrum as

$$P(k,t) \propto P_i(k)\, T^2(k)\, D^2(t). \tag{3.4}$$

The transfer function $T(k)$ contains the changes that have occurred to these fluctuations after entering the horizon.

For most cosmological models, T is approximately independent of time below a redshift $z \approx 100$. For the case of adiabatic fluctuations of cold dark matter (model used in this work), we can use a fitting function for T, which is

$$\begin{aligned} T(k) &= \frac{\ln(1+2,34q)}{2,34q}\left[1+3,89q+(16,1q)^2+(5,46)^3+(6,71q)^4\right]^{-1/4}, \\ q &= \frac{k}{\Omega_m h^2\,\mathrm{Mpc}^{-1}}, \end{aligned} \tag{3.5}$$

see [37]. In order to have a precise fit to the transfer function and to the initial spectrum we calculate the power spectrum today using CAMB [36]. To obtain $P(k,t)$, we multiply the square of linear growth factor $D(t)$ to the result obtaining with CAMB. See Figure 3.1.

3.3 The CMB angular power spectrum

Any scalar function can be expanded in spherical harmonics, anisotropies into CMB are not exception. Then, the map of CMB anisotropies Θ which is decomposed into spherical harmonics is given by

$$\frac{\delta T}{T}(\theta,\phi) = \Theta(\theta,\phi) = \sum_{l,m} a_{lm} Y_{lm}(\theta,\phi),$$

where Y_{lm} is a spherical harmonic and a_{lm} is a expansion coefficient. The statistical properties of the coefficients a_{lm} are translated into constraints on the underlying cosmological parameters which produce the fluctuations and determine their evolution. Theories for cosmic inflation, see [41], predict that the primordial density fields are Gaussian, in which case the average $\langle a_{lm} \rangle = 0$ and its variance is

$$\langle | a_{lm} a^{*}_{l'm'} | \rangle = \delta_{lm} \delta_{l'm'} C_l.$$

It is very important to note that, for a given l, each a_{lm} has the same variance. Therefore C_l contains the crucial cosmological information.

The values of C_l can be calculated from a 'Boltzmann code' such as CAMB [36], see Figure 3.1.

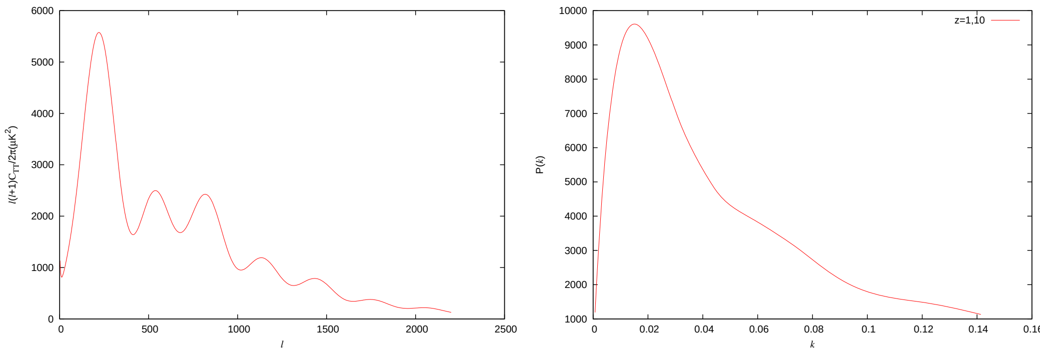


Figure 3.1: Using CAMB, we plot the CMB angular power spectrum, on the left–hand side, and the matter power spectrum in the redshift $z = 1, 1$, on the right side, see [36].

If one measures the components a_{lm} of the low multipoles, we do not get as much as information about the underlying variance as in the higher multipoles case. Thus, there is a fundamental uncertainty in the knowledge we may get about the C_l's. That uncertainty, which is most pronunced at low l, is called *cosmic variance*. The cosmic variance can be written as

$$\frac{\Delta C_l}{C_l} = \sqrt{\frac{2}{f_{sky}(2l+1)}},$$

where the factor $2l + 1$ is the number of a_{lm} modes measurable from the data which is proportional to the fractional area of the sky f_{sky} studied and the factor 2 reflects that the direction of the modes on the sky is unimportant.

From these coefficients one can compute the *rms* of the difference in temperature of two points separated by an angle θ as:

$$\left\langle \frac{\delta T}{T} \right\rangle_{rms} \approx \sqrt{\frac{l(l+1)C_l}{2\pi}}.$$

Different multipoles correspond to different angular separations and its relation is approximately $\theta \approx \frac{180}{l}$ degrees.

3.4 The Integrated Sachs-Wolfe effect

Photons propagating through the late-time Universe inevitably undergo to some gravitational potential Φ. As they descend into the potential, the photon is gravitationally blueshifted due to an increase in energy. Initially on departure it is redshifted resulting from a loss of energy in climbing from the potential well. The net effect would seem to be zero. However, this is only the case for a time independent potential. If it were to decay, for example, the photon would suffer a net blueshifting. Such decay is expected to occur in the presence of dark energy in a flat universe [13], which provides a cleaner test of this component of the Universe, or in presence of curvature, for the case of curved cosmologies see [11, 12]. The overall effect for a photon is the sum of all contributions along the line of sight, this effect is called Integrated Sachs-Wolfe effect (ISW) [10].

On large scales, the ISW effect will add power to CMB anisotropies, by:

$$\left(\frac{\Delta T}{T} \right)_{ISW} = 2 \int_{\eta_{LS}}^{\eta_0} \frac{\partial}{\partial \eta} \Phi[(\eta_0 - \eta)\hat{\mathbf{n}}, \eta] \, \mathrm{d}\eta, \tag{3.6}$$

where $\hat{\mathbf{n}}$ is the unit vector along the line of sight; η is the conformal time, and η_{LS} and η_0 are the conformal times today and at the surface of the last scattering respectively; T is the temperature; $\Phi(\mathbf{x}, \eta)$ is the gravitational potential at position $\mathbf{x}$ and at conformal time η.Its relative amplitude makes it difficult to distinguish from the primary anisotropies. However, it is possible to measure the ISW effect using the cross-correlation between the LSS and the CMB, independently from the intrinsic CMB fluctuations, see [14, 17, 20]. In the case where the gravitational potential decays, a positive correlation is expected. This means that on large scale hot spots in the CMB will correspond to over-dense regions in the galaxy distribution.

3.5 The Cross-Correlation as a tool for detecting the ISW effect

Once detecting the ISW effect from angular power spectrum is hard, we use the cross-correlation for this purpose. In this section, we will present the formalism for the calculation of cross-correlation between LSS and CMB, and LSS autocorrelation.

The predicted cross-correlation signal of the ISW effect in a spherical harmonic space is given by:

$$C_{gT}(l) = 4\pi b_g \int \mathrm{d}k \frac{\Delta^2(k)}{k} W_g(k) W_T(k). \tag{3.7}$$

For the angular autocorrelation we have:

$$C_{gg}(l) = 4\pi b_g^2 \int \mathrm{d}k \frac{\Delta^2(k)}{k} |W_g(k)|^2, \tag{3.8}$$

where:

$$W_g = \int \mathrm{d}x\, \Theta(x) j_l(kx)\, D,$$

$$W_T(k) = -\frac{3\Omega_m^0 H_0^2}{k^2 c^3} \int_0^{z_L} \mathrm{d}x\, j_l(kx) H(f-1)\, D,$$

$$\Theta(x) = \frac{r^2 n_c(x)}{\int \mathrm{d}x\, x^2 n_c(x)}.$$

We must remember that $\Delta^2(k)$ is the dimensionless power spectrum defined in the section 3.1; x is the comoving distance, which is implicitly used as a label for redshift epoch z; $D(z)$ is the linear growth factor, which is obtained solving the equation (2.22), in chapter 2 there are some solutions to this equation for different cosmologies; $j_l(kx)$ is the spherical bessel function of the 1st kind of order l.

Integrals are performed over the wavenumber k (expressed in $h\mathrm{Mpc}^{-1}$), and the comoving distance x (in $h^{-1}\,\mathrm{Mpc}$). $n_c(r)$ is called selection function and it represents the galaxies density distribution in the Universe. In $W_T(k)$, the redshift at the surface of the last scattering is $z_L \sim 1089$. The growth function f is given by

$$f(z) = \frac{\mathrm{d}\ln D}{\mathrm{d}\ln a}$$

and can be approximated by $f(z) \approx [\Omega_m(z)]^{0,6}$, see chapter 2.

The matter power spectrum $P(k)$, which was defined in the previous section, is related to the galaxy power spectrum through the *linear bias* b_g.

In a Gaussian model, Coles and Lucchin [38] described how the spatial two-point correlation functions in the underlying dark matter $\xi_D(r)$ and the galaxies $\xi_{gal}(r)$ could be related

$$\xi_{gal}(r) = b_g^2(z)\xi_D(r).$$

There are a number of more-or-less equivalent definitions of the linear bias b_g. From the relations between the two-point correlation function, the power spectrum and the linear growth factor, we see that

$$P_{gal}(k) = b_g^2 P_D(k) \quad \text{and} \quad \left(\frac{\Delta\rho}{\rho}\right)_{gal} = b_g(z)\left(\frac{\Delta\rho}{\rho}\right)_D \quad \text{or} \quad \delta_{gal}(r) = b_g(z)\delta_D(r).$$

Notice that, the linear bias is a function of redshift. Magliocchetti et al. [40] in the equation 2 show the dependence of b_g with z. In this equation we can see that with increasing of redshift on the sample galaxy the b_g would increase too. This model works well in the range $0 \leq z \leq 1$ for cold dark matter.

The linear bias could be derived directly from the 2MASS autocorrelation, see [20]. Given the model and the values of the cosmological parameters used in this work, we assume $b_g = 1,40$.

If we observe the equation (3.7), $C_{gT}(l)$ is zero if $f(z)$ is one for any redshift. This occurs only in the Einstein-de Sitter Universe ($\Omega_m = 1, \Omega_\Lambda = 0$). Then we can say that for flat cosmologies without Dark Energy and matter dominated, the ISW effect is null.

In this work we consider that the selection function is given by:

$$n_c(r) = \delta_{dirac}(r - r_0) \quad \text{so} \quad \Theta(r) = \frac{r^2\delta_{dirac}(r - r_0)}{r_0^2},$$

where $\delta_{dirac}(r - r_0)$ is Dirac delta function. This represents the density of a Universe with a localized punctual galaxy at r_0. Using the above expression, the function $W_g(k)$ can be written as

$$W_g(k) = j_l(kr_0)\delta(z_0),$$

recall that r_0, z_0 and a_0 are related, see chapter 1.

Since using equations (3.7) and (3.8) the calculation can be complicated and requires much computation time, in this work we replace them with the small angle approximation (reference [17] used that approximation to get Limber equation). Figure 3 in [20] shows the comparation between the exact theory and small angle approximation. The difference is less than 1% from $l = 5$ onwards, but it is considerably large for lower multipoles). This approximation arises from the Bessel function approximation:

$$\lim_{l\to\infty} j_l(x) = \sqrt{\frac{\pi}{2l+1}}\delta_{dirac}\left(l + \frac{1}{2} - x\right).$$

Equations (3.7) and (3.8) then simplify to

$$C_{gT}(l) = \frac{-3b_g H_0^2\Omega_m^0}{c^3(l+1/2)^2}\delta^2(z_0)H(z_0)(f(z_0) - 1)P\left(\frac{l+1/2}{r_0}\right), \tag{3.9}$$

$$C_{gg}(l) = \frac{b_g^2}{r_0^3}\sqrt{\frac{2(l+1/2)^3}{\pi}}\delta^2(z_0)P\left(\frac{l+1/2}{r_0}\right)j_l(l+1/2). \tag{3.10}$$

3.5.1 The error in the ISW detection and the likelihood function

The expected dispersion in the cross-correlation signal for harmonic multipole C_{gT} is given by

$$\Delta C_{gT}^2 = \frac{1}{f_{sky}(2l+1)}(C_{gT}^2 + C_{gg}C_{TT}), \tag{3.11}$$

where C_{TT} is the CMB angular power spectrum. In [17] we can see more details.

The cross-correlation is an important tool for restricting the parameters of different theories that try of explain the dark energy in the Universe. To this purpose we have used the likelihood function, which is defined as the probability that the parameters of a given theory fit to the data given by an experiment. This function allows to perform a statistical inference of parameters value via the observations, see [21]. In this case we do not have an experiment, we have two theories, $\mathcal{C}_0$ and $\mathcal{C}(\mathbf{p})$, where $\mathcal{C}_0$ is the standard cosmological model (in this case ΛCDM model) and $\mathcal{C}(\mathbf{p})$ is another model with $\mathbf{p}$ parameters. In this work $\mathcal{C}(\mathbf{p})$ corresponds to the parameterised Quintessence model, where the state equation varies.

The likelihood function is given by:

$$\mathcal{L} = \prod_l \left(\frac{1}{\sqrt{2\pi}\Delta C_{gT}(l;\mathcal{C})}\right)\exp\left(-\frac{\chi^2}{2}\right)$$

where χ^2 is

$$\chi^2 = \sum_l \frac{[C_{gT}(l;\mathcal{C}) - C_{gT}(l;\mathcal{C}_0)]^2}{\Delta C_{gT}^2(l;\mathcal{C})}. \tag{3.12}$$

Note that b_g is cancelled from the numerator and the denominator. The analysis using the function likelihood is summarized in the study of the behavior of χ^2.

Chapter 4

ISW: ΛCDM versus Quintessence model

So far we have presented the necessary topics to understand and analyze the ISW effect. In this chapter we present the numerical experiments results that we have made in this investigation. The plots were performed with points obtained with algorithms in C. They were specially developed using the routines of Numerical Recipes in C as tool, see [22].

4.1 Standard cosmological model

Since the objective in this work is to constrain the parameters via the ISW effect of a parameterised Quintessence model, we need to define a standard model $\mathcal{C}_0$ which is accepted by the observations. In this case, $\mathcal{C}_0$ is the ΛCDM, which describes a Universe dominated by the vacuum energy and dark matter.

Based on inflation, we assume that the Universe is flat, therefore $\Omega_\Lambda^0 = 1 - \Omega_m^0$. For the matter density parameter we take $\Omega_m^0 \approx 0,3$, in accord with supernovae Ia data combined with the flatness of the Universe. For the baryon density we assumed $\Omega_b^0 \approx 0,05$. Due to recent data of WMAP and some inflationary models, we consider that the spectral index have the value $n_s \approx 0,95$, see [7]. Based on the Hubble Space Telescope (HST) Key Project, we take for the Hubble parameter $h \approx 0,7$ [39]. The system employed in this work is described in the section 3.5.

4.2 Behavior of the ISW effect signal in different redshifts

Initially we observe the behavior of the ISW effect signal, which is observed in the cross-correlation between the galaxies and CMB, and autocorrelation C_{gg} in different redshift, using the standard cosmological model. In Figure 4.1 we can see that for higher redshifts, the signal is weaker than the signal to lower redshifts.

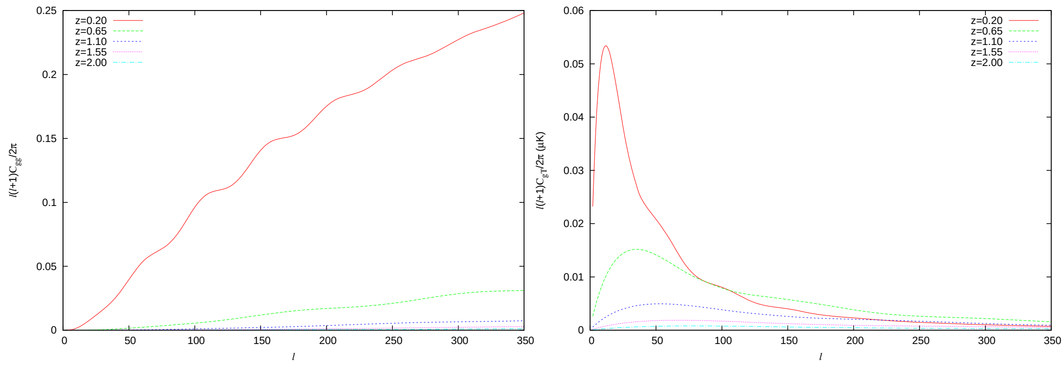


Figure 4.1: *Left*: Auto-correlation for a punctual galaxy localized at different distances $z_0 = 2,00$; $z_0 = 1,55$; $z_0 = 1,10$; $z_0 = 0,65$; $z_0 = 0,20$; respectively. *Right*: Cross-correlation between a punctual galaxy localized at different distances and the CMB $z_0 = 2,00$; $z_0 = 1,55$; $z_0 = 1,10$; $z_0 = 0,65$; $z_0 = 0,20$; respectively.

4.3 Constraints of Quintessence model parameters

As we have already said, using the ISW effect signal we can constrain the parameters of a cosmological model $\mathcal{C}(\mathbf{p})$ given a standard cosmological model $\mathcal{C}_0$, where $\mathcal{C}_0$ is described in the section 4.1, punctual galaxy is found in a redshift $z = 0,3$. The equation of state for dark energy in the cosmological model $\mathcal{C}(\mathbf{p})$ is given by

$$w(a) = w_0 + (a - a_0)w_a, \tag{4.1}$$

where w_a, w_0, a_0, are parameters. In this case, the Universe is flat with $\Omega_\Lambda^0 = 0,7$ and $\Omega_m^0 = 0,3$.

Considering the following settings in the equation of state, $w_a = 0$, we can observe the behavior of C_{gg} and C_{gT} for different values of w_0. In Figure 4.2 we observe that as the w_0 parameter decreases the ISW effect signal and the autocorrelation function also decrease, this plot is obtained using the equations (3.9) and (3.10).

Now we analyze the behavior of the error bars of the cross-correlation function for different values of w_0. Therefore we select $w_0 = -0,5$; $w_0 = -1,5$; $w_0 = -2,5$ and $w_0 = -3,5$. To perform this analysis we have done a bining with $\Delta_l = 8$ for each w_0. The $C_{gT}(l)$ values are averaged into bins of width $\Delta_l = 8$ using the following equation:

$$C_{gT}^{\Delta_l}(l) = \frac{\sum_{l'}^{l'+\Delta_l}(2l'+1)C_{gT}(l')}{\sum_{l'}^{l'+\Delta_l}(2l'+1)}.$$

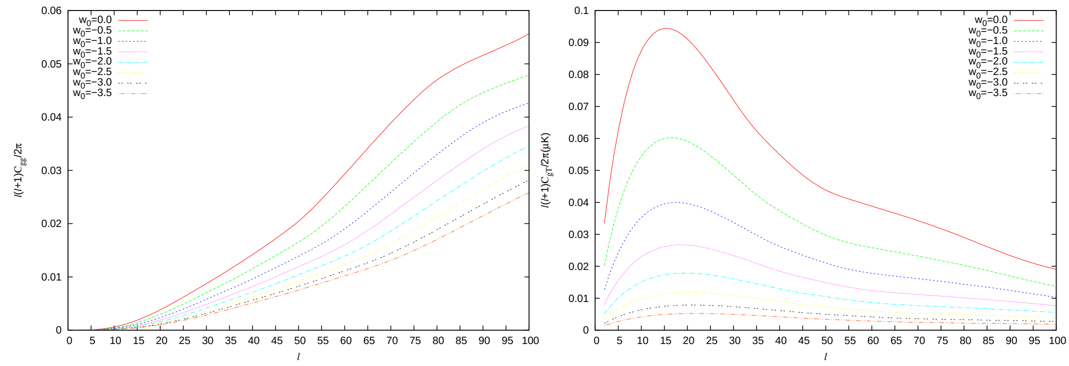


Figure 4.2: Auto-correlation and cross-correlation respectively for different values of w_0 in the punctual galaxy model; $w_0 = 0,0$ red; $w_0 = -0,5$ green; $w_0 = -1,0$ blue; $w_0 = -1,5$ magenta; $w_0 = -2,0$ lightblue; $w_0 = -2,5$ yellow; $w_0 = -3,0$ black; $w_0 = -3,5$ orange.

We find the error bars average using the following expression

$$\frac{1}{\Delta C_{gT}^2(l)} = \sum_{l'}^{l'+\Delta_l} \frac{1}{\Delta C_{gT}^2(l)}.$$

In Figure 4.3 we can see that the error bars decrease as the l multipole value increases. According to equation (3.12), the argument in χ^2 must tend to zero for higher multipoles. Although the error bars decrease, in the limit of higher multipoles, the cross-correlation function tends to zero for different values of w_0. Therefore, the summation in χ^2 goes to a multipole l_0 where the $C_{gT}(l)$ are insignificant, in this work $l_0 = 100$, since the ISW effect detection is given for low multipoles, see [17].

In addition to the study of the w_0 parameter behavior, we also can analyze the case where w_a varies, considering the following settings in the equation of state (4.1), $w_0 = -1,0$, $a_0 = 1,0$. In Figure 4.4 we compute χ^2 in $\mathcal{C}(w_0)$ and $\mathcal{C}(w_a)$. For bins of width $[-1,926\,,\,-0,323]$ in w_0 and $[-0,855\,,\,1,190]$ in w_a, we obtain one sigma level by taking the 68% of the sample which best fit to the standard cosmological model.

After studying the w_0 and w_a parameters behavior separately, we now study both parameters in a single $\mathcal{C}(w_0, w_a)$ cosmological model, considering $a_0 = 0$ in the equation of state (4.1). For this case, we calculated the χ^2 value which we have defined in the expression (3.12). Values set for w_0:

$$\{-0,2;\ -0,4;\ -0,6;\ -0,8;\ -1,0;\ -1,2;\ -1,4;\ -1,6;\ -1,8;\ -2,0\}$$

and for w_a:

$$\{+0,8;\ +0,6;\ +0,4;\ +0,2;\ 0;\ -0,2;\ -0,4;\ -0,6;\ -0,8;\ -1,0\}.$$

On the left-hand side Figure 4.5 we can see a map in which it is drawn the contour curve for χ^2. If we want to get one sigma level in the best fit, we need to consider the contour curve for $\chi^2 = 2,3$ (i.e., green line of this plot), since we have a cosmological model with two parameters, see page 697 in [22].

We can observe in Figure 4.5, the need to add more values of w_0 and w_a to get a good description in this case. Given that the designed algorithm in this work was made for a cosmological model with one parameter, the computation time in the case of two parameters is higher, hindering the data collection process. For future research cosmological models of two or more parameters will be need to improve routine to optimize the calculation time.

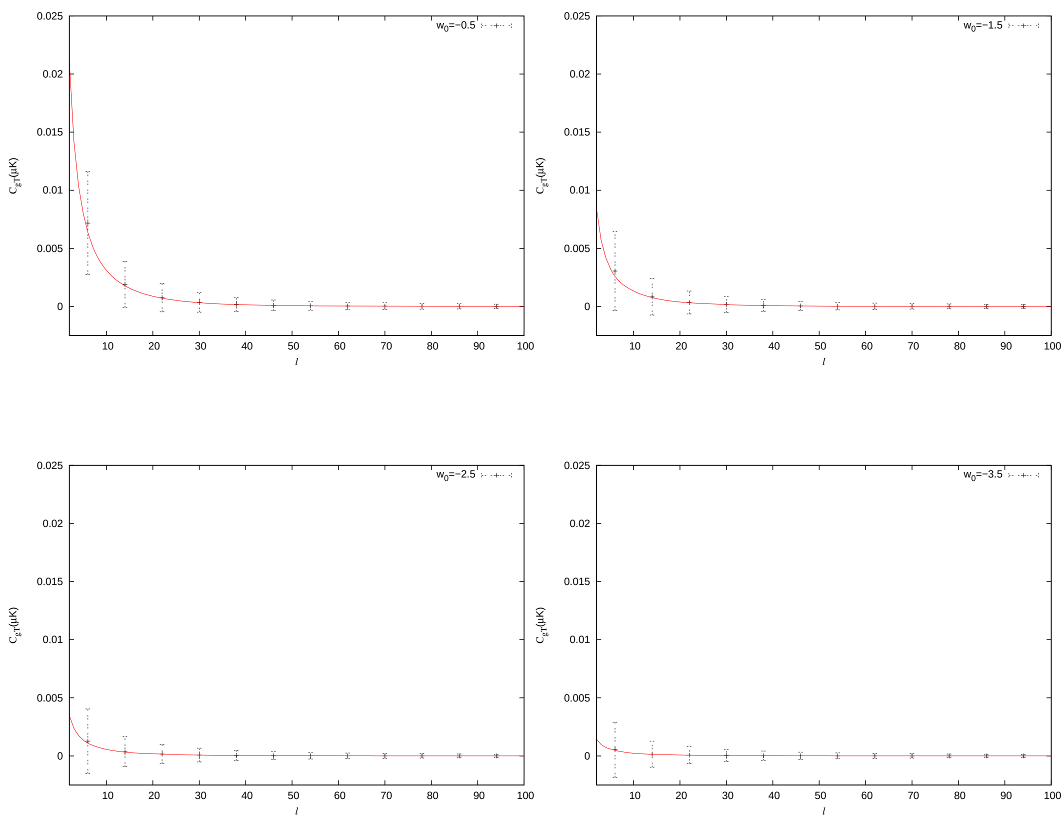


Figure 4.3: In this figure we show the cross-correlation for different w_0: *left top*: $w_0 = -0,5$, *right top*: $w_0 = -1,5$, *left bottom*: $w_0 = -2,5$, *right bottom*: $w_0 = -3,5$. Errors were calculated with the equation (3.11). We have done a bining with $\Delta_l = 8$ for each graph (see text for details).

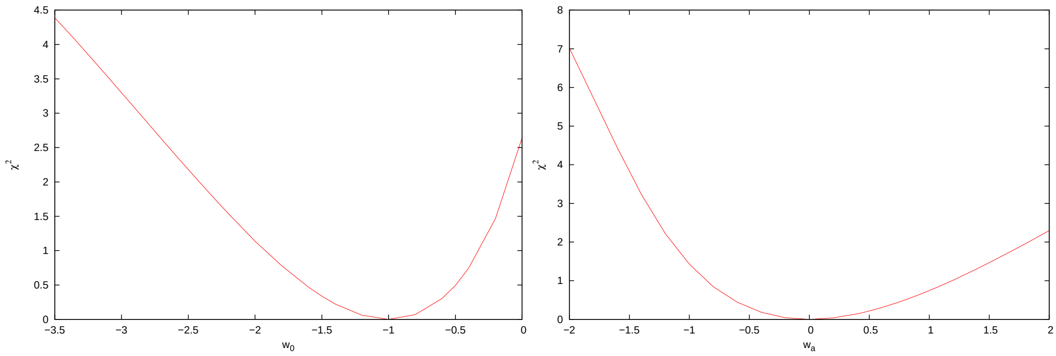


Figure 4.4: χ^2 values calculated using the equation (3.12) for the case of a cosmological model $\mathcal{C}(w_0)$ with $w_a = 0$ in the equation of state (4.1), on the left-hand side; and a cosmological model $\mathcal{C}(w_a)$ with $w_0 = -1,0$ and $a_0 = 1,0$ in the equation of state (4.1), on the right-hand side.

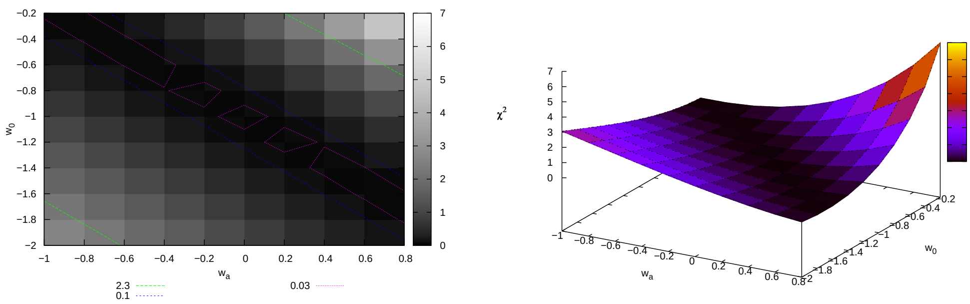


Figure 4.5: χ^2 values calculated using the equation (3.12) for the case of a cosmological model with two parameters, $\mathcal{C}(w_a, w_0)$, $a_0 = 1,0$ in the equation of state (4.1). On the left-hand side, we have a map with contour curves that represent the following χ^2 values: 0,03 magenta; 0,1 blue; 2,3 green. On the right-hand side, we have a representation 3D of χ^2. For the plot, we only consider 10 values for w_a and 10 values for w_0.

Future research can use this procedure to constrain other the parameters in other models Quintessence or constrain the cosmological parameters value as the spectral index n_s, the parameter of dark energy density Ω_Λ^0, etc.

Chapter 5

Conclusions

In this work, we study the ISW effect signal via cross-correlation between CMB and LSS. To accomplish this, we develop the necessary theory to explain the structure growth. In the chapter 2, we show the linear growth function and the growth function behavior for different cosmological models, among them, a cosmological model with dark energy where the equation of state varies, see Figures 2.2 and 2.3. The case of a Universe dominated by non-relativistic matter $\Omega_m^0 \approx 1,0$ Einstein-de Sitter Universe case, the ISW effect signal is null, given that the growth function is one, see Figure 2.2 and equation (3.9). In addition, we analyze the approximation $f(t) \approx (\Omega_m(t))^{0,6}$ for the growth function in a flat Universe with matter and dark energy. We show that it is a good approximation, see Figure 2.1. However, in this work, we do not use this approximation.

Independent measurements of the growth parameters as supernovae and Baryon Acoustic Oscillations (BAO) allow us an understanding of the distances as well as the growth getting a better understanding on the nature of dark energy. The main result in this work was to compute the best fit with the standard cosmological model until one sigma level via ISW effect for cases different of a parameterised Quintessence model. Which is decribed by the equation of state (4.1). In $\mathcal{C}(w_0)$ case, with $w_a = 0$ and $a_0 = 0$, we find a bins of width $[-1,926\,,\,-0,323]$, for the best fit. In $\mathcal{C}(w_a)$ case, with $w_0 = -1,0$ and $a_0 = 1,0$, we find a bins of width $[-0,855\,,\,1,190]$, for the best fit. With these two cases, we can observe that the ISW effect is a great tool, which allows us to constrain the cosmological parameter in a model with dark energy. But not only in the cosmogical models cases with one parameter, in the Figure 4.5 we show the cosmological model case with two parameters (w_0 and w_a, $a_0 = 1,0$), we obtain contour curves for one sigma level in the best fit with standard cosmological model. However, it is necessary to improve the algorithms used in this work to obtain best results. All cases considered here contain the standard cosmological model. The plots were made with Gnuplot.

Appendix A

Other Quintessence models

To develop this appendix we need to use the formalism of the general relativity, see [42]. As mentioned in chapter 1, to explain an accelerated expanding Universe, we have to introduce the concept of dark energy. The model based in the cosmological constant is used for this purpose, however we can generalize this model by introducing a scalar field which only interacts with other fields gravitationally. This cosmological model is part of Quintessence models. The dark energy field is supposed to slowly roll down the potential or is trapped in a local minimum. This leads to a vacuum dominated state of the Universe which hence leads to an accelerated expansion.

The stress energy tensor in a flat Universe is

$$T_{\mu\nu} = -\partial_\mu \phi \frac{\partial L}{\partial(\partial^\nu \phi)} + g_{\mu\nu} L, \tag{A.1}$$

where $g_{\mu\nu}$ is the FLRW metric. Let us write the lagrangian for a scalar field

$$L = -\frac{1}{2}\partial^\alpha \phi g_{\alpha\beta} \partial^\beta \phi - V(\phi),$$

where ϕ is the dark energy field. Introducing the above expression in (A.1) we get

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} L. \tag{A.2}$$

Assuming that the field is homogeneous on large scale, we can replace it into the lagrangian

$$\partial^\alpha \phi g_{\alpha\beta} \partial^\beta \phi = \dot{\phi}^2 g_{00} = -\dot{\phi}^2,$$

in addition, we have also

$$\partial_\mu \phi = g_{0\mu}\dot{\phi} \quad \text{and} \quad \partial_\nu \phi = g_{0\nu}\dot{\phi}.$$

Multiplying $g^{\lambda\mu}$ in (A.2) and by using the above expressions, we obtain

$$T^\lambda_\nu = \delta^\lambda_0 g_{0\nu}\dot{\phi}^2 + \delta^\lambda_\nu \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{A.3}$$

Given that we consider the Universe as a perfect fluid, its stress energy tensor is

$$T^{\lambda}_{\nu} = \rho u^{\lambda} u_{\nu} + p(\delta^{\lambda}_{\nu} + u^{\lambda} u_{\nu}),$$

where ρ is the energy density, p is the pressure and u_{ν} is the 4-velocity. Therefore, we can identify of the above expression

$$T^0_0 = -\rho_{\phi} = -\frac{1}{2}\dot{\phi}^2 - V(\phi) \quad \text{and} \quad T^i_i = p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

The proportionality factor

$$w_{\phi} \equiv \frac{p_{\phi}}{\rho_{\phi}},$$

in the equation of state, $p_{\phi} = w_{\phi}\rho_{\phi}$ is $w_{\phi} = -1$ if the kinetic term $\dot{\phi}^2/2$ is negligible, note that $c = 1$ in this case.

To find the evolution of the dark energy field, we replace p_{ϕ} and ρ_{ϕ} into the equation of state for this component (1.5)

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,$$

with $V'(\phi) = \mathrm{d}V/\mathrm{d}\phi$.

Friedmann's equation is given by

$$H^2 = \frac{\dot{a}}{a} = \frac{1}{3}\left[\rho_{other} + \frac{1}{2}\dot{\phi}^2 + V(\phi)\right],$$

where the Planck mass M_{pl} is equal to one, see [34], and ρ_{other} is the total energy density of the other contributing fields or energy components, like dark and baryonic matter and radiation.

If $V(\phi)$ is approximately constant and the other energy components are negligible, the solution for the scale factor is $a \approx \exp[\sqrt{V}t]$ and hence, the expansion of the Universe is accelerating, see subsection 1.7.4.

Appendix B

The cross-correlation power spectrum

Let us consider two random fields $A(\mathbf{x})$ and $B(\mathbf{x})$, with their Fourier transforms defined as

$$A_{\mathbf{k}} = \int \mathrm{d}^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} A(\mathbf{x}) \quad \text{and} \quad B_{\mathbf{k}} = \int \mathrm{d}^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} B(\mathbf{x}).$$

The cross-correlation power spectrum, $P_{AB}(k)$ is defined by

$$\langle A_{\mathbf{k}_1} B_{\mathbf{k}_2} \rangle = (2\pi)^3 \delta^3_{dirac}(\mathbf{k}_1 - \mathbf{k}_2) P_{AB}(k_1)$$

The project of A and B on the sky are defined using F_A and F_B projection kernels

$$\tilde{A}(\hat{\mathbf{n}}) = \int \mathrm{d}r\, F_A(r) A(r\hat{\mathbf{n}}) \quad \text{and} \quad \tilde{B}(\hat{\mathbf{n}}) = \int \mathrm{d}r F_B(r) B(r\hat{\mathbf{n}}).$$

For the ISW effect, the kernel is given by

$$\left(\frac{\Delta T}{T}\right)_{ISW} = 2 \int_{\eta_{LS}}^{\eta_0} \frac{\partial}{\partial \eta} \Phi[(\eta_0 - \eta)\hat{\mathbf{n}}, \eta]\, \mathrm{d}\eta.$$

For the projected galaxy overdensity, this kernel is

$$F_g(r) = \frac{r^2\, n_c(r)}{\int \mathrm{d}r'\, r'^2\, n_c(r')}.$$

Now, expanding $\tilde{A}$ and $\tilde{B}$ in terms of spherical harmonics, the cross-correlation coefficients, $C_{AB}(l)$ is defined as

$$C_{AB}(l) \equiv \langle \tilde{A}_{lm} \tilde{B}^*_{lm} \rangle =$$

$$\begin{aligned}
&= \int \mathrm{d}r_1 \mathrm{d}r_2 F_A(r_1) F_B(r_2) \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} P_{AB}(k) (4\pi)^2 j_l(kr_1) j_l(kr_2) Y_{lm}(\hat{\mathbf{k}}) Y^*_{lm}(\hat{\mathbf{k}}) \\
&= \int \mathrm{d}r_1 \mathrm{d}r_2 F_A(r_1) F_B(r_2) \int \frac{2k^2 \mathrm{d}k}{\pi} P_{AB}(k) j_l(kr_1) j_l(kr_2).
\end{aligned}$$

we can use the small angle (large l) approximation for the spherical Bessel function

$$j_l(x) = \sqrt{\frac{\pi}{2l+1}}[\delta_{dirac}(l + \frac{1}{2} - x)],$$

and obtain

$$C_{AB}(l) = \int \frac{\mathrm{d}r}{r^2} F_A(r) F_B(r) P_{AB}\left(\frac{l+1/2}{r}\right).$$

This is the so called Limber equation. Substitute the kernels for the ISW effect and for the projected galaxy overdensity, we have

$$C_{gT}(x) = -\frac{b_g}{\int \mathrm{d}r\, r^2 n_c(r)} \int \mathrm{d}r\, n_c(r)(3H_0^2)\Omega_m \frac{r^2}{(l+1/2)^2}\frac{D'}{D}(1+z)P\left(\frac{l+1/2}{r}\right),$$

where $P(k)$ is the matter power spectrum. Since the ISW effect is only important at large scales, the cross-power of the gravitational potential derivate with matter fluctuations can be expressed in terms of the matter power spectrum, using the Poisson equation and linear perturbation theory, we can find the kernel for ISW effect and so to obtain the above expression.

Appendix C

Acronyms and Notation

Acronym

WMAP	Wilkinson Microwave Anisotropy Probe
CMB	Cosmic Microwave Background
LSS	Large Scale Structure
DMR	Differential Microwave Radiometer
COBE	Cosmic Background Explorer
2MASS	2 Micron All-Sky Survey
ISW	Integrated Sachs-Wolfe
ΛCDM	Cosmological Constant with Cold Dark Matter
FLRW	Friedmann-Lemaître-Robertson-Walker metric
HST	Hubble Space Telescope
BAO	Baryon Acoustic Oscillations

Notation

ρ_γ	Energy density of photons
ρ_r	Energy density of all radiation
ρ_m	Matter energy density
ρ_Λ	Dark energy density
ρ_c	Critical energy density
ρ_k	Curvature density
Ω_i	Density parameter of the component i
Ω_{T_0}	Temperature of CMB
k_B	Boltzmann's constant
H_0	Hubble's constant today
H	Hubble's constant of expansion

h	Parameter for Hubble constant
G	Gravitational constant
w	Pressure to energy-density ratio
Λ	Cosmological constant
q	Deceleration parameter
a	Scale factor
η	Conformal time
d_c	Comoving distance
d_A	Angular distance
d_L	Luminosity distance
t_0	Age of the Universe
r_H	Particle horizon
r_E	Event horizon
d_h	Hubble's radius
z	Redshift
δ	Matter overdensity
D	Linear growth factor
f	Growth function
λ_J	Jeans' wavelength
$\xi(r)$	Correlation function
$P(k)$	Power spectrum
$\Delta^2(k)$	Dimensionless power spectrum
n_s	Spectral index
$T(k)$	Transfer function
Y_{lm}	Spherical harmonic
Φ	Gravitational potential
σ_8	Variance of the density field within a sphere of radius $8\,h^{-1}\,\mathrm{Mpc}$
b_g	Linear bias
$C_{gT}(l)$	Cross-correlation between CMB and LSS
$C_{gg}(l)$	Autocorrelation of galaxies
$C_{TT}(l)$	Autocorrelation of CMB
$\mathcal{C}_0$	Standard cosmological model
$\mathcal{L}$	likelihood function
χ^2	The expected significance level for ruling out a cosmological model

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