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# Switched Control via Static Output Feedback for Uncertain LTI Systems

Ilha Solteira - SP
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**Câmpus de Ilha Solteira**

## CERTIFICADO DE APROVAÇÃO

TÍTULO DA DISSERTAÇÃO: Switched Control via Static Output Feedback for Uncertain LTI Systems

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**ABSTRACT**

The switched static output feedback control applied to uncertain linear time-invariant (LTI) systems is addressed in this work.The approach chosen for the design of static output feedback (SOF) gains is based on the two-stage method, which consists in obtaining a state feedback gain matrix which is then used as an input parameter for the design of the desired static output feedback gain at the second stage. The solution for the investigated problems is presented in terms of linear matrix inequalities (LMIs), obtained by means of the application of the Finsler's Lemma. The method proposed was applied on the design of controllers for an active suspension system. In the practical experiments the dynamic performance achieved with the implementation of the derived controllers attests to the potential of the proposed strategy for designing switched static output feedback controllers. The obtained results are compared with a robust static output feedback controller design.

**Keywords:** Static output feedback control; Switching control; Uncertain linear time-invariant systems; Linear matrix inequalities (LMIs).

## RESUMO

Este trabalho aborda o controle chaveado via realimentação estática de saída aplicado à sistemas lineares incertos invariantes no tempo (no inglês, *linear time-invariant* - LTI). O projeto de controle chaveado via realimentação estática de saída (no inglês, static output feedback, SOF) apresentado neste trabalho é baseado no método dos dois estágios, o qual consiste em primeiramente obter um ganho de realimentação de estado ou das variáveis de estado, e então, utilizá-lo como entrada no segundo estágio para obter os ganhos de realimentação estática de saída desejado. As soluções para os problemas investigados são apresentadas na forma de desigualdades matriciais lineares (no inglês, linear matrix inequalities, LMIs), obtidas por meio da aplicação do Lema de Finsler. Adicionalmente, a solução proposta é aplicada no projeto de controladores para um sistema de suspensão ativa. Nos experimentos práticos a performance dinâmica alcançada com a aplicação dos controladores projetados atesta o potencial do método proposto de controladores chaveado via realimentação estática de saída. Os resultados obtidos foram comparados com técnicas de controle robusto via realimentação estática de saída.

**Palavras-chave:** Controle via realimentação estática de saída; Controle chaveado; Sistemas lineares incertos invariantes no tempo (LIT); Desigualdades matriciais lineares (LMIs).

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# 1 INTRODUCTION

The design of output feedback controller is important to deal with systems in which not all states are available for measurement, making impractical the use of the state feedback controller (KIMURA, 1975). There are two methods for development and control with output feedback. The first one is called dynamic output feedback (DOF) which is similar to of developing an observer to estimate the missing states for the control in the state space (ZHAI; LIU, 2021), as the feedback loop presents its own dynamics. On its turn, in the second method, the static output feedback (SOF), we use gains linked to the available states or the measured system output, which are directly in the control by means of an static feedback gain (BERNSTEIN, 1992). The static output feedback leads to a low-cost control design with relatively simple practical implementation since no additional sensors are needed and being based on the use of a single gain matrix (DONG; YANG, 2008). Furthermore, fixed-order DOF can be transformed into SOF by considering an augmented plant (BERNSTEIN, 1992; SYRMOS et al., 1997; GHAOUI; OUSTRY; AITRAMI, 1997).

In practical situations in control engineering one must consider the robustness to parameter uncertainty, which can be present due to imprecise system modeling, a failure during the application, sensors measurement errors or even disturbances in the system dynamics, that can generate unwanted outputs or create security risks for the system users (BARMISH, 1985). Many studies aiming at the design of robust controllers via output feedback to deal with these practical problems have been published so far. Among them we can mention Dong and Yang (2013) which synthesizes a robust static feedback controller with polytopic uncertainties, Manesco (2013) creates a robust controller design to handle structural failures using static output feedback using, $\mathcal{D}$-stability and parameter-dependent Lyapunov functions (PDLF), and Chang, Park and Zhou (2015) presents a robust $\mathcal{H}_\infty$ SOF controller.

Another control technique that has proven to be efficient, not only in stabilizing, but also in improving the transient response of the system is the switched control (SUN; GE, 2005), which consist to design a collection of controllers and a switching law such that the closed-loop system is asymptotically stable when orchestrating among the controllers under the switching law (XIAO et al., 2020). This strategy has been attracting the interest of the research community, as one can see from the development of interesting studies such as Pettersson (2004) which uses bilinear matrix inequalities (BMIs) to design the switched controllers, Lin and Antsaklis (2009) that presents an analysis and stabilization of a switched linear system. Additionally, quadratic stability to design state feedback controllers can be found in Souza et al. (2013), a switched state feedback robust control for continuous-time systems is presented in Geromel and Deaecto (2009), where is a procedure to find a set of state feedback gains and switching rule to coordinate them, for all time-varying uncertain parameters under consideration a guaranteed $\mathcal{H}_2$ cost. In this work will consider the practical engineering problems mentioned before where not all states are available, resulting in the necessity for the design of switched controllers via output feedback (YANG; LI; NIU, 2015; HE et al., 2019; HE; ZHU; SWEI, 2020; XIAO et al., 2020; LI et al., 2016; CARNIATO et al., 2020; OLIVEIRA et al., 2018; OLIVEIRA et al., 2014).

The literature of switched (SOF) control is relatively scarce, and the methods usually deal with this problem from the DOF controller perspective, in which it seeks the benefits of switched control by creating several gains or observers via output feedback and a switching law for the choosing the appropriate controller (YANG; LI; NIU, 2015). We can also mention a switched DOF controller applied in a highly maneuverable technology vehicle proposed in Yang, Li and Niu (2015), which uses linear matrix inequalities (LMIs) to create a problem model that combine Lyapunnov functions and the average dwell time method, and in He et al. (2019), which demonstrates a smooth-switching linear parameter-varying dynamic output feedback control, a DOF-switched controller that combines input covariance constraint (ICC) and $\mathcal{H}_\infty$ that avoids sudden variations between controllers and also minimizes $\mathcal{H}_\infty$ and $\mathcal{H}_2$ cost in the switching law, using parametric linear matrix inequalities (PLMIs) and applying the smooth-switching linear parameter-varying dynamic output feedback control for the vibration reduction of a flexible wing of an airplane (HE; ZHU; SWEI, 2020). Also, in Li et al. (2016) a switched controller DOF for nonlinear systems using the Takagi-Sugeno fuzzy modeling applied to a mass-spring-damping system, and a control through the construction of an observer and multiple Lyapunnov functions, estimating the states for switched state-feedback controller is discussed in Xiao et al. (2020). Furthermore, in Xiao et al. (2020) one can find two methods for synthesizing switched static output feedback controllers using LMIs obtained through Finsler's Lemma and another using a transformation matrix. In Carniato et al. (2020) the authors proposed the use of a hybrid metaheuristic technique, called DE–LMI (differential evolution – linear matrix inequality) to solve switched static output feedback for continuous-time uncertain switched linear systems and Bocca et al. (2022) propose a designing procedure for a robust guaranteed cost switched SOF that also uses DE–LMI and with parameter-dependent Lyapunov candidate function.

In this work, LMIs are used to describe the proposed strategy, as it consists of a powerful tool to solve linear control and optimization problems (SCHERER; GAHINET; CHILALI, 1997), and which can be easily programmed with the MATLAB® in interfaces such as the YALMIP (Yet Another LMI Parser) (LOFBERG, 2004) and solved with SeDuMi (STURM, 1998).

The results proposed in the present text are inspired by the works of Manesco

(2013) and Sereni (2019) that use the two-stage method to solve the SOF problem (AGULHARI; OLIVEIRA; PERES, 2010; MEHDI; BOUKAS; BACHELIER, 2004). This method consists in developing a state feedback gain $K$, then using this information as input for calculating the desired SOF gains. The proposed formulation was obtained using Finsler's Lemma (BOYD et al., 1994), and the control law was based on the one proposed in Mainardi Júnior et al. (2015) that considers the addition of switching matrices $Q_k$ on the controller design of switched systems with polytopic uncertainties. Considering the presented scope, this work investigates the proposition of new LMI conditions for design a switched control via static output feedback with polytopic uncertainties.

The switched control design technique via static output feedback is applied on an active suspension system to test and demonstrate the efficacy of our method. The chapters of this work are presented as follows:

- Chapter 2 addresses the synthesis of robust static output feedback controllers.
- Chapter 3 brings an initial presentation of proposed switched control design method via static output feedback gain without uncertainties in the output matrix, the use of minimum decay rate restrictions with the proposed theory, the switching law and the results of experiments for feasibility and performance analysis in an active suspension system in comparison with the robust SOF strategy proposed in Manesco (2013).
- Chapter 4 addresses a switched control design method via static output feedback gain with the a parameter-dependent state feedback in the first stage, relaxations methods and the experiments of feasibility and performance in an active suspension system in relation to the switched SOF proposed in Chapter 3.
- Finally, the concluding remarks and future perspectives on the subject at matter are presented in Chapter 5.

# 2 Robust Control via Static Output Feedback

This chapter is devoted to present some basic, concepts and definitions which will be considered as a basis for the development of the contributions presented in this text.

## 2.1 Finsler's Lemma

The approach proposed in this work is based on an application of the Finsler's Lemma to tackle the control via output feedback problem. Therefore, before properly addressing the issue, the Finsler's Lemma is formally presented in the sequence.

**Lemma 1.** *(Finsler's Lemma). Consider $\mathcal{W} \in \mathbb{R}^n$, $\mathcal{S} \in \mathbb{R}^{n\times n}$ and $\mathcal{R} \in \mathbb{R}^{m\times n}$ with (rank$(\mathcal{R}) < n$) where $\mathcal{R}^{\perp}$ is a basis for the null space of $\mathcal{R}$ $(i.e. \mathcal{R}\mathcal{R}^{\perp} = 0)$.*

*Then, the following conditions are equivalent:*

*(i)* $\mathcal{W}'\mathcal{S}\mathcal{W} < 0, \forall \mathcal{W} \neq 0, \mathcal{R}\mathcal{W} = 0$*,*

*(ii)* $\mathcal{R}'^{\perp}\mathcal{S}\mathcal{R}^{\perp} < 0$*,*

*(iii)* $\exists \eta \in \mathbb{R} : \mathcal{S} - \eta\mathcal{R}'\mathcal{R} < 0$*,*

*(iv)* $\exists \mathcal{X} \in \mathbb{R}^{2n\times n} : \mathcal{S} + \mathcal{X}\mathcal{R} + \mathcal{R}'\mathcal{X}' < 0$

*where $\eta$ and $\mathcal{X}$ are additional variables (or multipliers).*

*Proof.* See Skelton, Iwasaki and Grigoriadis (1997) and Oliveira and Skelton (2001) □

## 2.2 Decay Rate

Some systems may demand, besides stability, that also some transient performance requirements are meet. In that case, for ensuring such control requirements we might consider the decay rate, which is an index associated with system's transient duration, which can be defined according to Boyd et al. (1994), as the highest $\delta$ such that

$$\lim_{t\to\infty} e^{\delta t}||x(t)|| = 0 \tag{2.1}$$

holds for all trajectories of $x(t) \neq 0$ the system's states

Taking into account the Lyapunov's function $V(x(t)) = x'(t)Px(t)$, a low bound on the minimum decay rate can be established if

$$\dot{V}(x(t)) \leq -2\delta V(x(t)) \tag{2.2}$$

holds for all trajectories of the system's states $x(t)$ (BOYD et al., 1994).

## 2.3 System Description and Problem Formulation

Consider an uncertain linear time-invatiant (LTI) system such as:

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)u(t) \\ y(t) &= C(\alpha)x(t), \end{aligned} \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output vector, $u(t) \in \mathbb{R}^m$ is the control input vector. Moreover, the plant matrix $A(\alpha) \in \mathbb{R}^{n\times n}$, and the input matrix $B(\alpha) \in \mathbb{R}^{n\times m}$ and output matrix $C(\alpha) \in \mathbb{R}^{p\times n}$ are uncertain matrices that describe the system's dynamics, and can be represented in the polytopic domain $\mathfrak{D}$ defined as

$$\mathfrak{D} = \Big\{(A,B,C)(\alpha) : (A,B,C)(\alpha) = \sum_{i=1}^{N} \alpha_i (A,B,C)_i, \alpha \in \wedge_N \Big\}, \tag{2.4}$$

where $A_i$, $B_i$ and $C_i$ denote the i-th polytope vertex, and $N$ is the number of vertices of the polytope. Furthermore, $\mathfrak{D}$ is parameterized in terms of a vector $\alpha = (\alpha_1, ..., \alpha_N)$, whose parameters $\alpha$ are unknown constants belonging to the unitary simplex set $\wedge_N$, defined as

$$\wedge_N = \Big\{\alpha \in \mathbb{R}^n : \sum_{i=1}^{N} \alpha_i = 1; \alpha_i \geq 0\Big\}, \tag{2.5}$$

for $i \in \mathbb{K}_N$, where $\mathbb{K}_N$ is a set of positive integers $\{1, \dots, N\}$.

Supposing that the feedback loop is composed by the following control law

$$u(t) = Ly(t) \tag{2.6}$$

then the system (2.3) in closed-loop assumes the form

$$\dot{x}(t) = [A(\alpha) + B(\alpha)LC(\alpha)]x(t). \tag{2.7}$$

In these terms, the objective is to find a robust static output feedback gain $L \in \mathbb{R}^{m\times p}$, such that asymptotically stabilizes the system (2.7).

## 2.4 Robust Stabilization via Static Output Feedback

In this section the two-stage SOF control design method, presented in Sereni (2019) and in Manesco (2013) is described in terms of the robust stabilization problem.

The first stage is a preliminary state feedback control design. Therefore, considering the control law as

$$u(t) = Kx(t), \tag{2.8}$$

then the system (2.3) in closed-loop is represented by

$$\dot{x}(t) = [A(\alpha) + B(\alpha)K]x(t). \tag{2.9}$$

To obtain a robust state feedback gain $K$ that asymptotically stabilizes the system (2.9), we might consider the quadratic stability condition presented in Boyd et al. (1994), which states that if there are matrices $W \in \mathbb{R}^{n\times n}$ and $Z \in \mathbb{R}^{n\times m}$, such that

$$\begin{aligned} &W = W' > 0 \\ &A_iW + WA_i' + B_iZ + Z'B_i' < 0 \end{aligned} \tag{2.10}$$

for $i = 1, 2, ..., N$, then $K$ is given by $K = ZW^{-1}$.

The state-feedback gain ($K$) obtained in the first step then used as an input parameter for the composing the second step, where the switched output feedback $L \in \mathbb{R}^{m\times p}$ is designed as proposed in Theorem 1, which is given in terms of sufficient LMI conditions.

**Theorem 1.** *(MANESCO, 2013) Assuming that there exists a state feedback gain $K$, such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, then there exists a stabilizing static output feedback gain, $L$ , such that $A(\alpha) + B(\alpha)LC(\alpha)$ is asymptotically stable, if there exist the symmetric matrix $P \in \mathbb{R}^{n\times n}$ with $P > 0$ and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J \in \mathbb{R}^{m\times p}$ such that*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{2.11}$$

*holds for $\forall i \in \mathbb{K}_N$.*

*In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.*

*Proof.* See Manesco (2013). □

## 2.5 Decay Rate Bounding in Robust controller via Static Output Feedback

Considering additional systems performance requirements, Manesco (2013) proposes the use of decay rate bounding in robust controller synthesis via SOF.

The concept of the minimum decay rate is applied on both design stages of the method. For avoiding confusion, the minimum decay rate associated to each design stage is referred to as $\beta$ and $\gamma$, respectively. Even though they can assumes different values, the decay rate in first stage $(\beta)$ acts as a limiting factor in the design of the second stage (SERENI, 2019).

For the first stage, we need to obtain a gain $K$ that asymptotically stabilizes the system (2.9). The strategy chosen is based on the quadratic stability condition presented in Boyd et al. (1994), which states that if there are matrices $W \in \mathbb{R}^{n\times n}$, $Z \in \mathbb{R}^{n\times m}$, and a positive scalar $\beta$ such that

$$\begin{aligned} &W = W' > 0 \\ &A_iW + WA_i' + B_iZ + Z'B_i' + 2\beta W < 0 \end{aligned} \tag{2.12}$$

hold for $i = 1, 2, ..., N$, then $K$ is given by $K = ZW^{-1}$. In the sequence, the $K$ obtained in the first step is used as an input parameter for the second stage design according to Theorem 2.

**Remark 1.** The LMIs (2.10) and (2.12) compose the strategy chosen for the first stage, but it is important to emphasise that as each design stage is performed separately, any other state feedback control design can be implemented to derive the first stage gain matrix $K$.

**Theorem 2.** *(MANESCO, 2013) Assuming that there exists a state feedback gain K, such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, with a minimum decay rate greater or equal to $\gamma > 0$, then there exists a stabilizing robust static output feedback gain, $L$, such that $A(\alpha) + B(\alpha)LC(\alpha)$ is asymptotically stable, if the symmetric matrix $P \in \mathbb{R}^{n\times n}$ with $P > 0$ and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J \in \mathbb{R}^{m\times p}$ such that*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{2.13}$$

*holds for $\forall i \in \mathbb{K}_N$. In the affirmative case, the robust static output feedback gain is given by $L = H^{-1}J$.*

*Proof.* See Manesco (2013). □

## 2.6 Relaxation Strategies for the Design of Robust controllers via Static Output Feedback

In this section more relaxed LMI conditions for obtaining the robust SOF controller, as proposed in Manesco (2013) and Sereni (2019) are presented. The conservatism in the robust SOF control problem can be reduced by considering the use of a parameter-dependent Lyapunov matrix, and Finsler Lemma's additional variables.

### 2.6.1 Parameter-Dependent Lyapunov Function

Theorem 3 presents the LMI formulation to the robust SOF problem with PDLV as proposed by Manesco (2013). This result was obtained by considering a parameter-dependent matrix $P(\alpha)$ in order to achieve less conservative restrictions.

**Theorem 3.** *(MANESCO, 2013) Assuming that there exists a state feedback gain K, such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, with a minimum decay rate greater or equal to $\gamma > 0$, then there exists a stabilizing robust static output feedback gain, $L$, such that $A(\alpha) + B(\alpha)LC(\alpha)$ is asymptotically stable, if the symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ with $P_i > 0$ and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P_i & * & * \\ P_i - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + JC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0 \qquad (2.14)$$

*for $\forall i \in \mathbb{K}_N$, in the affirmative case, the robust static output feedback gains is given by $L = H^{-1}J$.*

*Proof.* See Manesco (2013). □

### 2.6.2 Parameter-Dependent Finler's Variables

In Sereni (2019) even more relaxed conditions are proposed, considering matrices $F(\alpha)$ and $G(\alpha)$ in order to achieve less conservative restrictions, such conditions are stated in Theorem 4.

**Theorem 4.** *(SERENI, 2019) Assuming that there exists a state feedback gain K, such that $A(\alpha) + B(\alpha)K$ is asymptotically stable, with a minimum decay rate greater or equal to $\gamma > 0$, then there exists a stabilizing static output feedback gain, $L$, such that $A(\alpha) + B(\alpha)LC(\alpha)$ is asymptotically stable, if the symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ with $P_i > 0$ and matrices $F_i$, $G_i \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F_i' + F_iA_i + K'B_i'F_i' + F_iB_iK + 2\gamma P_i & * & * \\ P_i - F_i' + G_iA_i + G_iB_iK & -G_i - G_i' & * \\ B_i'F_i' + JC_i - HK & B_i'G_i' & -H - H' \end{bmatrix} < 0 \qquad (2.15)$$

*for $\forall i \in \mathbb{K}_N$, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -G_i - G_i' - G_j - G_j' & * \\ B_i'F_j' + B_j'F_i' + JC_i + JC_j - 2HK & B_i'G_j' + B_j'G_i' & -2H - 2H' \end{bmatrix} < 0 \qquad (2.16)$$

*for $i = 1, 2, \ldots, N-1$ and $j = i+1, i+2, \ldots, N$ where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = A_i'F_j' + F_iA_j + K'B_i'F_j' + F_iB_jK + 2\gamma P_i + A_j'F_i' + F_jA_i \\ + K'B_j'F_i' + F_jB_iK + 2\gamma P_j \end{aligned} \qquad (2.17)$$

*and*

$$\Pi_{ij} + \Pi_{ji} = P_i - F_i' + G_iA_j + G_iB_jK + P_j - F_j' + G_jA_i + G_jB_iK \tag{2.18}$$

*in the affirmative case, the robust static output feedback gains is given by* $L = H^{-1}J$.

*Proof.* See Sereni (2019). □

# 3 Switched Control via SOF

In this chapter, a design methodology for switched controllers via static output feedback will be proposed considering uncertain linear time-invariant (LTI) systems defined in (2.3), where the plant matrix $A(\alpha) \in \mathbb{R}^{n\times n}$, and the input control matrix $B(\alpha) \in \mathbb{R}^{n\times m}$ and output matrix $C(\alpha) \in \mathbb{R}^{p\times n}$ are uncertain matrices that describe the system's dynamics, and can be represented in the polytopic domain $\mathfrak{D}$ defined in (2.4).

Furthermore, $\mathfrak{D}$ is parameterized in terms of a vector $\alpha = (\alpha_1, ..., \alpha_N)$, whose parameters $\alpha$ are unknown constants belonging to the unitary simplex set $\wedge_N$, defined in (2.5) for $i \in \mathbb{K}_N$, where $\mathbb{K}_N$ is a set of positive integers $\{1, \ldots, N\}$.

Likewise, $\mathbb{K}_S$ is a set of positive integers $\{1, \ldots, S\}$, and the set of all vectors $\lambda = (\lambda_1, ..., \lambda_S)$ such that $1 \geq \lambda_k \geq 0$ and $\sum_{k=1}^{S} \lambda_k = 1$ is denoted by $\wedge_S$. At last, the convex combination of a set of matrices $(Q_1, \ldots, Q_S)$ is denoted by $Q(\lambda) = \sum_{k=1}^{S} \lambda_k Q_k$.

Supposing that the feedback loop is composed by the following control law, presented by Mainardi Júnior et al. (2015)

$$u(t) = L_\sigma y(t) \tag{3.1}$$

where $\sigma$ is the switching strategy defined by

$$\sigma = \arg\min_{k\in\mathbb{K}_S} (y'(t) Q_k y(t)) = \arg\min_{k\in\mathbb{K}_S} (x'(t) C'(\alpha) Q_k C(\alpha) x(t)) \tag{3.2}$$

where $Q_k$ are switching matrices and $L_\sigma$ is the switched gain, which is selected among a set of constant gains $L_k \in \mathbb{R}^{m\times p}$, $\forall k \in \mathbb{K}_S$, such that

$$L_\sigma \in \{L_1, L_2, \ldots, L_S\}, \tag{3.3}$$

therefore considering these terms, the system (2.3) in closed-loop assumes the form

$$\dot{x}(t) = [A(\alpha) + B(\alpha) L_\sigma C(\alpha)] x(t). \tag{3.4}$$

The objective is to find $L_k$ and $Q_k$, $\forall k \in \mathbb{K}_S$, such that when the gain $L_\sigma$ is selected according to (3.2), it asymptotically stabilizes (3.4).

## 3.1 Stabilization via Switched SOF

In this section it is proposed a design method for switched static output feedback controllers. This approach is based upon the strategy presented in Sereni (2019), Manesco (2013) and in Mehdi, Boukas and Bachelier (2004), which consists of a two-stage control design.

The state-feedback gain ($K$) is obtained in the first step as in the Section 2.4, and regarding Remark 1, and then used as an input parameter for composing the second step, where the switched static output feedback gains $L_k \in \mathbb{R}^{m\times p}$, for $k \in \mathbb{K}_S$, are designed as proposed in Theorem 5, which consists of sufficient LMI conditions for designing the desired controller.

However, in this method, considering the switching law (3.2), will at some point imply in a product between two parameter-dependent matrices in the mathematical formulation of this control problem, which can be exemplified with terms such as

$$T(\alpha)U(\alpha) \tag{3.5}$$

where $(T, U)(\alpha)$ are generic parameter-dependent matrices, will produce crossed-products between the $\alpha_i$ as evidenced bellow.

$$\begin{gathered}\textstyle\sum_{i=1}^{N}\alpha_i T_i \sum_{i=1}^{N}\alpha_i U_i = \sum_{i=1}^{N}\alpha_i \sum_{j=1}^{N}\alpha_j T_i U_j = \\ \alpha_1\alpha_1 T_1 U_1 + \alpha_1\alpha_2 T_1 U_2 + \dots + \alpha_1\alpha_N T_1 U_N + \dots \\ \alpha_2\alpha_1 T_2 U_1 + \alpha_2\alpha_2 T_2 U_2 + \dots + \alpha_2\alpha_N T_2 U_N + \dots \\ \alpha_N\alpha_1 T_N U_1 + \alpha_N\alpha_2 T_N U_2 + \dots + \alpha_N\alpha_N T_N U_N.\end{gathered} \tag{3.6}$$

In order to derive an equivalent representation for (3.6) one can observe the following property.

**Property 1.** *If the following LMIs*

$$\Upsilon_{ii} < 0,\ i = 1, 2, 3, \dots, N, \tag{3.7}$$

*and,*

$$\Upsilon_{ij} + \Upsilon_{ji} < 0\ ,\ 0 \le i < j \le N \tag{3.8}$$

*holds, then it is true that*

$$\sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j\Upsilon_{ij} < 0. \tag{3.9}$$

*Proof.* See Tanaka, Ikeda and Wang (1998). □

**Theorem 5.** *Assuming that there exists a state feedback gain $K$, such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P > 0, \tag{3.10}$$

$$Q_{0i} + C_i' Q(\lambda) C_i < 0 \tag{3.11}$$

*for* $\forall\, i \in \mathbb{K}_N$*, and*

$$Q_{0i} + C_i' Q(\lambda) C_j + Q_{0j} + C_j' Q(\lambda) C_i < 0 \tag{3.12}$$

*for* $i = 1, 2, \ldots, N-1$*,* $j = i+1, i+2, \ldots, N$*, and matrices* $F$*,* $G \in \mathbb{R}^{n\times n}$*,* $H \in \mathbb{R}^{m\times m}$ *and* $J_k \in \mathbb{R}^{m\times p}$ *such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK - Q_{0i} - C_i'Q_kC_i & * & * \\ P - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + J_kC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0, \tag{3.13}$$

*for* $\forall i \in \mathbb{K}_N$*,* $\forall\, k \in \mathbb{K}_S$*, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F_i' + J_k(C_i + C_j) - 2HK & (B_i' + B_j')G_i' & -2H - 2H' \end{bmatrix} < 0, \tag{3.14}$$

*for* $i = 1, 2, \ldots, N-1$*,* $j = i+1, i+2, \ldots, N$ *and* $\forall k \in \mathbb{K}_S$ *where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = {} & (A_i' + A_j')F' + F(A_i + A_j) + K'B_i'F' + FB_iK \\ & -Q_{0i} - C_i'Q_kC_j + K'B_j'F' + FB_jK - Q_{0j} - C_j'Q_kC_i \end{aligned} \tag{3.15}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = 2P - 2F' + G(A_i + A_j) + GB_iK + GB_jK \tag{3.16}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$*,* $\forall k \in \mathbb{K}_S$*.*

*Proof.* Assuming that (3.13) and (3.14) hold, we can see that $H$ is invertible, since according to Boyd et al. (1994), a non-symmetric matrix $M$ is invertible, if $M + M' < 0$.

Now, splitting the matrix in (3.14) as follows:

$$\begin{aligned} & \begin{bmatrix} \Omega_{ij} & * & * \\ \Pi_{ij} & -G - G' & * \\ B_i'F' + J_kC_i - HK & B_i'G' & -H - H' \end{bmatrix} \\ + & \begin{bmatrix} \Omega_{ji} & * & * \\ \Pi_{ji} & -G - G' & * \\ B_j'F' + J_kC_j - HK & B_j'G' & -H - H' \end{bmatrix} < 0 \end{aligned} \tag{3.17}$$

for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$, one can observe that (3.13), (3.17), (3.11) and (3.12) can be represented as $\Upsilon_{ii}$ and $\Upsilon_{ij} + \Upsilon_{ji}$, so regarding Property 1 and considering the switching law (3.2), we have

$$\begin{bmatrix} \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (A_i'F' + FA_i + K'B_i'F' + FB_iK - Q_{0i} - C_i'Q_\sigma C_j) \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (P - F' + GA_i + GB_iK) \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'F' + J_\sigma C_i - HK) \end{bmatrix.$$
$$\left. \begin{matrix} * & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-G - G') & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'G') & \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-H - H') \end{matrix} \right] < 0, \tag{3.18}$$

$$\sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (Q_{0i} + C_i'Q(\lambda)C_j) < 0. \tag{3.19}$$

Expanding the terms in (3.18) and (3.19) and regarding that $\sum_{i=1}^N \alpha_i = \sum_{j=1}^N \alpha_j = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) \\ P - F' + GA(\alpha) + GB(\alpha)K \\ B'(\alpha)F' + J_\sigma C(\alpha) - HK \end{bmatrix.$$
$$\left. \begin{matrix} * & * \\ -G - G' & * \\ B'(\alpha)G' & -H - H' \end{matrix} \right] < 0 \tag{3.20}$$

and

$$Q_0(\alpha) + C'(\alpha)Q(\lambda)C(\alpha) < 0. \tag{3.21}$$

Using the idea presented by Mehdi, Boukas and Bachelier (2004), pre- and post-multiplying (3.20) by $T_\sigma$ and $T_\sigma'$, where $T_\sigma$ is

$$T_\sigma = \begin{bmatrix} I & 0 & S_\sigma'(\alpha) \\ 0 & I & 0 \end{bmatrix} \tag{3.22}$$

it follows that

$$\begin{bmatrix} \psi_\sigma(\alpha) & \phi_\sigma(\alpha) \\ \phi_\sigma(\alpha) & -G - G' \end{bmatrix} < 0, \tag{3.23}$$

where

$$\begin{aligned} \psi_\sigma(\alpha) = A'(\alpha)F' + K'B'(\alpha)F' + S_\sigma'(\alpha)B'(\alpha)F' + S_\sigma'(\alpha)J_\sigma C(\alpha) - S_\sigma'(\alpha)HK \\ +FA(\alpha) + FB(\alpha)K + FB(\alpha)S_\sigma(\alpha) + C'(\alpha)J_\sigma'S_\sigma(\alpha) - K'H'S_\sigma(\alpha) \\ +S_\sigma'(\alpha)(-H - H')S_\sigma(\alpha) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) \end{aligned} \tag{3.24}$$

and,

$$\phi_\sigma(\alpha) = P - F + A'(\alpha)G' + K'B'(\alpha)G' + S_\sigma'(\alpha)B'(\alpha)G'. \tag{3.25}$$

Replacing $S_\sigma(\alpha) = H^{-1}J_\sigma C(\alpha) - K$, in (3.24) and (3.25), then $\psi(\alpha)$ and $\phi_\sigma(\alpha)$ can be rewritten as

$$\begin{aligned}\psi(\alpha) = A'(\alpha)F' + K'B'(\alpha)F' - C'(\alpha)Q_\sigma C(\alpha) + (C'(\alpha)J'_\sigma H'^{-1} - K')B'(\alpha)F' + FA(\alpha)\\ +(C'(\alpha)J'_\sigma H'^{-1} - K')J_\sigma C(\alpha) - (C'(\alpha)J'_\sigma H'^{-1} - K')HK + FB(\alpha)K - Q_0(\alpha)\\ +FB(\alpha)(H^{-1}J_\sigma C(\alpha) - K) + C'(\alpha)J'_\sigma(H^{-1}J_\sigma C(\alpha) - K) - K'H'(H^{-1}J_\sigma C(\alpha) - K)\\ +(C'(\alpha)J'_\sigma H'^{-1} - K')(-H - H')(H^{-1}J_\sigma C(\alpha) - K)\end{aligned} \tag{3.26}$$

and

$$\phi_\sigma(\alpha) = P - F + A'(\alpha)G' + K'B'(\alpha)G' + (C'(\alpha)J'_\sigma H'^{-1} - K')B'(\alpha)G'. \tag{3.27}$$

Expanding the products in (3.26)

$$\begin{aligned}\psi_\sigma(\alpha) = A'(\alpha)F' + C'(\alpha)J'_\sigma H'^{-1}B'(\alpha)F' + C'(\alpha)J'_\sigma H'^{-1}J_\sigma C(\alpha)\\ -K'J_\sigma C(\alpha) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) - C'(\alpha)J'_\sigma H'^{-1}HK + FA(\alpha)\\ +FB(\alpha)H^{-1}J_\sigma C(\alpha) + C'(\alpha)J'_\sigma H^{-1}J_\sigma C(\alpha) - C'(\alpha)J'_\sigma K - K'H'H^{-1}J_\sigma C(\alpha)\\ -C'(\alpha)J'_\sigma H'^{-1}HH^{-1}J_\sigma C(\alpha) + C'(\alpha)J'_\sigma H'^{-1}HK - C'(\alpha)J'_\sigma H'^{-1}H'H^{-1}J_\sigma\\ +C'(\alpha)J'_\sigma H'^{-1}H'K + K'HH^{-1}J_\sigma C(\alpha) + K'H'H^{-1}J_\sigma C(\alpha).\end{aligned} \tag{3.28}$$

Considering the following equivalent relation

$$H^{-1}H = HH^{-1} = I = H'^{-1}H' = H'H'^{-1}, \tag{3.29}$$

then, (3.28) assumes the below form

$$\begin{aligned}\psi_\sigma(\alpha) = (A(\alpha) + B(\alpha)H^{-1}J_\sigma C(\alpha))'F' + F(A(\alpha) + B(\alpha)H^{-1}J_\sigma C(\alpha))\\ -Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha).\end{aligned} \tag{3.30}$$

Now, making $L_\sigma = H^{-1}J_\sigma$ in (3.27) and (3.30), we obtain

$$\psi_\sigma(\alpha) = (A(\alpha) + B(\alpha)L_\sigma C(\alpha))'F' + F(A(\alpha) + B(\alpha)L_\sigma C(\alpha)) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) \tag{3.31}$$

and,

$$\phi_\sigma(\alpha) = P - F + (A(\alpha) + B(\alpha)L_\sigma C(\alpha))'G'. \tag{3.32}$$

Considering (3.31) and (3.32), we can rewrite (3.23) in terms of a sum of matrices as follows:

$$\begin{aligned}\begin{bmatrix} -Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) + F(A(\alpha) + B(\alpha)L_\sigma C(\alpha)) & P - F \\ P + G(A(\alpha) + B(\alpha)L_\sigma C(\alpha)) & -G \end{bmatrix}\\ + \begin{bmatrix} (A(\alpha) + B(\alpha)L_\sigma C(\alpha))'F' & (A(\alpha) + B(\alpha)L_\sigma C(\alpha))'G' \\ -F' & -G' \end{bmatrix} < 0\end{aligned} \tag{3.33}$$

then, splitting the first matrix in (3.33), and rearranging properly, we have

$$\begin{aligned}\begin{bmatrix}-Q_0(\alpha)-C'(\alpha)Q_\sigma C(\alpha) & P\\ P & 0\end{bmatrix}+\begin{bmatrix}F\\G\end{bmatrix}\begin{bmatrix}(A(\alpha)+B(\alpha)L_\sigma C(\alpha)) & -I\end{bmatrix}\\ +\begin{bmatrix}(A(\alpha)+B(\alpha)L_\sigma C(\alpha))'\\ -I\end{bmatrix}\begin{bmatrix}F' & G'\end{bmatrix}<0.\end{aligned} \tag{3.34}$$

Considering the following definitions:

$$\begin{bmatrix}F\\G\end{bmatrix}=\begin{bmatrix}\mathcal{X}_1\\ \mathcal{X}_2\end{bmatrix}=\mathcal{X}, \tag{3.35}$$

$$\mathcal{S}_\sigma(\alpha)=\begin{bmatrix}-Q_0(\alpha)-C'(\alpha)Q_\sigma C(\alpha) & P\\ P & 0\end{bmatrix} \tag{3.36}$$

and

$$\mathcal{R}_\sigma(\alpha)=\begin{bmatrix}(A(\alpha)+B(\alpha)L_\sigma C(\alpha)) & -I\end{bmatrix} \tag{3.37}$$

we can rewrite (3.34) as

$$\exists\mathcal{X}\in\mathbb{R}^{2n\times n}, \mathcal{S}_\sigma(\alpha)+\mathcal{X}\mathcal{R}_\sigma(\alpha)+\mathcal{R}'_\sigma(\alpha)\mathcal{X}'<0. \tag{3.38}$$

Futhermore, note that (3.38) corresponds to the condition, (iv), of Finsler's Lemma as stated on Lemma 1. Thus, considering the condition, (i), of Finsler's Lemma, which is $\mathcal{W}'\mathcal{S}_\sigma(\alpha)\mathcal{W}<0$, $\forall\mathcal{W}\neq 0$, $\mathcal{R}_\sigma(\alpha)\mathcal{W}=0$, and assuming that $\mathcal{W}=\begin{bmatrix}x'(t) & \dot{x}'(t)\end{bmatrix}'$, we can derive regarding (3.37), (3.36) and (3.35) the following expressions:

$$\begin{bmatrix}(A(\alpha)+B(\alpha)L(\alpha)C(\alpha)) & -I\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}=0 \tag{3.39}$$

$$\begin{bmatrix}x'(t) & \dot{x}'(t)\end{bmatrix}\begin{bmatrix}-Q_0(\alpha)-C'(\alpha)Q_\sigma C(\alpha) & P\\ P & 0\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix}<0. \tag{3.40}$$

Therefore, according to (3.39) we have

$$\dot{x}(t)=(A(\alpha)+B(\alpha)L_\sigma C(\alpha))x(t), \tag{3.41}$$

which corresponds to the closed-loop system equation (3.4).

Furthermore, (3.40) leads to

$$\dot{x}'(t)Px(t)+x'(t)P\dot{x}(t)<x'(t)(Q_0(\alpha)+C'(\alpha)Q_\sigma C(\alpha))x(t). \tag{3.42}$$

Making $V(x(t))=x'(t)Px(t)$ and rearranging, we can conclude that (3.42) becomes

$$\dot{V}(x(t))<x'(t)(Q_0(\alpha)+C'(\alpha)Q_\sigma C(\alpha))x(t) \tag{3.43}$$

Pre- and post-multiplying (3.21) by $x'(t)$ and $x(t)$, we have by initial assumption that

$$x'(t)(Q_0(\alpha)+C'(\alpha)Q(\lambda)C(\alpha))x(t)<0. \tag{3.44}$$

As shown in Souza et al. (2013), the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers. Therefore, we have that

$$\begin{aligned} x'(t)(Q_0(\alpha)+C'(\alpha)Q_\sigma C(\alpha))x(t) &= \min_{\forall k\in\mathbb{K}_S}(x'(t)(Q_0(\alpha)+C'(\alpha)Q_k C(\alpha))x(t)) \\ &\leq x'(t)(Q_0(\alpha)+C'(\alpha)Q(\lambda)C(\alpha))x(t)<0 \end{aligned} \tag{3.45}$$

so, finally, we can conclude that

$$\dot{V}(x(t))<0, \tag{3.46}$$

for $x(t)\neq 0$, which is Lyapunov's equation for stability (BOYD et al., 1994). □

## 3.2 Decay Rate Bounding in Switched Output Feedback Controller

Some systems may demand, besides stability, that also some transient performance requirements are meet. In that case, we see that Theorem 5, presented in Section 3.1, is not able to properly address this issue. For ensuring such control requirements we might consider the minimum decay rate.

In this section, we apply the concept of the minimum decay rate on the Theorem 5 presented in Section 3.1, in the both of the stages of the proposed method. For avoiding confusion, the minimum decay rate associated to each design stage is referred to as $\beta$ and $\gamma$, respectively.

The state-feedback gain ($K$) obtained in the first step as described in Section 2.5 is then used as an input parameter for the composing the second step, where the switched output feedback gains $L_k\in\mathbb{R}^{m\times p}$, for $k\in\mathbb{K}_S$, are designed according to Theorem 6, in which sufficient LMI conditions for deriving the desired controller are proposed.

**Theorem 6.** *Assuming that there exists a state feedback gain $K$, such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha)+B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a minimum decay rate higher than or equal to $\gamma>0$, considering the switching rule* (3.2)*, if there exist $\lambda\in\wedge_S$ and symmetric matrices, $P$, $Q_{0i}\in\mathbb{R}^{n\times n}$ and $Q_k\in\mathbb{R}^{p\times p}$, with*

$$P>0, \tag{3.47}$$

$$Q_{0i}+C_i'Q(\lambda)C_i<0 \tag{3.48}$$

*for $\forall\, i\in\mathbb{K}_N$, and*

$$Q_{0i}+C_i'Q(\lambda)C_j+Q_{0j}+C_j'Q(\lambda)C_i<0 \tag{3.49}$$

*for* $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, *and matrices* $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ *and* $J_k \in \mathbb{R}^{m\times p}$ *such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK - Q_{0i} + 2\gamma P - C_i'Q_kC_i & * & * \\ P - F' + GA_i + GB_iK & -G-G' & * \\ B_i'F' + J_kC_i - HK & B_i'G' & -H-H' \end{bmatrix} < 0, \tag{3.50}$$

*for* $\forall\, i \in \mathbb{K}_N$, $\forall\, k \in \mathbb{K}_S$, *and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F_i' + J_k(C_i + C_j) - 2HK & (B_i' + B_j')G_i' & -2H - 2H' \end{bmatrix} < 0, \tag{3.51}$$

*for* $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ *and* $\forall k \in \mathbb{K}_S$ *where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = {} & (A_i' + A_j')F' + F(A_i + A_j) + K'B_i'F' + FB_iK + 4\gamma P \\ & -Q_{0i} - C_i'Q_kC_j + K'B_j'F' + FB_jK - Q_{0j} - C_j'Q_kC_i \end{aligned} \tag{3.52}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = 2P - 2F' + G(A_i + A_j) + GB_iK + GB_jK \tag{3.53}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.

*Proof.* Given the similarity between the Theorem 5 and 6, and considering the Lemma 1 (Finsler's Lemma), we have such that

$$\begin{bmatrix} (A(\alpha) + B(\alpha)L_\sigma C(\alpha)) & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = 0 \tag{3.54}$$

$$\begin{bmatrix} x'(t) & \dot{x}'(t) \end{bmatrix} \begin{bmatrix} 2\gamma P - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0. \tag{3.55}$$

Therefore, according to (3.54) we have

$$\dot{x}(t) = (A(\alpha) + B(\alpha)L_\sigma C(\alpha))x(t), \tag{3.56}$$

which corresponds to the closed-loop system equation (3.4).

Furthermore, (3.55) leads to

$$\dot{x}'(t)Px(t) + x'(t)P\dot{x}(t) < x'(t)(Q_0(\alpha) + C'(\alpha)Q_\sigma C(\alpha) - 2\gamma P)x(t). \tag{3.57}$$

Making $V(x(t)) = x'(t)Px(t)$ and rearranging, we can conclude that (3.57) becomes

$$\dot{V}(x(t)) + 2\gamma V(x(t)) < x'(t)(Q_0(\alpha) + C'(\alpha)Q_\sigma C(\alpha))x(t). \tag{3.58}$$

Pre- and post-multiplying (3.48) by $x'(t)$ and $x(t)$, we have by initial assumption that

$$x'(t)(Q_0(\alpha) + C'(\alpha)Q(\alpha)C(\alpha))x(t) < 0. \tag{3.59}$$

As shown in Souza et al. (2013), the minimum of a set of real numbers is less than or equal to an arbitrary convex combination of these numbers. Therefore, we have that

$$\begin{aligned} x'(t)(Q_0(\alpha) + C'(\alpha)Q_\sigma C(\alpha))x(t) &= \min_{\forall k \in \mathbb{K}_S} (x'(t)(Q_0(\alpha) + C(\alpha)'Q_k C(\alpha))x(t)) \\ &\leq x'(t)(Q_0(\alpha) + C'(\alpha)Q(\lambda)C(\alpha))x(t) < 0 \end{aligned} \tag{3.60}$$

then we can affirm that

$$x'(t)(Q_0(\alpha) + C'(\alpha)Q_\sigma C(\alpha))x(t) < 0 \tag{3.61}$$

so, finally, we can conclude that

$$\dot{V}(x(t)) < -2\gamma V(x(t)), \tag{3.62}$$

for $x(t) \neq 0$, which is Lyapunov's equation for stability, considering the minimum decay rate as defined in (2.2) (BOYD et al., 1994).

$\square$

## 3.3 Relaxation Strategies for the Design of Switched SOF Controllers

In this section, the development of more relaxed LMI conditions for obtaining the switched SOF controller is presented. The conservatism in the switched SOF control problem can be reduced by considering the use of a parameter-dependent Lyapunov matrix, and parameter-dependent Finsler Lemma's additional variables.

### 3.3.1 Parameter-dependent Lyapunov Functions

The strategy in Sections 3.1 and 3.2 is based on the existence of a common quadratic Lyapunov function (CQLF), i.e. the Lyapunov's matrix $P$ is assumed to be fixed in Theorems 5 and 6. As discussed in Chapter 1, CDLF is an efficient way to solve many convex optimization problems, however its use has restrictive effects when dealing with uncertain systems (OLIVEIRA; PERES, 2006).

With the intend of obtaining more relaxed LMI conditions, it is proposed a formulation based on parameter-dependent Lyapunov functions (PDLFs). Basically, the Lyapunov's matrix $P$ is considered to be dependent on the system's uncertain parameter $\alpha$, and thus, belongs to the unitary simplex given by (2.5), as proposed in Theorem 7.

**Theorem 7.** *Assuming that there exists a state feedback gain $K$, such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha)+B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a decay rate higher than or equal to $\gamma > 0$, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P_i$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P_i > 0, \tag{3.63}$$

$$Q_{0i} + C_i' Q(\lambda) C_i < 0 \tag{3.64}$$

*for $\forall\, i \in \mathbb{K}_N$, and*

$$Q_{0i} + C_i' Q(\lambda) C_j + Q_{0j} + C_j' Q(\lambda) C_i < 0 \tag{3.65}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J_k \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K'B_i'F' + FB_iK - Q_{0i} + 2\gamma P_i - C_i'Q_kC_i & * & * \\ P_i - F' + GA_i + GB_iK & -G - G' & * \\ B_i'F' + J_kC_i - HK & B_i'G' & -H - H' \end{bmatrix} < 0, \tag{3.66}$$

*for $\forall\, i \in \mathbb{K}_N$, $\forall\, k \in \mathbb{K}_S$, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F_i' + J_k(C_i + C_j) - 2HK & (B_i' + B_j')G_i' & -2H - 2H' \end{bmatrix} < 0, \tag{3.67}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$ where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = {} & (A_i' + A_j')F' + F(A_i + A_j) + K'B_i'F' + FB_iK + 2\gamma P_i \\ & + 2\gamma P_j - Q_{0i} - C_i'Q_kC_j + K'B_j'F' + FB_jK - Q_{0j} - C_j'Q_kC_i \end{aligned} \tag{3.68}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = P_i + P_j - 2F' + G(A_i + A_j) + GB_iK + GB_jK. \tag{3.69}$$

*In the affirmative case, the switched output feedback gains are given by $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.*

*Proof.* As mentioned in Theorem's 5 demonstration, the matrix $H$ is invertible if (3.66) and (3.67) have a solution.

Regarding Property 1 and considering the switching law (3.2), we have

$$\begin{bmatrix} \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (A_i'F' + FA_i + K'B_i'F' + FB_iK + 2\gamma P_i - Q_{0i} - C_i'Q_\sigma C_j) & * & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (P_i - F' + GA_i + GB_iK) & \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-G - G') & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'F' + J_\sigma C_i - HK) & \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'G') & \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-H - H') \end{bmatrix} < 0, \tag{3.70}$$

$$\sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j (Q_{0i} + C_i' Q(\lambda) C_j) < 0. \tag{3.71}$$

Expanding the terms in (3.70) and (3.71) and regarding that $\sum_{i=1}^{N} \alpha_i = \sum_{j=1}^{N} \alpha_j = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F' + FA(\alpha) + K'B'(\alpha)F' + FB(\alpha)K + 2\gamma P(\alpha) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) & * & * \\ P(\alpha) - F' + GA(\alpha) + GB(\alpha)K & -G - G' & * \\ B'(\alpha)F' + J_\sigma C(\alpha) - HK & B'(\alpha)G' & -H - H' \end{bmatrix} < 0 \tag{3.72}$$

and

$$Q_0(\alpha) + C'(\alpha)Q(\lambda)C(\alpha) < 0. \tag{3.73}$$

The remaining part of the proof follows similarly as for Theorem 6, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (2.2). □

### 3.3.2 Parameter-Dependent Finsler's Variables

In order to obtain even less conservative conditions for the design of Switched SOF controllers, a second relaxation strategy is proposed in this work. In this new approach, the additional variables introduced by Finsler's Lemma are considered dependent on the uncertain parameter $\alpha$, as is the Lyapunov matrix $(P(\alpha))$.

Therefore, the Finsler's Lemma additional variables are assumed to be defined as

$$\begin{bmatrix} F(\alpha) \\ G(\alpha) \end{bmatrix} = \begin{bmatrix} \mathcal{X}_1(\alpha) \\ \mathcal{X}_2(\alpha) \end{bmatrix} = \mathcal{X}(\alpha) \tag{3.74}$$

In Theorem 8, based on Sereni (2019), the new and more relaxed LMI conditions for the design of Switched SOF controllers are proposed, as a result of the assumption made in (3.74).

**Theorem 8.** *Assuming that there exists a state feedback gain $K$, such that $A(\alpha)+B(\alpha)K$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a decay rate higher than $\gamma > 0$, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P_i$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P_i > 0, \tag{3.75}$$

$$Q_{0i} + C_i'Q(\lambda)C_i < 0 \tag{3.76}$$

*for* $\forall\, i \in \mathbb{K}_N$*, and*

$$Q_{0i} + C_i'Q(\lambda)C_j + Q_{0j} + C_j'Q(\lambda)C_i < 0 \tag{3.77}$$

*for* $i = 1, 2, \ldots, N-1$*,* $j = i+1, i+2, \ldots, N$*, and matrices* $F_i$*,* $G_i \in \mathbb{R}^{n\times n}$*,* $H \in \mathbb{R}^{m\times m}$ *and* $J_k \in \mathbb{R}^{m\times p}$ *such that,*

$$\begin{bmatrix} A_i'F_i' + F_iA_i + K'B_i'F_i' + F_iB_iK + 2\gamma P_i - Q_{0i} - C_i'Q_kC_i & * & * \\ P_i - F_i' + G_iA_i + G_iB_iK & -G_i - G_i' & * \\ B_i'F_i' + J_kC_i - HK & B_i'G_i' & -H - H' \end{bmatrix} < 0 \tag{3.78}$$

*and* $\forall\, i \in \mathbb{K}_N$ *and* $\forall k \in \mathbb{K}_S$*, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -G_i - G_i' - G_j - G_j' & * \\ B_i'F_j' + B_j'F_i' + J_kC_i + J_kC_j - 2HK & B_i'G_j' + B_j'G_i' & -2H - 2H' \end{bmatrix} < 0, \tag{3.79}$$

*for* $i = 1, 2, \ldots, N-1$*,* $j = i+1, i+2, \ldots, N$ *and* $\forall k \in \mathbb{K}_S$ *where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = A_i'F_j' + F_iA_j + K'B_i'F_j' + F_iB_jK + 2\gamma P_i - Q_{0i} \\ + A_j'F_i' + F_jA_i + K'B_j'F_i' + F_jB_iK + 2\gamma P_j - Q_{0j} - C_j'Q_kC_i - C_i'Q_kC_j \end{aligned} \tag{3.80}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = P_i - F_i' + G_iA_j + G_iB_jK + P_j - F_j' + G_jA_i + G_jB_iK \tag{3.81}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$*,* $\forall k \in \mathbb{K}_S$*.*

*Proof.* As mentioned in Theorem's 5 demonstration, the matrix $H$ is invertible if (3.75), (3.78), (3.76) and (3.79) have a solution.

Now, splitting the matrix in (3.79)

$$\begin{aligned} &\begin{bmatrix} \Omega_{ij} & * & * \\ \Pi_{ij} & -G_i - G_i' & * \\ B_i'F_j' + J_kC_i - HK & B_i'G_j' & -H - H' \end{bmatrix} \\ + &\begin{bmatrix} \Omega_{ji} & * & * \\ \Pi_{ji} & -G_j - G_j' & * \\ B_j'F_i' + J_kC_j - HK & B_j'G_i' & -H - H' \end{bmatrix} < 0 \end{aligned} \tag{3.82}$$

for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$,

Regarding Property 1 and considering the switching law (3.2), we have

$$\begin{bmatrix} \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(A_i'F_j'+F_iA_j+K'B_i'F_j'+F_iB_jK+2\gamma P_i-Q_{0i}-C_i'Q_\sigma C_j) & * & * \\ \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(P_i-F_j'+G_iA_j+G_iB_jK) & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(-G_i-G_i') & * \\ \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(B_i'F_j'+J_\sigma C_i-HK) & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(B_i'G_j') & \sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j(-H-H') \end{bmatrix} < 0, \quad (3.83)$$

$$\sum_{i=1}^{N}\alpha_i(Q_{0i}+C_i'Q(\lambda)C_j) < 0. \quad (3.84)$$

Then, developing the terms in (3.83) and (3.84), and regarding that $\sum_{i=1}^{N}\alpha_i = \sum_{j=1}^{N}\alpha_j = 1$, we obtain

$$\begin{bmatrix} A'(\alpha)F'(\alpha)+F(\alpha)A(\alpha)+K'B'(\alpha)F'(\alpha)+F(\alpha)B(\alpha)K+2\gamma P(\alpha)-Q_0(\alpha)-C'(\alpha)Q_\sigma C(\alpha) & * & * \\ P(\alpha)-F'(\alpha)+G(\alpha)A(\alpha)+G(\alpha)B(\alpha)K & -G(\alpha)-G'(\alpha) & * \\ B'(\alpha)F'(\alpha)+J_\sigma C(\alpha)-HK & B'(\alpha)G'(\alpha) & -H-H' \end{bmatrix} < 0 \quad (3.85)$$

and

$$Q_0(\alpha)+C'(\alpha)Q(\lambda)C(\alpha) < 0. \quad (3.86)$$

The remaining part of the proof follows similarly as Theorem's 6, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (2.2). □

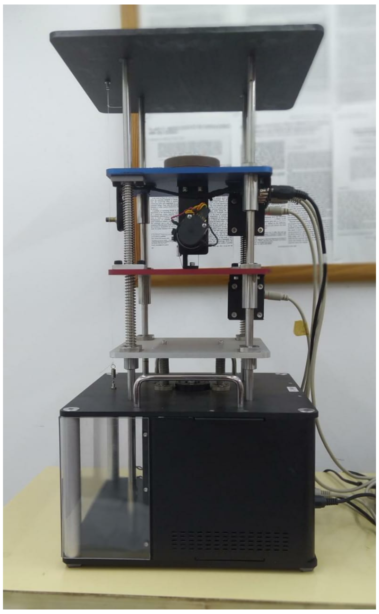
## 3.4 Analysis

Intending to investigate the efficiency of the proposed theorems and to compare with the strategy proposed in Manesco (2013), some practical experiments were executed. In this section, feasibility and performance analysis are presented using the active suspension system.

### 3.4.1 Active Suspension System

Intending to study the practical applicability of the proposed method, the Active Suspension System is presented, which represents a quarter-car model, formed by three parts: the vehicle body, that is suspended over the tire assembly by springs and the active suspension mechanism. The tire assembly is in contact with the tire through springs, and the tire has to be able to pass through different types of terrain without the compromising of the passenger's comfort. In the performed experiment the equipment used was a Quanser® Active Suspension system as shown in Figure 1, in which the parts of the car are replaced by plates, or floors, and the active suspension mechanism is emulated by a DC motor, and the road profiles are simulated by another motor DC (QUANSER, 2009).

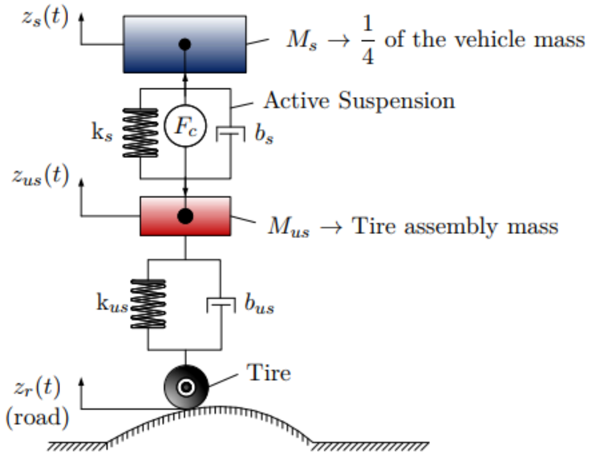
Figure 1 – QUANSER® Active Suspension. Property of the Laboratory of Research in Control at FEIS-UNESP.



**Source:** Own Autor.

In Figure 2 the schematic diagram of the system is presented, where $M_s$ is representing 1/4 of the vehicle body mass, $M_{us}$ is acting as the mass of the tire, and $k_s$, $k_{us}$, $b_s$ and $b_{us}$, are the springs and dampers in the model assembly. $z_s(t)$ and $z_{us}(t)$ are the related position of the body floor and tire assembly floor. Finally, $z_r(t)$ is the input of the system, which represents as the surface profile of the road, and $F_c(t)$ is the active suspension control command (SERENI, 2019).

Figure 2 – Schematic diagram of an active suspension system.




**Source:** Adapted from Silva (2012).

The system illustrated in the Figure 2 can be described in a state-space model presented by QUANSER (2009) as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-k_s}{M_s} & \frac{-b_s}{M_s} & 0 & \frac{b_s}{M_s} \\ 0 & 0 & 0 & -1 \\ \frac{k_s}{M_{us}} & \frac{b_s}{M_{us}} & \frac{-k_{us}}{M_{us}} & \frac{-(b_s+b_{us})}{M_{us}} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho}{M_s} \\ 0 \\ \frac{-\rho}{M_{us}} \end{bmatrix} u(t), \tag{3.87}$$

where $0 < \rho \leq 1$ is an uncertain parameter that acts as a possible fault in the actuator. Furthermore, the state and the input vectors in (3.87) are defined as

$$\dot{x}(t) = \begin{bmatrix} z_s(t) - z_{us}(t) \\ \dot{z}_s(t) \\ z_{us}(t) - z_r(t) \\ \dot{z}_{us}(t) \end{bmatrix} \tag{3.88}$$

And using the parameters presented in Table 1 to work on the experiments (QUANSER, 2009).

Table 1 – Active Suspension Parameters.

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $M_s$ | 2.45 Kg | $M_{us}$ | 1.0 Kg |
| $k_s$ | 900 N/m | $k_{us}$ | 2500 N/m |
| $b_s$ | 7.5 Ns/m | $b_{us}$ | 5.0 Ns/m |

**Source:** Adapted from QUANSER (2009).

**Experiment 3.1.** In this experiment we consider that the Active Suspension System may present a fault of up to 75% power loss $(i.e. \, 0.25 \leq \rho \leq 1)$. Also, it is supposed that only the measurement of $(z_s(t) - z_{us}(t))$ and $\dot{z}_s(t)$ are available. Regarding that the control input signal is $F_c(t)$, the active suspension may be described in terms of a polytope with two vertices:

- Vertex 1 (without fault)

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.408 \\ 0 \\ -1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t), \end{aligned} \tag{3.89}$$

- Vertex 2 (with fault - 75% of power loss)

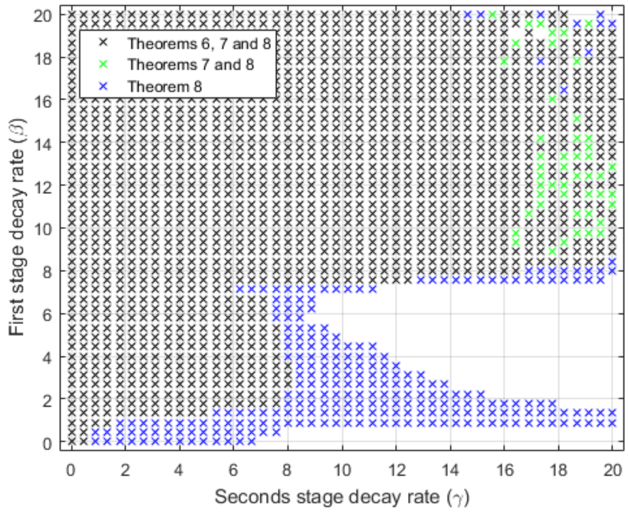
$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.102 \\ 0 \\ -0.25 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) \end{aligned} \tag{3.90}$$

The objective of this experiment is compare the feasibility of the Theorem 6, which is based on a fixed matrix $P$, Theorem 7, which considers the matrix $P$ as dependent on the uncertain parameter $\alpha$ and, Theorem 8, which considers the Finsler's variables $(F$ and $G)$ as dependent on the uncertain parameter $\alpha$, considering the specification of a minimum decay in both stages.

This analysis was performed considering a range for the minimum decay rates in the first and second stage of the project as $0 \leq \beta \leq 20$ and $0 \leq \gamma \leq 20$, and requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

Programming the LMIs via MATLAB® software, and solving via YALMIP interface (LOFBERG, 2004) and the solver SeDuMi (STURM, 1998), one can obtain the results presented in Figure 3, which represents the feasibility region obtained by each of the three considered theorems.

Figure 3 – Feasibility region obtained with Theorems 6, 7 and 8 changing the minimum decay rate.




**Source:** Own Autor.

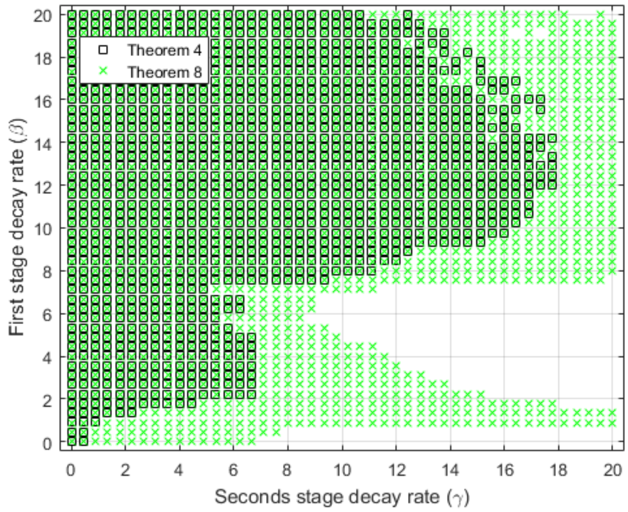
In the Figure 3 is possible to observe that Theorem 7, which considers the matrix $P(\alpha)$, have a slightly better feasibility of the theorem 6, considering the specification of a minimum decay. Also, Figure 3 shows that Theorem 8 provides an even larger result than Theorem 7.

**Experiment 3.2.** An additional analysis with same conditions of the Experiment 3.1 is compare the feasibility of Theorems 4 and 8, considering the specification of a minimum decay in both stages. This experiment was performed considering a range for the minimum decay rates in the first and second stage of the project as $0 \leq \beta \leq 20$ and $0 \leq \gamma \leq 20$, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

The conditions were solved similarly as in previous analysis, we can obtain the results presented in Figure 4, which represents the feasibility region obtained by each of the two considered theorems.

In the Figure 4 is possible to observe that Theorem 8, which is the switched SOF that considers the Finsler's variables $(F$ and $G)$ as dependent on the uncertain parameter $\alpha$, has a have a better feasibility than Theorem 4, which is the robust SOF that contemplate $F(\alpha)$ and $G(\alpha)$, considering the specification of a minimum decay, this must have occurred due to the use of switching matrices $Q_k$ and matrices $Q_{0i}$ which brought a larger area of feasibility, making the switched SOF more feasible.

Figure 4 – Feasibility region obtained with Theorems 4 and 8 changing the minimum decay rate.




**Source:** Own Autor.

**Experiment 3.3.** In this experiment the Active Suspension System will be examined from the politopic uncertain perspective, evaluating the performance of the Theorems 6, 7 and 8 when the power loss and vehicle body mass may have an uncertain, thus the system can be represented in therms of convex combination of the following vertices

- Vertex 1 (without fault and vehicle maximum mass)

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-900}{M_{max}} & \frac{-7.5}{M_{max}} & 0 & \frac{7.5}{M_{max}} \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{M_{max}} \\ 0 \\ -1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t),\end{aligned} \tag{3.91}$$

- Vertex 2 (with fault and vehicle maximum mass)

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-900}{M_{max}} & \frac{-7.5}{M_{max}} & 0 & \frac{7.5}{M_{max}} \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho}{M_{max}} \\ 0 \\ -\rho \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)\end{aligned} \tag{3.92}$$

- Vertex 3 (without fault and vehicle minimum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-900}{M_s} & \frac{-7.5}{M_s} & 0 & \frac{7.5}{M_s} \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{M_s} \\ 0 \\ -1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t), \tag{3.93}$$

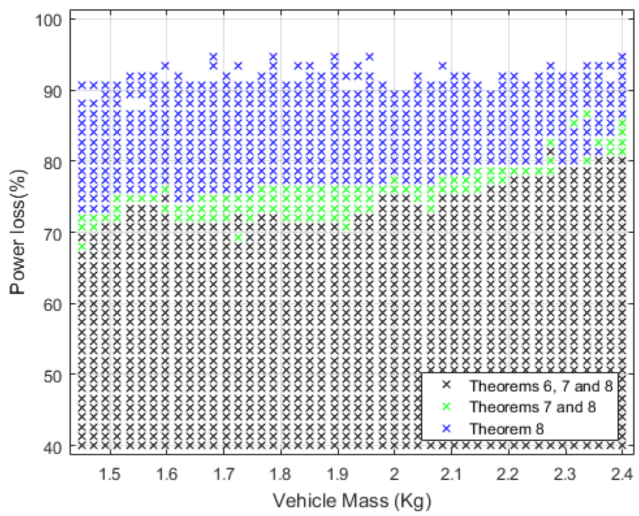
- Vertex 4 (with fault and vehicle minimum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ \frac{-900}{M_s} & \frac{-7.5}{M_s} & 0 & \frac{7.5}{M_s} \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{\rho}{M_s} \\ 0 \\ -\rho \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t) \tag{3.94}$$

where $M_{max} = 2.45$, $0 \leq \rho \leq 0.60$ and $1.45 \leq M_s \leq 2.4$.

The proposed LMIs were solved similarly as in previous analysis, but now fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 3$, respectively, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$. Regarding the values of $\rho$ and $M_s$ in the specified ranges, the obtained feasibility regions wit each theorem is presented in Figure 5.

Figure 5 – Feasibility region obtained with Theorems 6, 7 and 8 changing the polytope's vertex.

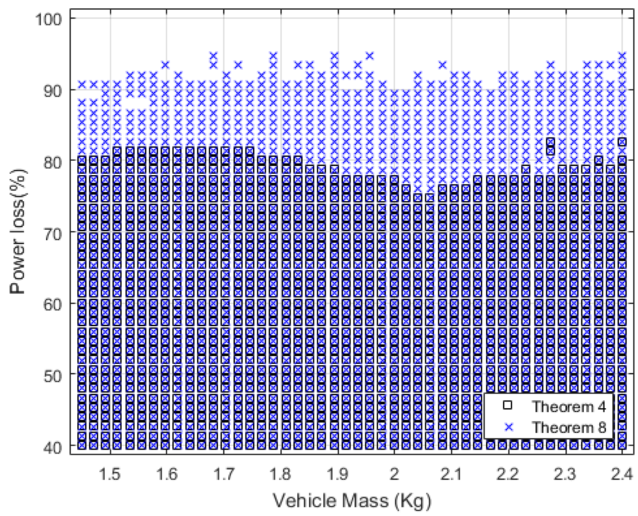



**Source:** Own Autor.

One can observe that by using Theorem 7, which considers the matrix $P$ as dependent on the uncertain parameter $\alpha$, results in a larger feasibility region when compared to Theorem 6, which is based on a fixed matrix $P$. Also, Figure 5 shows that Theorem 8 provides an even better result than Theorem 7.

**Experiment 3.4.** To compare the Theorems 4 and 8 this test with the same conditions from the Experiment 3.3. Regarding the values of power loss and $M_s$ in the specified ranges, the obtained feasibility regions from each theorem is presented in Figure 6.

Figure 6 – Feasibility region obtained with Theorems 4 and 8 changing the polytope's vertex.




**Source:** Own Autor.

In the Figure 6 is possible to observe that Theorem 8, which is the switched SOF that considers the Finsler's variables ($F$ and $G$) as dependent on the uncertain parameter $\alpha$, has a have a better feasibility than the Theorem 4, which is the robust SOF that contemplate $F(\alpha)$ and $G(\alpha)$, changing one of the polytope's vertex.

**Experiment 3.5.** Intending to compare the performance of Theorem 4 and the Theorem 8, in this experiment the Active Suspension System is considered to present a fault of up to 50% power loss (i.e. $0.5 \leq \rho \leq 1$) and vehicle body mass may vary between $1.455 \leq M_s \leq 2.45$. Also, it is suppose that only the measurement of state variable $\dot{z}_s(t)$ is available, the active suspension may be described in terms of a polytope with four vertices:

- Vertex 1 (without fault and vehicle maximum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.408 \\ 0 \\ -1 \end{bmatrix} u(t), \tag{3.95}$$
$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t),$$

- Vertex 2 (with fault and vehicle maximum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -367.347 & -3.061 & 0 & 3.061 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.204 \\ 0 \\ -0.5 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \tag{3.96}$$

- Vertex 3 (without fault and vehicle minimum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.557 & -5.155 & 0 & 5.155 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.687 \\ 0 \\ -1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t), \tag{3.97}$$

- Vertex 4 (with fault and vehicle minimum mass)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -618.557 & -5.155 & 0 & 5.155 \\ 0 & 0 & 0 & -1 \\ 900 & 7.5 & -2500 & -12.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.344 \\ 0 \\ -0.5 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} x(t) \tag{3.98}$$

The LMIs were solved similarly as in previous analysis, but now fixing the decay rates of the first and second stages of the project as $\beta = \gamma = 0.75$, we obtain the state feedback gain

$$K = \begin{bmatrix} -189.76 & -78.178 & -201.25 & -17.994 \end{bmatrix} \tag{3.99}$$

And, in the second stage, using the Theorem 4 proposed by Manesco (2013) and the first-stage state feedback gain, the designed static output feedback gain is

$$L_{Robust} = -109.94 \tag{3.100}$$

On its turn, conditions in Theorem 8, the first-stage state feedback gain and design two gains in second stage ($S = 2$), with $\lambda_1 = \lambda_2 = 0.5$, the designed switched static output feedback gains were

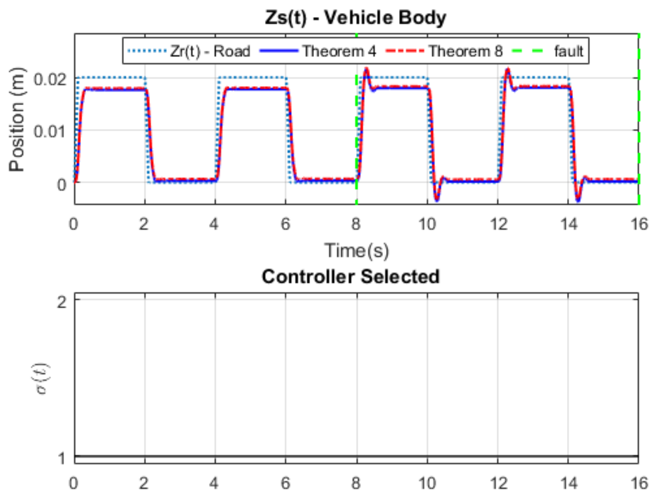
$$L_1 = -112.09481 \\ L_2 = -112.09479 \tag{3.101}$$

with $Q_1$ and $Q_2$:

$$Q_1 = -0.014098 \\ Q_2 = -0.014096 \tag{3.102}$$

The observed behavior of the active suspension system with the designed controllers is shown in Figure 7. During the experiment, the road $z_r(t)$ was set to square wave with 0.02 m in amplitude and a period of 4s. In the first 8 seconds of experiment, works in nominal conditions, then the system is subject to a 50% power loss in the actuator. It can be observed that the switched controller proposed in this work has a similar response, even when we have the experience the actuator failure.

Figure 7 – Active suspension system behavior using Theorem 4 and 8 controllers with fault: 50% power loss in the actuator $(8-16s)$, for $M_s = 2.45kg$ and $p = 1$.




**Source:** Own Autor.

However in the Figure 7 it is possible to see that there was no switching during this experiment. Due to the relationship between the size of the output matrix C and the size of the Q switching matrices which, as in this experiment, can be considered as scalar, there will be no switching along the process. Therefore, the number of outputs available for static output feedback must be considered for controller design, because if $p = 1$, the only matrix Q possible to be chosen will be the

$$\min_{\forall k \in \mathbb{K}_S} Q(k). \tag{3.103}$$

**Experiment 3.6.** To analyze whether there is improvement in behavior when switching is possible, we considered in this experiment the Active Suspension System is considered to present a fault of up to 50% power loss (i.e. $0.5 \leq \rho \leq 1$) and vehicle body mass may vary between $1.455 \leq M_s \leq 2.45$. Also, it is supposed that only the measurement of $(z_s(t) - z_{us}(t))$ and $\dot{z}_s(t)$ are available. Taking that into account, and the parameters presented in Table 1 (QUANSER, 2009), the active suspension may be described in terms of a polytope with four vertices where the matrices $A_i$ and $B_i$ are similar as in the Experiment 3.5 and the output matrix in both vertex is :

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.104}$$

The LMI (2.12) was solved similarly as in previous analysis, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0.75$, we obtain the same state feedback gain (3.99). And, in the second stage, using the LMIs in Theorem 4 and the first-stage state feedback gain, the designed static output feedback gain is

$$L_{Robust} = \begin{bmatrix} -197.99 & -89.124 \end{bmatrix} \tag{3.105}$$

Finally, in the second stage, using the LMIs in Theorem 8, the first-stage state feedback gain and design two gains in second stage $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$, the designed switched static output feedback gains were

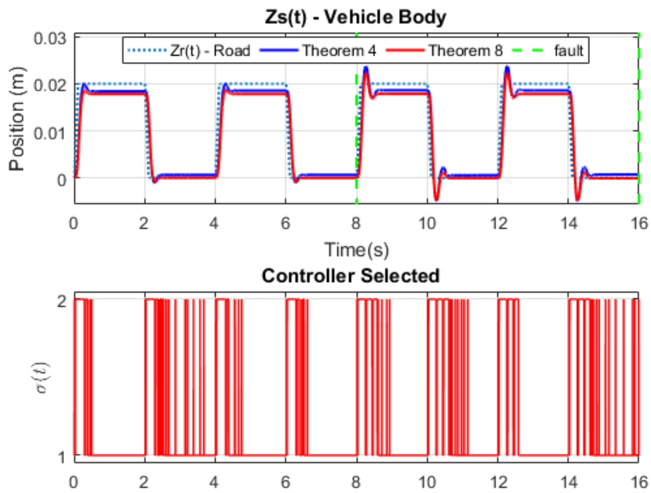
$$\begin{aligned} L_1 &= \begin{bmatrix} -218.8597 & -101.45606 \end{bmatrix} \\ L_2 &= \begin{bmatrix} -218.8676 & -101.45625 \end{bmatrix} \end{aligned} \tag{3.106}$$

with $Q_1$ and $Q_2$:

$$\begin{aligned} Q_1 &= \begin{bmatrix} 1.9571 & 0.41453 \\ 0.41453 & 0.099156 \end{bmatrix} \\ Q_2 &= \begin{bmatrix} 1.9577 & 0.41458 \\ 0.41458 & 0.099148 \end{bmatrix} \end{aligned} \tag{3.107}$$

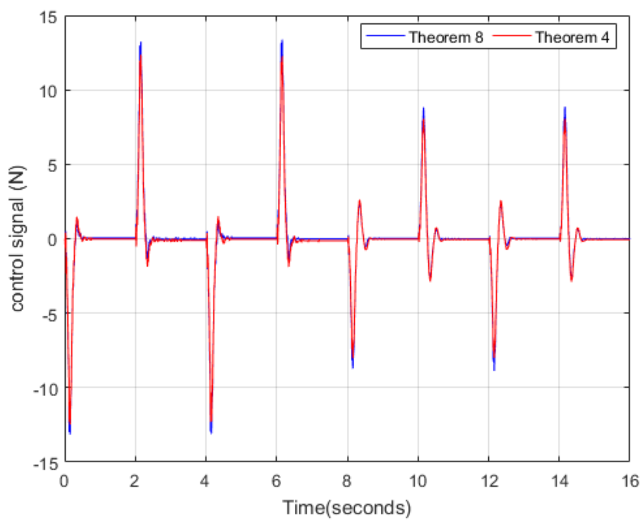
According to Figure 8 the designed switched controller from Theorem 8, which considers the Finsler's variables $(F$ and $G)$ as dependent on the uncertain parameter $\alpha$, offered an improvement when compared with the controllers from Theorem 4, which is the robust SOF that contemplate $F(\alpha)$ and $G(\alpha)$, the improvement can be seen in the values of overshoot and the suppression of the oscillations in the system, and it is possible to see that there was switching during this experiment. Figure 8 shows an increase in control command $F_c(t)$ with the switched SOF control. However the amplitude of the control signal is less than $20N$, which does not lead to the saturation of the control of the system.

Figure 8 – Active suspension system behavior using Theorem 4 and 8 controllers with fault: 50% power loss in the actuator $(8 - 16s)$, for $M_s = 2.45kg$ and $p = 2$.




**Source:** Own Autor.

Figure 9 – Control command $F_c(t)$ generated by the designed switched SOF controllers during the executed experiments with using Theorem 4 and 8 for $M_s = 2.45kg$.




**Source:** Own Autor.

**Experiment 3.7.** Another analysis were made aiming to evaluate the system performance with the controllers design from the Theorems 8 with different minimum decay rate, we considered in this experiment that the Active Suspension System may present the same conditions from the last experiment.

The LMI (2.12) was solved similarly as in previous analysis, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0.75$, we obtain the same state feedback gain (3.99). And, in the second stage, using the LMIs in Theorem 8 and the first-stage state feedback gain, the designed static output feedback gain are expressed in (3.106) and with the matrix $Q_1$ and $Q_2$ are described in (3.107).

Finally, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0$, we obtain the state feedback gain:

$$K = \begin{bmatrix} 86.351 - 31.163 - 55.768 - 8.104 \end{bmatrix} \tag{3.108}$$

and, in the second stage, using the LMIs in Theorem 8, and the first-stage state feedback gain, the designed switched static output feedback gain was

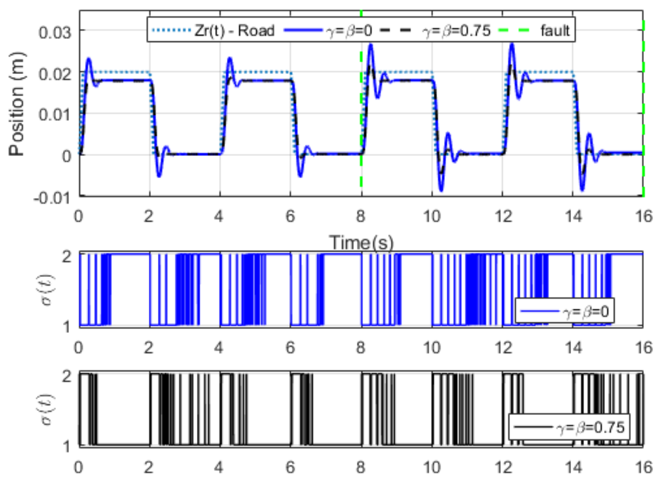
$$\begin{aligned} L_1 &= \begin{bmatrix} 87.36978 & -38.26167 \end{bmatrix} \\ L_2 &= \begin{bmatrix} 87.36933 & -38.2627 \end{bmatrix} \end{aligned} \tag{3.109}$$

with the matrix $Q_1$ and $Q_2$:

$$\begin{aligned} Q_1 &= \begin{bmatrix} -0.14426 & 0.41542 \\ 0.41542 & -0.0048393 \end{bmatrix} \\ Q_2 &= \begin{bmatrix} -0.14441 & 0.4154 \\ 0.4154 & -0.0047983 \end{bmatrix}. \end{aligned} \tag{3.110}$$

According to Figure 10 the controller with higher minimum decay rate a better dynamic response was achieved. Also, Figure 10 shows that the $\sigma$ presented less switching when compared to the controller with $\gamma = \beta = 0$.

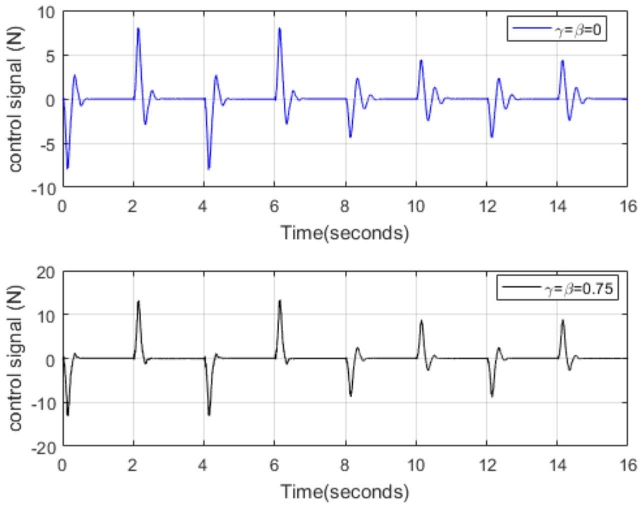
Figure 10 – Active suspension system behavior using Theorem 8 controllers with $\gamma = 0$ and $\gamma = 0.75$ fault: 50% power loss in the actuator $(8 - 16s)$, for $M_s = 2.45kg$.




**Source:** Own Autor.

Figure 11 shows that the control input $F_c(t)$ with $\beta = \gamma = 0.75$ presented higher amplitudes when compared to the results of controller with lower minimum decay rate.

Figure 11 – Control command $F_c(t)$ generated by the designed switched SOF controllers during the executed experiments with decay rates $\beta = \gamma = 0$ (top) and $\beta = \gamma = 0.75$ (bottom) for $M_s = 2.45kg$.




**Source:** Own Autor.

## 3.5 Partial Conclusion

In this chapter, a new design strategy for switched SOF controllers was proposed, using the two-stage method and for the control law, a relationship between the measured output vector $y$ and the switching matrix $Q_k$ was used. During feasibility experiments it was shown that the Theorem 8 has a better feasibility region than Theorem 4 found in Sereni (2019). Furthermore, it showed a superior performance, that can be credited the use of the switching matrix and the switching law, when compared to use of a controller with a single feedback gain.

# 4 Parameter-dependent state-feedback gain in the first stage

In this chapter, the development of LMI conditions for obtaining the switched SOF controller considering the design with pole allocation in circle by parameter-dependent state-feedback gain in the fist stage is presented.

As in Chapter 3 the objective is to find $L_k$ and $Q_k, \forall k \in \mathbb{K}_S$, such that when the gain $L_\sigma$ is selected according to (3.2), it asymptotically stabilizes (2.3) and (3.4).

## 4.1 Stabilization via switched SOF considering a parameter-dependent state-feedback design

This section presents the Theorem 9, in which we propose new LMIs conditions for deriving a switched SOF controller. The results obtained are based on the studies presented in Sereni (2019), Manesco (2013) and in Mehdi, Boukas and Bachelier (2004), which have been intensively discussed in Chapters 2 and 3. Moreover, Theorem 9 has as input a state-feedback $K(\alpha)$ is considered to be dependent on the system's uncertain parameter $\alpha$, and thus, belongs to the unitary simplex given by (2.5).

For this, we must consider the first stage control law as

$$u(t) = K(\alpha)x(t), \tag{4.1}$$

then the system (2.3) in closed-loop is represented by

$$\dot{x}(t) = [A(\alpha) + B(\alpha)K(\alpha)]x(t). \tag{4.2}$$

To obtain a parameter-dependent state feedback gain $K(\alpha)$, we might consider the condition presented by Leite, Montagner and Peres (2002), which states that if there are matrices $W \in \mathbb{R}^{n\times n}$, $Z_i$ and r such that

$$\begin{gathered} W = W' > 0 \\ \begin{bmatrix} A_iW + WA_i' + B_iZ_i + Z_i'B_i' + 2\beta W & * \\ WA_i' + Z_i'B_i' + \beta W & \mathrm{r}W \end{bmatrix} < 0 \end{gathered} \tag{4.3}$$

for $i = 1, 2, ..., N$, and

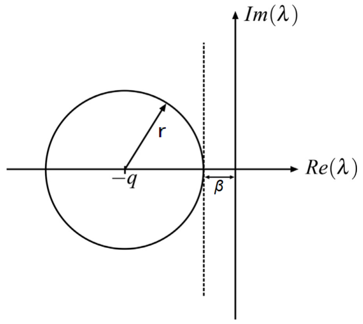
$$\begin{bmatrix} A_iW + WA_j' + B_iZ_j + Z_i'B_j' + A_jW + WA_i' + B_jZ_i + Z_j'B_i' + 4\beta W & * \\ WA_i' + WA_j' + Z_i'B_j' + Z_j'B_i' + 2\beta W & 2\mathrm{r}W \end{bmatrix} < 0 \tag{4.4}$$

for $i = 1, 2, ..., N$ and $j = i + 1, i + 2, ..., N$

In the affirmative case, a parameter-dependent state feedback vertices are given by $K_i = Z_iW^{-1}\ \forall i \in \mathbb{K}_N$, which is contained within a circular region represented by

Figure 12. Note that the region in Figure 12 is a circle defined in terms of its centre $(-q, 0)$ and radius r, where $q = \mathrm{r} + \beta$.

Figure 12 – Region circular for pole placement of $K(\alpha)$.




**Source:** Adapted from Sereni et al. (2022).

The parameter-dependent state feedback gains obtained in the first step are used as an input parameter for the second step in Theorem 9, where the switched output feedback gains $L_k \in \mathbb{R}^{m\times p}$, for $k \in \mathbb{K}_S$ are designed, which consists of sufficient LMI conditions for designing the desired controller.

**Theorem 9.** *Assuming that there exists a state feedback gain $K(\alpha)$, such that $A(\alpha) + B(\alpha)K(\alpha)$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P > 0, \tag{4.5}$$

$$Q_{0i} + C_i' Q(\lambda) C_i < 0 \tag{4.6}$$

*for $\forall\, i \in \mathbb{K}_N$, and*

$$Q_{0i} + C_i' Q(\lambda) C_j + Q_{0j} + C_j' Q(\lambda) C_i < 0 \tag{4.7}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J_k \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K_i'B_i'F' + FB_iK_i - Q_{0i} - C_i'Q_kC_i & * & * \\ P - F' + GA_i + GB_iK_i & -G - G' & * \\ B_i'F' + J_kC_i - HK_i & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{4.8}$$

*for $\forall\, i \in \mathbb{K}_N$ and $\forall k \in \mathbb{K}_S$, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F' + J_k(C_i + C_j) - H(K_i + K_j) & (B_i' + B_j')G' & -2H - 2H' \end{bmatrix} < 0, \tag{4.9}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$ where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = (A_i' + A_j')F' + F(A_j + A_i) + (K_i'B_j' + K_j'B_i')F' + F(B_jK_i + B_iK_j) \\ -Q_{0i} - Q_{0j} - C_j'Q_kC_i - C_i'Q_kC_j \end{aligned} \tag{4.10}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = 2P - 2F' + G(A_i + A_j) + G(B_jK_i + B_iK_j) \tag{4.11}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.

*Proof.* As mentioned for Theorem 5 demonstration, the matrix $H$ is invertible if (4.8) and (4.9) have a solution.

Now, splitting the matrix in (4.9)

$$\begin{aligned} &\begin{bmatrix} \Omega_{ij} & * & * \\ \Pi_{ij} & -G-G' & * \\ B_i'F' + J_kC - HK_j & B_i'G' & -H-H' \end{bmatrix} \\ &+ \begin{bmatrix} \Omega_{ji} & * & * \\ \Pi_{ji} & -G-G' & * \\ B_j'F' + J_kC - HK_i & B_j'G' & -H-H' \end{bmatrix} < 0 \end{aligned} \tag{4.12}$$

for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_N$,

Regarding Property 1 and considering the switching law (3.2), we have

$$\begin{aligned} &\left[\begin{matrix} \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (A_i'F' + FA_j + K_i'B_j'F' + FB_iK_j - Q_{0i} - C_i'Q_\sigma C_j) \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (P - F' + GA_j + GB_iK_j) \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'F' + J_\sigma C_i - HK_j) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} * & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-G-G') & * \\ \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_i'G') & \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-H-H') \end{matrix}\right] < 0, \end{aligned} \tag{4.13}$$

$$\sum_{i=1}^N \alpha_i (Q_{0i} + C_i'Q(\lambda)C_j) < 0. \tag{4.14}$$

Then, developing the terms in (4.13) and (4.14), and regarding that $\sum_{i=1}^N \alpha_i = \sum_{j=1}^N \alpha_j = 1$, we obtain

$$\begin{aligned} &\left[\begin{matrix} A'(\alpha)F' + FA(\alpha) + K'(\alpha)B'(\alpha)F' + FB(\alpha)K(\alpha) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha) \\ P - F' + GA(\alpha) + GB(\alpha)K(\alpha) \\ B'(\alpha)F' + J_\sigma C(\alpha) - HK(\alpha) \end{matrix}\right. \\ &\qquad \left.\begin{matrix} * & * \\ -G-G' & * \\ B'(\alpha)G' & -H-H' \end{matrix}\right] < 0 \end{aligned} \tag{4.15}$$

and

$$Q_0(\alpha) + C'(\alpha)Q(\lambda)C(\alpha) < 0. \tag{4.16}$$

The remaining part of the proof follows similarly as for Theorem 5. □

## 4.2 Decay Rate Bounding via switched SOF considering a parameter-dependent state-feedback design

Similarly as presented for the switched or robust SOF case, the inclusion of a performance improvement requirement via decay rate restriction may also be achieved in the design of the switched SOF considering a parameter-dependent state-feedback design, with a slight change on the problem's LMIs, as state Theorem 10.

**Theorem 10.** *Assuming that there exists a state feedback gain $K(\alpha)$, such that $A(\alpha) + B(\alpha)K(\alpha)$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a decay rate higher than $\gamma > 0$, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P > 0, \tag{4.17}$$

$$Q_{0i} + C_i' Q(\lambda) C_i < 0 \tag{4.18}$$

*for $\forall\, i \in \mathbb{K}_N$, and*

$$Q_{0i} + C_i' Q(\lambda) C_j + Q_{0j} + C_j' Q(\lambda) C_i < 0 \tag{4.19}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J_k \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K_i'B_i'F' + FB_iK_i + 2\gamma P - Q_{0i} - C_i'Q_kC_i & * & * \\ P - F' + GA_i + GB_iK_i & -G - G' & * \\ B_i'F' + J_kC_i - HK_i & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{4.20}$$

*for $\forall i \in \mathbb{K}_N$ and $\forall k \in \mathbb{K}_S$, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F' + J_k(C_i + C_j) - H(K_i + K_j) & (B_i' + B_j')G' & -2H - 2H' \end{bmatrix} < 0, \tag{4.21}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$ where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = {} & (A_i' + A_j')F' + F(A_j + A_i) + (K_i'B_j' + K_j'B_i')F' + F(B_jK_i + B_iK_j) \\ & + 4\gamma P - Q_{0i} - Q_{0j} - C_j'Q_kC_i - C_i'Q_kC_j \end{aligned} \tag{4.22}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = 2P - 2F' + G(A_i + A_j) + G(B_jK_i + B_iK_j) \tag{4.23}$$

*In the affirmative case, the switched output feedback gains are given by $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.*

*Proof.* The demonstration follows similarly as for Theorem 9, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (2.2). □

## 4.3 Relaxation Strategies for design switched SOF controller considering a parameter-dependent state-feedback design

As a consequence of the similarities in the LMI formulation, the strategies based on the PDLF and PDFV, as proposed on Section 3.3, can be extended.

### 4.3.1 Parameter-Dependent Lyapunov's Function

In Theorem 11 the LMI formulation to the switched SOF problem with problem with PDLF approach is proposed. This result was obtained by considering matrix $P(\alpha)$ in order to achieve less conservative restrictions.

**Theorem 11.** *Assuming that there exists a state feedback gain $K_\sigma$, such that $A(\alpha) + B(\alpha)K(\alpha)$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$, such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a decay rate higher than $\gamma > 0$, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P_i$, $Q_{0i} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P_i > 0, \tag{4.24}$$

$$Q_{0i} + C_i' Q(\lambda) C_i < 0 \tag{4.25}$$

*for $\forall\, i\ \in \mathbb{K}_N$, and*

$$Q_{0i} + C_i' Q(\lambda) C_j + Q_{0j} + C_j' Q(\lambda) C_i < 0 \tag{4.26}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, and matrices $F$, $G \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J_k \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A_i'F' + FA_i + K_i'B_i'F' + FB_iK_i + 2\gamma P - Q_{0i} - C_i'Q_kC_i & * & * \\ P_i - F' + GA_i + GB_iK_i & -G - G' & * \\ B_i'F' + J_kC_i - HK_i & B_i'G' & -H - H' \end{bmatrix} < 0 \tag{4.27}$$

*for $\forall\, i \in \mathbb{K}_N$ and $\forall k \in \mathbb{K}_S$, and*

$$\begin{bmatrix} \Omega_{ij} + \Omega_{ji} & * & * \\ \Pi_{ij} + \Pi_{ji} & -2G - 2G' & * \\ (B_i' + B_j')F' + J_k(C_i + C_j) - H(K_i + K_j) & (B_i' + B_j')G' & -2H - 2H' \end{bmatrix} < 0, \tag{4.28}$$

*for $i = 1, 2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$ and $\forall k \in \mathbb{K}_S$ where*

$$\begin{aligned} \Omega_{ij} + \Omega_{ji} = (A_i' + A_j')F' + F(A_j + A_i) + (K_i'B_j' + K_j'B_i')F' + F(B_jK_i + B_iK_j) \\ +2\gamma P_i + 2\gamma P_j - Q_{0i} - Q_{0j} - C_j'Q_kC_i - C_i'Q_kC_j \end{aligned} \tag{4.29}$$

*and*

$$\Pi_{ij} + \Pi_{ji} = P_i + P_j - 2F' + G(A_i + A_j) + G(B_jK_i + B_iK_j) \tag{4.30}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.

*Proof.* The demonstration of Theorem 11 is omitted since it follows similarly as the proof of Theorem 9, regarding the parameter-dependent Lyapunov's function $P(\alpha)$ and taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (2.2). □

### 4.3.2 Parameter-Dependent Finsler's Variables

In order to obtain even less conservative conditions with PDFV as in Chapter 3 for the design of Switched SOF controllers, the conditions will at some point imply the product between three parameter-dependent matrices in the mathematical formulation of this control problem, which can be exemplified with terms such as

$$T(\alpha)U(\alpha)V(\alpha) \tag{4.31}$$

where $(T, U, V)(\alpha)$ are generic parameter-dependent matrices, will produce crossed-products between the $\alpha_i$ similarly as evidenced in (3.6).

In order to derive an equivalent representation one can observe the following property.

**Property 2.** *If the following LMIs*

$$\Upsilon_{iii} < 0,\ i = 1, 2, 3, \ldots, N, \tag{4.32}$$

$$\Upsilon_{iij} + \Upsilon_{iji} + \Upsilon_{jii} < 0,\ \begin{cases} i, j = 1, 2, 3, \ldots, N \\ i \neq j \end{cases} \tag{4.33}$$

*and,*

$$\Upsilon_{hij} + \Upsilon_{hji} + \Upsilon_{ijh} + \Upsilon_{ihj} + \Upsilon_{jhi} + \Upsilon_{jih} < 0,\ \begin{cases} h = 1, 2, 3, \ldots, N-2 \\ i = h+1, h+2, \ldots, N-1 \\ j = i+1, i+2, \ldots, N \end{cases} \tag{4.34}$$

*hold, then it is true that*

$$\sum_{h=1}^{N} \alpha_h \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \Upsilon_{hij} < 0. \tag{4.35}$$

*Proof.* See OLIVEIRA et al. (2004). □

In Theorem 12 the new and more relaxed LMI conditions for the design of switched SOF controllers considering a parameter-dependent state-feedback design are proposed, as a result of the assumption made in (3.74) and the application of Property 2.

**Theorem 12.** *Assuming that there exists a state feedback gain $K(\alpha)$, such that $A(\alpha) + B(\alpha)K(\alpha)$ is asymptotically stable, then there exists a stabilizing switched static output feedback controller, $L_\sigma$ , such that $A(\alpha) + B(\alpha)L_\sigma C(\alpha)$ is asymptotically stable with a decay rate higher than $\gamma > 0$, considering the switching rule* (3.2)*, if there exist $\lambda \in \wedge_S$ and symmetric matrices, $P_h$, $Q_{0h} \in \mathbb{R}^{n\times n}$ and $Q_k \in \mathbb{R}^{p\times p}$, with*

$$P_h > 0, \tag{4.36}$$

$$Q_{0h} + C'_h Q(\lambda) C_h < 0 \tag{4.37}$$

*for $\forall\, h \in \mathbb{K}_N$, and*

$$Q_{0i} + C'_h Q(\lambda) C_i + Q_{0h} + C'_i Q(\lambda) C_h < 0 \tag{4.38}$$

*for $h = 1, 2, \ldots, N-1$, $i = h+1, h+2, \ldots, N$, and matrices $F_h$, $G_h \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times m}$ and $J_k \in \mathbb{R}^{m\times p}$ such that,*

$$\begin{bmatrix} A'_hF'_h + F_hA_h + K'_hB'_hF'_h + F_hB_hK_h + 2\gamma P - Q_{0h} - C'_hQ_kC_h & * & * \\ P_h - F'_h + G_hA_h + G_hB_hK_h & -G_h - G'_h & * \\ B'_hF'_h + J_kC_h - HK_h & B'_hG'_h & -H - H' \end{bmatrix} < 0 \tag{4.39}$$

*for $\forall\, i \in \mathbb{K}_N$ and $\forall k \in \mathbb{K}_S$, and*

$$\begin{bmatrix} \Omega^{hi}_{11} & * & * \\ \Omega^{hi}_{21} & -2(G_h + G'_h) - G_i - G'_i & * \\ \Omega^{hi}_{31} & B'_h(G'_i + G'_h) + B'_iG'_h & -3(H + H') \end{bmatrix} < 0 \tag{4.40}$$

*for $h, i = 1, 2, \ldots, N$, and $h \neq i$, and $\forall k \in \mathbb{K}_S$ where*

$$\begin{aligned} \Omega^{hi}_{11} = {} & A'_h(F'_h + F'_i) + A'_iF'_h + K'_h(B'_hF'_i + B'_iF'_h) + K'_iB'_hF'_h \\ & + F_h(A_h + A_i) + F_iA_h + F_h(B_hK_i + B_iK_h) + F_iB_hK_h \\ & 4\gamma P_h + 2\gamma P_i - 2Q_{0h} - Q_{0i} - C'_hQ_k(C_h + C_i) - C'_iQ_kC_h, \end{aligned} \tag{4.41}$$

$$\begin{aligned} \Omega^{hi}_{21} = {} & 2P_h + P_i - (2F'_h + F'_i) + G_h(A_h + A_i) + G_iA_h \\ & + G_h(B_hK_i + B_iK_h) + G_iB_hK_h \end{aligned} \tag{4.42}$$

*and*

$$\Omega^{hi}_{31} = B'_h(F'_h + F'_i) + B'_iF'_h + 2J_kC_h + J_kC_i - 2HK_h - HK_i \tag{4.43}$$

*and,*

$$\begin{bmatrix} \Theta^{hij}_{11} & * & * \\ \Theta^{hij}_{21} & -2(G_h + G'_h) - 2(G_i + G'_i) - 2(G_j + G'_j) & * \\ \Theta^{hij}_{31} & (B'_h + B'_i)G'_j + (B'_i + B'_j)G'_h + (B'_j + B'_h)G'_i & -6(H + H') \end{bmatrix} < 0 \tag{4.44}$$

*for* $h = 1, 2, \ldots, N-2$, $i = h+1, i+2, \ldots, N-1$, $j = i+1, i+2, \ldots, N$, *and* $\forall k \in \mathbb{K}_S$ *where*

$$\begin{aligned}
\Theta_{11}^{hij} &= (A_h' + A_i')F_j' + (A_i' + A_j')F_h' + (A_h' + A_i')F_j' \\
&+F_h(A_i + A_j) + F_i(A_j + A_h) + F_j(A_h + A_i) + (K_h'B_i' + K_i'B_h')F_j' \\
&+(K_i'B_j' + K_j'B_i')F_h' + (K_j'B_h' + K_h'B_j')F_i' + F_h(B_iK_j + B_jK_i) \\
&+F_i(B_jK_h + B_hK_j) + F_j(B_hK_i + B_iK_h)4\gamma P_h + 4\gamma P_i + 4\gamma P_j - 2Q_{0h} \\
&-2Q_{0i} - 2Q_{0j} - C_h'Q_k(C_i + C_j) - C_i'Q_k(C_j + C_h) - C_j'Q_k(C_h + C_i),
\end{aligned} \tag{4.45}$$

$$\begin{aligned}
\Theta_{21}^{hij} &= 2(P_h + P_i + P_j) - 2(F_h' + F_i' + F_i') + (G_h + G_i)A_j \\
&+(G_i + G_j)A_h + (G_j + G_h)A_i + G_h(B_iK_j + B_jK_i) \\
&+G_i(B_jK_h + B_hK_j) + G_j(B_hK_i + B_iK_h)
\end{aligned} \tag{4.46}$$

*and*

$$\begin{aligned}
\Theta_{31}^{hij} &= (B_h' + B_i')F_j' + (B_i' + B_j')F_h' + (B_j' + B_h')F_i' \\
&+2J_kC_h + 2J_kC_i + 2J_kC_j - 2HK_h - 2HK_i - 2HK_j
\end{aligned} \tag{4.47}$$

*In the affirmative case, the switched output feedback gains are given by* $L_k = H^{-1}J_k$, $\forall k \in \mathbb{K}_S$.

*Proof.* As mentioned for Theorem 5 demonstration, the matrix $H$ is invertible if (4.39), (4.40) and (4.44) have a solution.

One can observe that (4.39), (4.40) and (4.38) are in the form of (4.32), (4.33) and (4.34), from Property 1 and 2 and considering the switching law (3.2), we have

$$\begin{bmatrix}
\sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (A_h'F_i' + F_hA_i + K_h'B_i'F_j' + F_hB_iK_j + 2\gamma P_h - Q_{0h} - C'Q_\sigma C) & * & * \\
\sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (P_h - F_i' + G_hA_i + G_hB_iK_j) & \sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-G_h - G_h') & * \\
\sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_h'F_i' + J_\sigma C - HK_i) & \sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (B_h'G_i') & \sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i \sum_{j=1}^N \alpha_j (-H - H')
\end{bmatrix} < 0, \tag{4.48}$$

and

$$\sum_{h=1}^N \alpha_h \sum_{i=1}^N \alpha_i (Q_{0i} + C_h'Q(\lambda)C_i) < 0. \tag{4.49}$$

Then, developing the terms in (4.48) and (4.49), and regarding that $\sum_{h=1}^N \alpha_h =$

$\sum_{i=1}^{N} \alpha_i = \sum_{j=1}^{N} \alpha_j = 1$, we obtain

$$\begin{bmatrix} \Gamma_\sigma(\alpha) & * & * \\ P(\alpha) - F'(\alpha) + G(\alpha)A(\alpha) + G(\alpha)B(\alpha)K(\alpha) & -G(\alpha) - G'(\alpha) & * \\ B'(\alpha)F'(\alpha) + J_\sigma C(\alpha) - HK(\alpha) & B'(\alpha)G'(\alpha) & -H - H' \end{bmatrix} < 0 \tag{4.50}$$

where

$$\begin{aligned}\Gamma_\sigma(\alpha) = A'(\alpha)F'(\alpha) + F(\alpha)A(\alpha) + K'(\alpha)B'(\alpha)F'(\alpha) + F(\alpha)B(\alpha)K(\alpha) \\ +2\gamma P(\alpha) - Q_0(\alpha) - C'(\alpha)Q_\sigma C(\alpha)\end{aligned} \tag{4.51}$$

and

$$Q_0(\alpha) + C'(\alpha)Q(\lambda)C(\alpha) < 0. \tag{4.52}$$

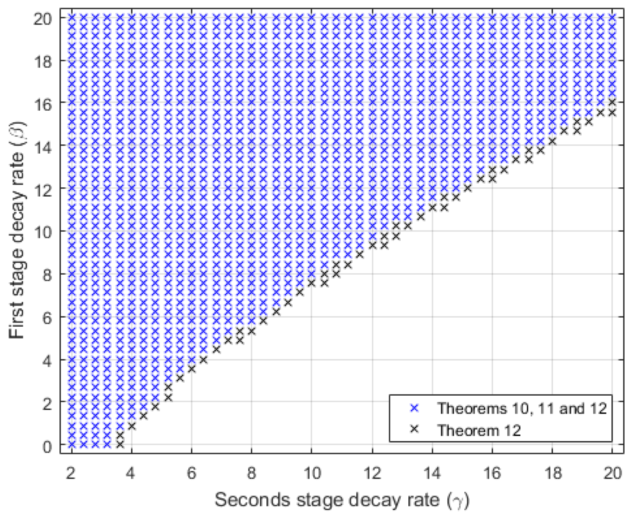
The demonstration follows similarly as for Theorem 5, taking into account that the quadratic stability condition considering a lower bound $\gamma$ for the system decay rate is as defined in (2.2). $\square$

## 4.4 Analysis

Intending to investigate the efficiency of the proposed theorems and to compare with the strategy proposed in Chapter 3, some practical experiments were executed. In this section, feasibility and performance analysis are presented using the active suspension system.

**Experiment 4.1.** In this experiment with same conditions of the Experiment 3.1 is compare the feasibility of Theorems 10, 11 and 12, considering the specification of a minimum decay in both stages. This experiment was performed considering a range for the minimum decay rates in the first and second stage of the project as $0 \leq \beta \leq 20$, $2 \leq \gamma \leq 20$ and r $= 1000$, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$. The proposed LMIs were solved similarly as in previous analysis, one can obtain the results presented in Figure 13, which represents the feasibility region obtained by each of theorems.

Figure 13 – Feasibility region obtained with Theorems 10, 11 and 12 changing the minimum decay rate.



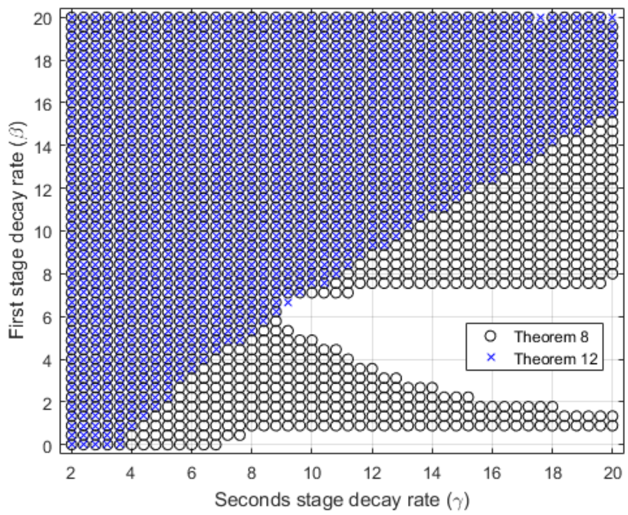

**Source:** Own Autor.

In the Figure 13 is possible to observe that Theorem 12, which is the switched SOF with $K(\alpha)$ that contemplate $F(\alpha)$ and $G(\alpha)$, has a bigger feasibility area than Theorem 10, which is the switched SOF with $K(\alpha)$ that is based on a fixed matrix $P$, when considering the specification of a minimum decay in both stages. Theorem 11 showed no improvement in Figure 13 compared to Theorem 10.

**Experiment 4.2.** Another analysis with same conditions of the Experiment 3.1 is compare the feasibility of Theorems 8 and 12, considering the specification of a minimum decay in both stages. This experiment was performed considering a range for the minimum decay rates in the first and second stage of the project as $0 \leq \beta \leq 20$, $2 \leq \gamma \leq 20$ and r $= 1000$, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

The proposed LMIs were solved similarly as in previous analysis, one can obtain the results presented in Figure 14, which represents the feasibility region obtained by each of the two considered theorems.

Figure 14 – Feasibility region obtained with Theorems 8 and 12 changing the minimum decay rate.




**Source:** Own Autor.

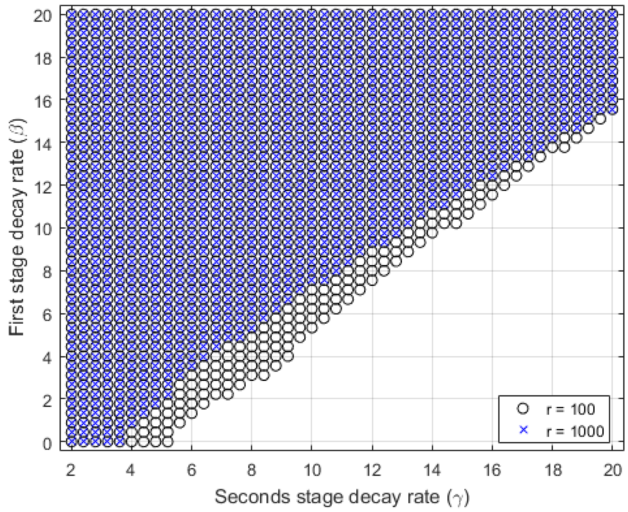
One can observe in Figure 14 that Theorem 8 has a better feasibility region when compared to Theorem 12, given the distance of the values of $K(\alpha)$, having higher restrictions.

**Experiment 4.3.** An additional analysis with same conditions of the Experiment 3.1 is compare the feasibility of Theorem 12 with $\mathsf{r} = 100$ and $\mathsf{r} = 1000$ in the first stage design, considering the specification of a minimum decay in both stages. This experiment was performed considering a range for the minimum decay rates in the first and second stage of the project as $0 \leq \beta \leq 20$, $2 \leq \gamma \leq 20$, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

The proposed LMIs were solved similarly as in previous analysis, one can obtain the results presented in Figure 15, which represents the feasibility region obtained by each of the two considered theorems.

One can observe in Figure 15 that Theorem 12 with $\mathsf{r} = 100$ in the first stage design has a better feasibility region, given the proximity of the values of $K(\alpha)$ reducing the restrictions, allowing larger values in the minimum decay rate.

Figure 15 – Feasibility region obtained with r $= 100$ and r $= 1000$ in the first stage design changing the minimum decay rate.



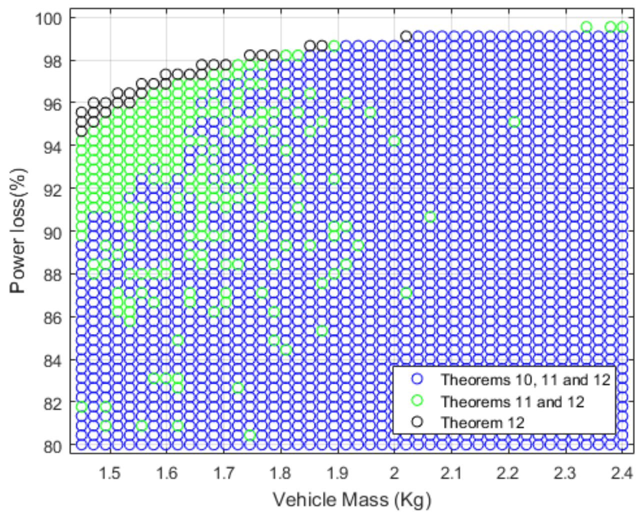

**Source:** Own Autor.

**Experiment 4.4.** In this experiment the Active Suspension System will be examined from the politopic uncertain perspective, evaluating the perform of the Theorems 10, 11 and 12 with same conditions of the Experiment 3.3, however the power loss has range 80 to 100% and r $= 1000$, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

The proposed LMIs were solved similarly as in previous analysis, regarding the values of $\rho$ and $M_s$ in the specified ranges, the obtained feasibility regions wit each theorem is presented in Figure 16.

One can observe that by using Theorem 11, which considers the matrix $P$ as dependent on the uncertain parameter $\alpha$, results in a larger feasibility region when compared to Theorem 10, which is based on a fixed matrix $P$. Also, Figure 16 shows that Theorem 12 provided a better result than Theorem 11 in this experiment.

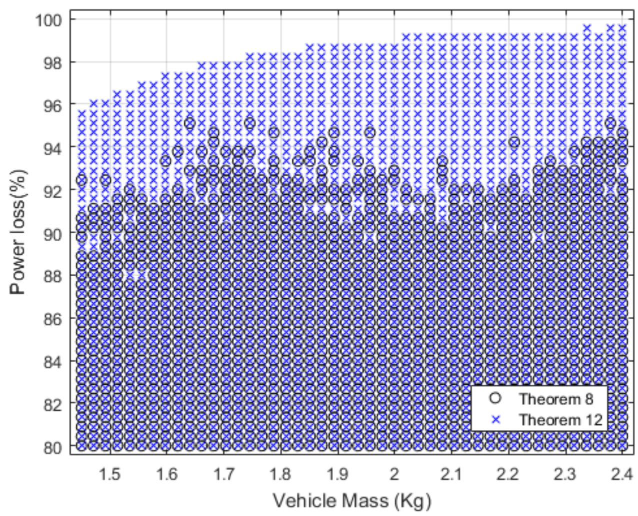
Figure 16 – Feasibility region obtained with Theorems 10, 11 and 12 changing the polytope's vertex.




**Source:** Own Autor.

**Experiment 4.5.** To compare the Theorems 8 and 12 this test with the same conditions from the Experiment 3.3, however the power loss has range 80 to 100% and r $= 1000$. Regarding the values of power loss and $M_s$ in the specified ranges, the obtained feasibility regions with each theorem is presented in Figure 17, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.

Figure 17 – Feasibility region obtained with Theorems 8 and 12 changing the polytope's vertex.



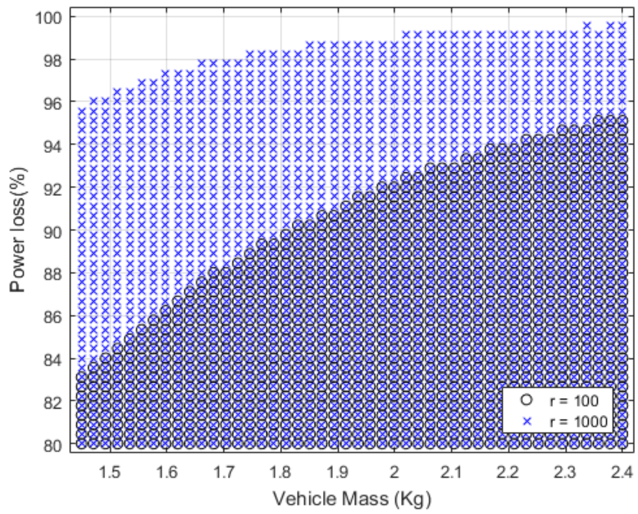

**Source:** Own Autor.

One can observe in Figure 17 that Theorem 12 has a better feasibility region when compared to Theorem 8, $K(\alpha)$ allowed larger uncertainties.

**Experiment 4.6.** An additional comparison the Theorem 12 with r $= 100$ and r $= 1000$

in the first stage design this test with the same conditions from the Experiment 3.3, however the power loss has range 80 to 100%, and still requiring a design of two gains for the switched SOF $(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$.Regarding the values of power loss and $M_s$ in the specified ranges, the obtained feasibility regions wit each theorem is presented in Figure 18.

Figure 18 – Feasibility region obtained with r $= 100$ and r $= 1000$ in the first stage changing the polytope's vertex.




**Source:** Own Autor.

One can observe in Figure 18 that r $= 1000$ has a better feasibility region when compared to r $= 100$, given the distance of the values of $K(\alpha)$ allowed larger uncertainties.

**Experiment 4.7.** To analyze whether there is improvement in behavior when switched SOF considering a parameter-dependent state-feedback design, we considered in this experiment with the same conditions from the Experiment 3.6.

The LMI (2.12) was solved similarly as in previous analysis, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0.75$ and r $= 100$, we obtain the same state feedback gain (3.99). And, in the second stage, using the LMIs in Theorem 8 and the first-stage state feedback gain, the designed static output feedback gain are expressed in (3.106) and with the matrix $Q_1$ and $Q_2$ are described in (3.107).

The LMIs (4.3) and (4.4) was solved similarly as the robust state feedback, we obtain the state feedback gains:

$$\begin{aligned} K_1 &= \begin{bmatrix} 658.347 - 29.385 & 42.051 & 19.945 \end{bmatrix} \\ K_2 &= \begin{bmatrix} 1032 & -80.293 & 448.79 & 52.629 \end{bmatrix} \\ K_3 &= \begin{bmatrix} 657.31 & -26.096 & 78.079 & 18.331 \end{bmatrix} \\ K_4 &= \begin{bmatrix} 1231.6 & -52.911 & 229.1 & 45.062 \end{bmatrix} \end{aligned} \tag{4.53}$$

Finally, in the second stage, using the LMIs in Theorem 12, the first-stage state feedback gain and design two gains in second stage$(S = 2)$, with $\lambda_1 = \lambda_2 = 0.5$, the designed switched static output feedback gains were

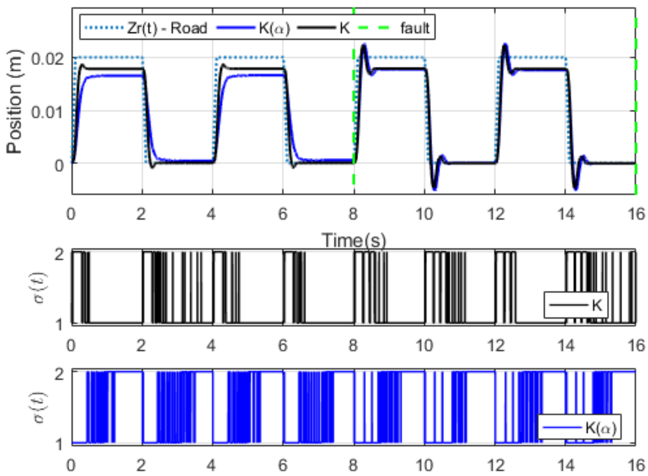
$$\begin{aligned} L_1 &= \begin{bmatrix} 739.63 & -65.196 \end{bmatrix} \\ L_2 &= \begin{bmatrix} 739.49 & -65.221 \end{bmatrix} \end{aligned} \tag{4.54}$$

with the matrix $Q_1$ and $Q_2$:

$$\begin{aligned} Q_1 &= \begin{bmatrix} 2.9802 \times 10^{10} & -2.4932 \times 10^{9} \\ -2.4932 \times 10^{9} & 2.006 \times 10^{10} \end{bmatrix} \\ Q_2 &= \begin{bmatrix} 2.9794 \times 10^{10} & -2.4927 \times 10^{9} \\ -2.4927 \times 10^{9} & 2.0079 \times 10^{10} \end{bmatrix} \end{aligned} \tag{4.55}$$

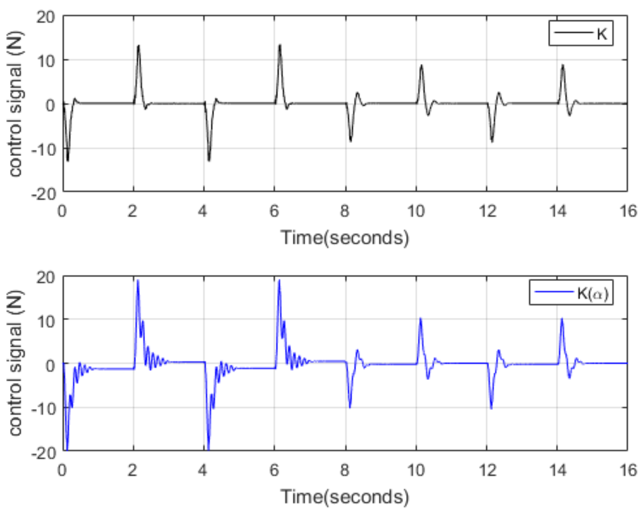
According to Figure 19 the controller designed with Theorem 12 offered an improvement when compared with Theorem 8. The Figure 20 illustrates that during the experiment the controller with the Theorem 12 have higher $F_c$ than Theorem 8.

Figure 19 – Active suspension system behavior and control signal using Theorems 8 and 12 controllers with fault: 50% power loss in the actuator $(8 - 16s)$, for $M_s = 2.45kg$




**Source:** Own Autor.

Figure 20 – Control command $F_c(t)$ generated by the designed switched SOF controllers during the executed experiments with using Theorem 8 (top) and 12 (bottom) for $M_s = 2.45kg$.




**Source:** Own Autor.

**Experiment 4.8.** Another analysis were made aiming to evaluate the system performance with the controllers design from the Theorems 12 with different r in the first stage, we considered in this experiment that the Active Suspension System may present the same conditions from the last experiment.

The LMI (2.12) was solved similarly as in previous analysis, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0.75$ and r $= 100$, we obtain the same state feedback gain (4.53). And, in the second stage, using the LMIs in Theorem

12 and the first-stage state feedback gain, the designed static output feedback gain are expressed in (4.54) and with the matrix $Q_1$ and $Q_2$ are described in (4.55).

Finally, fixing the decay rates of the first and second stage of the project as $\beta = \gamma = 0.75$ and r $= 1000$, we obtain the state feedback gains:

$$\begin{aligned} K_1 &= \begin{bmatrix} -114.69 & -276.81 & 3419.2 & 171.83 \end{bmatrix} \\ K_2 &= \begin{bmatrix} -506.4 & -507.45 & 6282.1 & 319.69 \end{bmatrix} \\ K_3 &= \begin{bmatrix} 140.98 & -196.79 & 2431.4 & 134.88 \end{bmatrix} \\ K_4 &= \begin{bmatrix} -154.9 & -434.55 & 5594.8 & 314.71 \end{bmatrix} \end{aligned} \tag{4.56}$$

and, in the second stage, using the LMIs in Theorem 12, and the first-stage state feedback gain, the designed switched static output feedback gain was

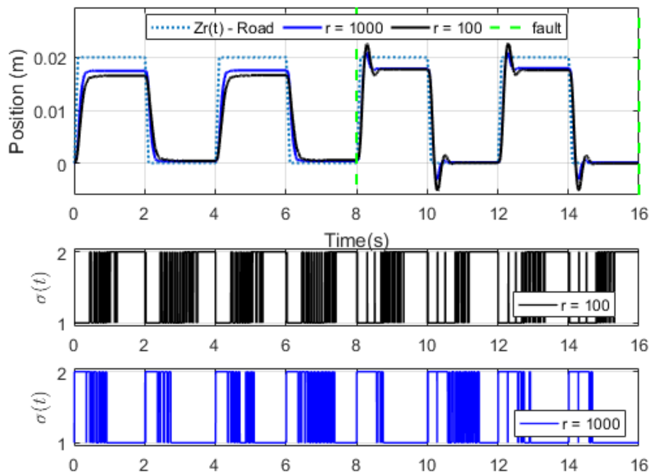
$$\begin{aligned} L_1 &= \begin{bmatrix} -10.665 & -122.9 \end{bmatrix} \\ L_2 &= \begin{bmatrix} -10.454 & -122.86 \end{bmatrix} \end{aligned} \tag{4.57}$$

with the matrix $Q_1$ and $Q_2$:

$$\begin{aligned} Q_1 &= \begin{bmatrix} -6.6658 \times 10^{10} & -8.201 \times 10^{10} \\ -8.201 \times 10^{10} & 1.8315 \times 10^{11} \end{bmatrix} \\ Q_2 &= \begin{bmatrix} -6.6658 \times 10^{10} & -8.2035 \times 10^{10} \\ -8.2035 \times 10^{10} & 1.8308 \times 10^{11} \end{bmatrix}. \end{aligned} \tag{4.58}$$

According to Figure 19 the controller with r $= 1000$ had a better dynamic response. Also, Figure 19 shows that the $\sigma$ presented less switching when compared to the controller with r $= 100$.

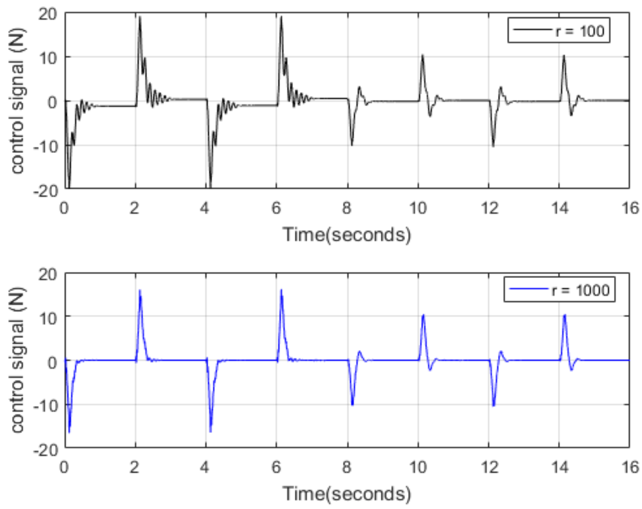
Figure 21 – Active suspension system behavior using Theorem 12 controllers with r $= 100$ and r $= 1000$ fault: 50% power loss in the actuator $(8 - 16s)$, for $M_s = 2.45kg$.




**Source:** Own Autor.

Figure 20 shows that the control input $F_c(t)$ with r $= 100$ presented higher amplitudes when compared to the results of controller with higher r.

Figure 22 – Control command $F_c(t)$ generated by the designed switched SOF controllers during the executed experiments with decay rates r $= 100$ (top) and r $= 1000$ (bottom) for $M_s = 2.45kg$.




**Source:** Own Autor.

## 4.5 Partial Conclusion

In this chapter, a new design strategy for switched SOF controllers was proposed, using the two-stage method, with pole allocation in circle by parameter-dependent state feedback, which bring less conservatism in the second stage. In addition, the controllers developed in the second stage with the Theorem 12 brought a superior performance in the dynamics of the system during the experiments, note that the control input was higher, where we can observe a rapid change between the controllers (Shattering). Another factor when using a with pole allocation in circle by parameter-dependent state-feedback in the first stage is a parameter $\mathrm{r}$, that could influence the feasibility, control input and performance of the controller designed with Theorems, and a more extensive theoretical study to assess whether there really is a relationship between SOF performance and design specifications in the first stage, can help to understand the results.

# 5 CONCLUSIONS AND PERSPECTIVES

This work proposes the study of switching control via static output feedback control. As an overall conclusion, the outcome of the performed studies corroborates with the fact that this line of research can provide interesting and useful contributions to control theory.

The Chapter 3 compiled the LMI conditions for design switched SOF controllers for uncertain LTI systems. The feasibility tests shown that the conditions in Theorem 8, which involves the design of switched SOF that considers the Finsler's variables ($F$ and $G$) as dependent on the uncertain parameter $\alpha$, has an improvement if compared with Theorem 4, which is the robust SOF that contemplate $F(\alpha)$ and $G(\alpha)$, presented by Sereni (2019). The results of a practical implementation of a switched SOF controller conditions for the QUANSER® Active Suspension in Experiment 3.5 highlighted a limitation in switching law used on the proposed theorem, which is the size of the switching matrices $Q_i$ that should not be smaller than 2 to be able to change between the gains. Although, in Experiment 3.6 showed that the controller was able to suppress the oscillations, even during the occurrence of 50% power loss on the system actuator, and has a switching between the gains during the experiment. Additionally the system dynamic performance was slightly better using the switched controllers, when compared with the robust strategy, however such performance improvement comes with increases the control input.

In Chapter 4 a new design strategy for switched SOF controllers was proposed, using the two-stage method, with parameter-dependent state-feedback, which the controllers developed in the second stage with the Theorem 12, which is the switched SOF with $K(\alpha)$ that contemplate $F(\alpha)$ and $G(\alpha)$, brought a superior performance in the dynamics of the system during the experiments. Note the control input was higher, where we can observe a shattering between the gains. However this results is empirical, there is no mathematical prove yet, to say that switched SOF with $K(\alpha)$ in the first stage can brought a superior performance. Another factor when using a pole allocation in circle by parameter-dependent state feedback in the first stage is a parameter $\mathbf{r}$, that could influence the feasibility, control input and performance of the controller designed with Theorems from this Chapter, and a more extensive theoretical study to assess whether there really is a relationship between SOF performance and design specifications in the first stage, can help to understand the results.

**Research perspectives**

As future perspectives we can mention:

- Include requirements of $\mathcal{H}_\infty$ in the LMIs conditions, so that the controllers can work with disturbance robustness.
- Search for a smooth switching law to avoid cases of increased cost of control due to shattering.
- Insertion of D-stability into LMIs to determine a controller that best fits for more practical design requirements.
- Expand to more practical systems, that is, more complex, such as those with saturation or nonlinearity.
- Expand the results for discrete-time systems.
- A more in-depth theoretical study to assess whether there really is a relationship between SOF performance and design specifications in the first stage, can help to understand the results.
- Improve switching conditions also so that it is possible to switch between gains when only one system output variable is available.

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