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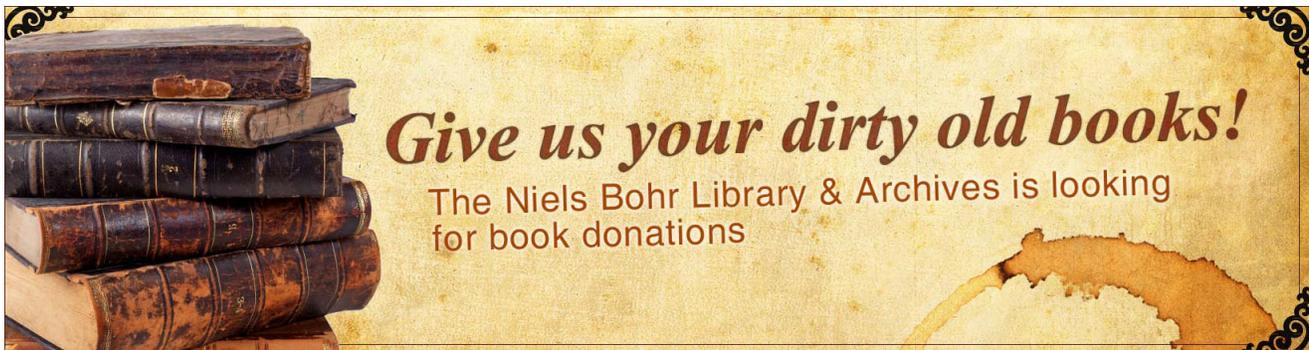
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Construction of time-dependent dynamical invariants: A new approach

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We propose a new way to obtain polynomial dynamical invariants of the classical and quantum time-dependent harmonic oscillator from the equations of motion. We also establish relations between linear and quadratic invariants, and discuss how the quadratic invariant can be related to the Ermakov invariant. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.3702824>]

Dedicated to Professor Ruben Aldrovandi on the occasion of his 70th birthday.

I. INTRODUCTION

Dynamical invariants are constants of motion related to the temporal evolution of dynamical systems. They constitute a very important chapter in the study of classical and quantum dynamics, since they can be associated with special symmetries of the physical systems, as in the case of Noether invariants.¹ In most systems the equations of motion reveal dynamical invariants related to fundamental group structures that may also define their own integrability. In general, this is the case of conservative systems, whose equations of motion are not explicitly time-dependent.

Time-dependent systems, specially in quantum mechanics, also provide important applications for the theory of dynamical invariants. Some examples are the Paul traps,² which are ion traps where charged particles are confined with time-varying potentials. Because the Hamiltonian is explicitly time-dependent, a well defined spectrum cannot be found. This problem is dealt with by finding the dynamical invariants of the system and studying the relationship between these quantities and time evolution of the quantum states of the particles within the trap,³ or by finding its coherent states, looking for the most classical behavior.^{4,5} Other important applications take place in optical cavities with moving mirrors, such as the quantization of the electromagnetic radiation,^{6,7} the study of dynamical Casimir effect,⁸ and laser physics.^{9,10} In general, dynamical invariants are of fundamental importance in solving problems in optics and condensed matter physics.

Non-stationary quantum systems face problems in time operator ordering, and for these systems eigenvalue problems are not well defined. However, the knowledge of their dynamical invariants can still provide information of their fundamental algebraic structure and solutions. The theoretical developments on this field were performed by many authors using fundamental tools introduced back to the 19th century by Ermakov¹¹ in the study of integrability of second-order differential equations. In his works, Ermakov introduced the so-called Ermakov invariant, which is a fundamental quadratic conserved quantity used since then to analyze classical¹² and quantum systems.^{13,14}

In this paper we present a new approach to classical and quantum polynomial invariants of the one-dimensional time-dependent harmonic oscillator, using nothing more than its equation of motion. The use of this system is not only a matter of simplicity, since all the above cited physical applications, and others, can be modeled by this simple system. We begin in Sec. II, where we perform

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the derivation of general linear and quadratic invariants of the classical oscillator, proceeding in Sec. III to the related quantum system. In Sec. IV we build general relations between first and second order invariants in both classical and quantum case. At last, we discuss the relation of the general second-order quadratic invariant to the Ermakov's invariant.

II. THE CLASSICAL TIME-DEPENDENT HARMONIC OSCILLATOR

Let us start with the equation of motion of the one-dimensional harmonic oscillator,

$$\frac{d^2q}{dt^2} + \omega^2(t)q = 0, \quad (1)$$

whose frequency is time dependent. This is a second-order ordinary differential equation (ODE), and we may redefine it as a set of two first-order ODEs by introducing the variable

$$p = \frac{dq}{dt}. \quad (2a)$$

In this case (1) becomes

$$\frac{dp}{dt} = -\omega^2(t)q. \quad (2b)$$

Equations (2) represent the motion of the oscillator in a state-space spanned by the variables (q, p) . To find a unique solution to this problem it is necessary and sufficient to provide two initial conditions: the values of the coordinates (q, p) at a given instant of time in the case of (2), or the values of the coordinates q and velocities \dot{q} at a given instant of time, in the case of (1). In any case, the general theory of ordinary differential equations tells us that any complete solution must contain two independent constants of integration, also called first integrals of motion.

A. Linear invariants

Now let us introduce two arbitrary time-dependent functions $\alpha(t)$ and $\beta(t)$. Multiplying (2a) by α and (2b) by β , and making the summation of both equations, we obtain

$$\beta \frac{dp}{dt} + \alpha \frac{dq}{dt} = \alpha p - \beta \omega^2 q.$$

Collecting terms with total time derivatives we have

$$\frac{d}{dt}(\beta p + \alpha q) = \left(\alpha + \frac{d\beta}{dt}\right)p + \left(\frac{d\alpha}{dt} - \beta \omega^2\right)q.$$

Suppose that the functions α and β obey the following set of ODE

$$\alpha + \frac{d\beta}{dt} = 0, \quad (3a)$$

$$\frac{d\alpha}{dt} - \beta \omega^2 = 0. \quad (3b)$$

In this case the quantity

$$I_L = \beta p + \alpha q$$

is a dynamical invariant, i.e., $dI_L/dt = 0$.

Derivating (3a) and subtracting (3b) from the result we obtain

$$\frac{d^2\beta}{dt^2} + \omega^2\beta = 0. \quad (4)$$

Therefore, we may describe the linear invariant with a single parameter, let it be β , and in this case we write

$$I_L = \beta p - \frac{d\beta}{dt} q. \quad (5)$$

The equation for α is second-order, but its form is more complicated, what makes β the simplest choice of parameter. We notice that β has the same general solution of the equation of motion (1), but without the requirement of being a real valued function. Therefore, the invariant I_L becomes two independent functions

$$I_L = \beta p - \frac{d\beta}{dt} q, \quad (6a)$$

$$I_L^* = \beta^* p - \frac{d\beta^*}{dt} q, \quad (6b)$$

where β^* is the complex conjugation of β .

B. Quadratic invariants

At the same way, quadratic invariants can be derived as linear combinations of the equations of motion. However, in this case we may consider superposition of quadratic products of Eqs. (2a) and (2b).

Let us consider the following combinations:

$$\begin{aligned} p \frac{dp}{dt} &= -\omega^2 pq, \\ q \frac{dq}{dt} &= qp, \\ \frac{dq}{dt} p + \frac{dp}{dt} q &= p^2 - \omega^2 q^2. \end{aligned}$$

Other combinations are possible, but these ones are sufficient to give us the most general second-order invariant. Multiplying these equations by arbitrary time-dependent functions γ , ϵ , and ζ , and taking the sum

$$\gamma p \frac{dp}{dt} + \epsilon \left(\frac{dq}{dt} p + \frac{dp}{dt} q \right) + \zeta q \frac{dq}{dt} = \epsilon (p^2 - \omega^2 q^2) + (\zeta - \gamma \omega^2) pq.$$

Making explicit total time derivatives, we find the following structure:

$$\begin{aligned} \frac{d}{dt} (\gamma p^2 + 2\epsilon qp + \zeta q^2) &= \left(2\epsilon + \frac{d\gamma}{dt} \right) p^2 + \left(\frac{d\zeta}{dt} - 2\epsilon\omega^2 \right) q^2 \\ &\quad + 2 \left(\frac{d\epsilon}{dt} + \zeta - \gamma\omega^2 \right) pq. \end{aligned}$$

Therefore, if the coefficients obey the set of equations

$$2\epsilon + \frac{d\gamma}{dt} = 0, \quad (7a)$$

$$\frac{d\zeta}{dt} - 2\epsilon\omega^2 = 0, \quad (7b)$$

$$\frac{d\epsilon}{dt} + \zeta - \gamma\omega^2 = 0, \quad (7c)$$

we have that the time derivative of the quadratic polynomial

$$I_Q = \gamma p^2 + 2\epsilon qp + \zeta q^2 \quad (8)$$

is identically zero, then I_Q is also an invariant. Equations (7) can be written by the single third-order ODE

$$\frac{d^3\gamma}{dt^3} + 4\omega^2 \frac{d\gamma}{dt} + 2\frac{d\omega^2}{dt} \gamma = 0. \quad (9)$$

This equation is equivalent to

$$\frac{1}{2} \frac{d^2\gamma}{dt^2} + \omega^2 \gamma = \frac{W^2}{\gamma} + \frac{1}{4\gamma} \left(\frac{d\gamma}{dt} \right)^2, \quad (10)$$

in which W^2 is the integration constant.

III. THE QUANTUM TIME-DEPENDENT HARMONIC OSCILLATOR

The equation of motion for the one-dimensional quantum harmonic oscillator with time-dependent frequency has the same functional form of the classical case

$$\frac{d^2\hat{q}}{dt^2} + \omega^2(t) \hat{q} = 0,$$

where \hat{q} is the linear operator related to the generalized coordinate q . The equivalent first-order equations are also the same as (2), but with the variables (q, p) substituted by the operators (\hat{q}, \hat{p}) . In the quantum case we have to observe the ordering problem of the operators, since $\hat{q}\hat{p} - \hat{p}\hat{q} \neq 0$. However, the derivation of the linear invariants is not modified by this restriction. Then we have the functional operators

$$\hat{I}_L = \beta \hat{p} - \frac{d\beta}{dt} \hat{q}, \quad (11a)$$

$$\hat{I}_L^\dagger = \beta^* \hat{p} - \frac{d\beta^*}{dt} \hat{q}, \quad (11b)$$

as the linear invariants of the system, if the function β is solution of the ODE (4).

The quadratic invariant, on the other hand, needs some caution. In this case all possible quadratic combinations of products of the first-order equations are given by

$$\begin{aligned} \left\{ \hat{p}, \frac{d\hat{p}}{dt} \right\} &= \frac{d\hat{p}^2}{dt} = -\omega^2 \{ \hat{q}, \hat{p} \}, \\ \frac{d}{dt} \{ \hat{q}, \hat{p} \} &= 2\hat{p}^2 - 2\omega^2 \hat{q}^2, \\ \left\{ \hat{q}, \frac{d\hat{q}}{dt} \right\} &= \frac{d\hat{q}^2}{dt} = \{ \hat{q}, \hat{p} \}, \end{aligned}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anti-commutator. Taking the set of c-functions $(\gamma, \epsilon, \zeta)$ we build the following expression:

$$\gamma \frac{d\hat{p}^2}{dt} + \epsilon \frac{d}{dt} \{ \hat{q}, \hat{p} \} + \zeta \frac{d\hat{q}^2}{dt} = (\zeta - \gamma\omega^2) \{ \hat{q}, \hat{p} \} + \epsilon (2\hat{p}^2 - 2\omega^2 \hat{q}^2).$$

Making explicit total time derivatives yield

$$\begin{aligned} \frac{d}{dt} (\gamma \hat{p}^2 + \epsilon \{ \hat{q}, \hat{p} \} + \zeta \hat{q}^2) &= \left(\frac{d\gamma}{dt} + 2\epsilon \right) \hat{p}^2 + \left(\frac{d\zeta}{dt} - 2\epsilon\omega^2 \right) \hat{q}^2 \\ &\quad + \left(\frac{d\epsilon}{dt} + \zeta - \gamma\omega^2 \right) \{ \hat{q}, \hat{p} \}. \end{aligned}$$

Therefore, we see that the same conditions (7) must be satisfied for the function

$$\hat{I}_Q \equiv \gamma \hat{p}^2 + \epsilon \{\hat{q}, \hat{p}\} + \zeta \hat{q}^2 \quad (12)$$

to be a dynamical invariant of the quantum system.

IV. THE ALGEBRA OF THE DYNAMICAL INVARIANTS

A. The classical mechanical case

In Sec. II we showed that simple linear combinations of the first-order equations of motion of the classical time dependent harmonic oscillator yield the linear dynamical invariants

$$I_L = \beta p - \frac{d\beta}{dt} q, \quad (13a)$$

$$I_L^* = \beta^* p - \frac{d\beta^*}{dt} q, \quad (13b)$$

for a time dependent complex function β (and β^*) that obeys the second-order equation

$$\left(\frac{d^2}{dt^2} + \omega^2 \right) \begin{pmatrix} \beta \\ \beta^* \end{pmatrix} = 0. \quad (14)$$

On the other hand, the quadratic form

$$I_Q = \left(\gamma \omega^2 + \frac{1}{2} \frac{d^2 \gamma}{dt^2} \right) q^2 - \frac{d\gamma}{dt} qp + \gamma p^2 \quad (15)$$

is a second-order invariant if γ obeys

$$\frac{d^3 \gamma}{dt^3} + 4\omega^2 \frac{d\gamma}{dt} + 2\frac{d\omega^2}{dt} \gamma = 0. \quad (16)$$

We may rise the question if the quadratic invariant is somehow related to the linear ones. In other words, can we generate the quadratic form from the linear ones, and still have consistent equations for the parameters? In the classical case, where the ordering problem is absent, we may build all the quadratic products, but it is sufficient to consider

$$I_L^* I_L = I_L I_L^* = \beta^* \beta p^2 - \frac{d}{dt} (\beta^* \beta) qp + \frac{d\beta^*}{dt} \frac{d\beta}{dt} q^2. \quad (17)$$

This product is a quadratic invariant since I_L and I_L^* are invariants. It coincides with I_Q if we identify

$$\gamma = \beta^* \beta, \quad (18a)$$

$$\gamma \omega^2 + \frac{1}{2} \frac{d^2 \gamma}{dt^2} = \frac{d\beta^*}{dt} \frac{d\beta}{dt}. \quad (18b)$$

Equation (18b) has no new information. It is identically satisfied since β and β^* obey (14). Therefore, the quadratic invariant I_Q can be naturally generated by the linear invariants I_L when (18a) is obeyed.

Let us consider the polar form $\beta = \rho e^{i\phi}$, so we may write $\gamma = \rho^2$. In this case, (14) results in

$$\frac{d^2 \rho}{dt^2} + \omega^2 \rho = \rho \dot{\phi}^2, \quad (19)$$

also entailing

$$\frac{d}{dt} [\rho^2 \dot{\phi}] = 0 \quad \Rightarrow \quad \dot{\phi} = \frac{W}{\rho^2}, \quad (20)$$

where W is the same constant that appears in (10). Therefore,

$$\frac{d^2\rho}{dt^2} + \omega^2\rho = \frac{W^2}{\rho^3}. \quad (21)$$

This is the Ermakov equation.

On the other hand, Eq. (18a) implies that γ is a real non-negative number. Equation (16) then yields

$$\frac{d}{dt} \left[\rho^3 \left(\frac{d^2\rho}{dt^2} + \omega^2\rho \right) \right] = 0,$$

which also gives the Ermakov equation (21). Taking (15) it is straightforward to see that

$$I_Q = I_E = \frac{W^2}{\rho^2} q^2 + \left(\frac{d\rho}{dt} q - \rho p \right)^2. \quad (22)$$

This is known as the Ermakov invariant.

B. The quantum case

Considering the quantum mechanical time-dependent harmonic oscillator we may perform the products

$$\hat{I}_L^\dagger \hat{I}_L = \beta^* \beta \hat{p}^2 - \beta \frac{d\beta^*}{dt} \hat{q} \hat{p} - \beta^* \frac{d\beta}{dt} \hat{p} \hat{q} + \frac{d\beta^*}{dt} \frac{d\beta}{dt} \hat{q}^2, \quad (23a)$$

$$\hat{I}_L \hat{I}_L^\dagger = \beta^* \beta \hat{p}^2 - \beta^* \frac{d\beta}{dt} \hat{q} \hat{p} - \beta \frac{d\beta^*}{dt} \hat{p} \hat{q} + \frac{d\beta}{dt} \frac{d\beta^*}{dt} \hat{q}^2. \quad (23b)$$

We have two possible combinations: the antisymmetric case

$$\hat{I}_A \equiv \frac{1}{2} (\hat{I}_L \hat{I}_L^\dagger - \hat{I}_L^\dagger \hat{I}_L) = \left(\beta \frac{d\beta^*}{dt} - \frac{d\beta}{dt} \beta^* \right) (\hat{q} \hat{p} - \hat{p} \hat{q}),$$

and the symmetric product

$$\hat{I}_S \equiv \frac{1}{2} (\hat{I}_L \hat{I}_L^\dagger + \hat{I}_L^\dagger \hat{I}_L) = \beta^* \beta \hat{p}^2 - \frac{1}{2} \frac{d\beta^* \beta}{dt} \{\hat{q}, \hat{p}\} + \frac{d\beta^*}{dt} \frac{d\beta}{dt} \hat{q}^2. \quad (24)$$

The invariant \hat{I}_A is not really a quadratic form. Aside the fact that the relation $[\hat{q}, \hat{p}] \equiv \hat{q} \hat{p} - \hat{p} \hat{q} = i\hbar$ is postulated in quantum mechanics, it can be shown by the quantum equations of motion

$$\hat{p} = \frac{d\hat{q}}{dt}, \quad \frac{d\hat{p}}{dt} = -\omega^2(t) \hat{q}$$

that

$$\frac{d}{dt} (\hat{q} \hat{p} - \hat{p} \hat{q}) = 0,$$

therefore the commutation $(\hat{q} \hat{p} - \hat{p} \hat{q})$ is actually a c-number.

On the other hand, \hat{I}_S is a legitimate quadratic invariant. Let us compare it with (12),

$$\hat{I}_Q = \left(\gamma \omega^2 + \frac{1}{2} \frac{d^2\gamma}{dt^2} \right) \hat{q}^2 - \frac{1}{2} \frac{d\gamma}{dt} \{\hat{q}, \hat{p}\} + \gamma \hat{p}^2, \quad (25)$$

where (7) are explicitly used. It is straightforward to see that $\hat{I}_S = \hat{I}_Q$ if $\gamma = \beta^* \beta$, and

$$\gamma \omega^2 + \frac{1}{2} \frac{d^2\gamma}{dt^2} = \frac{d\beta^*}{dt} \frac{d\beta}{dt}.$$

These conditions are the same ones of the classical case. Then, the Ermakov equation (21) and the Ermakov invariant (22) follow from the same calculations.

V. FINAL REMARKS

In this work we showed how linear and quadratic dynamical invariants of the classical and quantum one-dimensional harmonic oscillator can be derived exclusively from the equations of motion.

In the case of the linear invariants (6) and (11), they arise directly from linear combinations of the first-order equations, and are parameterized by a β function that obeys a second-order ODE (4). Quadratic invariants, on the other hand, arise with linear combinations of quadratic products of the equations of motion. In this case the ordering problem must be taken account in the quantum case. They have the general forms (8) and (12) in the classical and quantum systems, respectively, with the functions γ , ϵ , and ζ obeying the ODE (7). These equations define dependency between these functions, so the most general quadratic invariant takes the form (15) in the classical case, and (25) in the quantum system. The function γ is then solution of the ODE (9).

In Sec. IV we used the linear invariants to build quadratic ones, and analyze under which conditions these invariants coincide with the quadratic ones derived in Secs. II and III. In the classical case, the usual product of functions of q and p is sufficient, and the product of two linear invariants becomes the quadratic invariant (15) if (18a) is satisfied.

In the quantum case, the ordering problem of the operators \hat{q} and \hat{p} forced us to consider antisymmetric and symmetric products of the linear invariants (11). We saw that the antisymmetric product is not really a quadratic invariant, so we are left with the symmetric product (24). This invariant becomes equal to (25) under the same conditions derived in the classical case.

In both classical and quantum systems, since the equations of the parameter γ are the same, the Ermakov invariant arise by making γ a positive real function ($\gamma = \beta^* \beta = \rho^2$). Therefore, the Ermakov invariant becomes a special case of the quadratic invariants (15) and (25).

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- ¹E. Noether, "Invariant variational problems," (translated by M. A. Travel), e-print [arXiv:physics/0503066v1](http://arxiv.org/abs/physics/0503066v1) [physics.hist-ph].
- ²W. Paul, "Electromagnetic traps for charged and neutral particles," *Rev. Mod. Phys.* **62**, 531–540 (1990).
- ³D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, "Quantum dynamics of single trapped ions," *Rev. Mod. Phys.* **75**, 281–324 (2003).
- ⁴I. A. Malkim, V. I. Man'ko, and D. A. Trifonov, "Coherent states and transition probabilities in a time-dependent electromagnetic field," *Phys. Rev. D* **2**, 1371 (1970).
- ⁵I. A. Malkim and V. I. Man'ko, "Coherent states and excitation of N-dimensional non-stationary forced oscillator," *Phys. Lett. A* **32**, 243 (1970).
- ⁶G. T. Moore, "Quantum theory of the electromagnetic field in a variable-length one-dimensional cavity," *J. Math. Phys.* **11**, 2679 (1970).
- ⁷H. Johnston and S. Sarkar, "A re-examination of the quantum theory of optical cavities with moving mirrors," *J. Phys. A* **29**, 1741–1746 (1996).
- ⁸V. V. Dodonov, A. B. Klimov, and V. I. Man'ko, "Nonstationary Casimir effect and oscillator energy level shift," *Phys. Lett. A* **142**, 511–513 (1989).
- ⁹J. P. Bernardin and N. M. Lawandy, "Stable mode locking of a coupled cavity moving mirror ring dye laser," *J. Mod. Opt.* **39**, 457–460 (1992).
- ¹⁰V. A. Alekseev and D. D. Krylova, "Quantum theory of laser cooling in a Paul trap," *Laser Phys.* **6**, 268–273 (1996).
- ¹¹V. P. Ermakov, "Second-order differential equations: conditions of complete integrability," (translated by A. O. Harin), *Appl. Anal. Discr. Math.* **2**, 123–145 (2008).
- ¹²R. S. Kaushal, "Construction of exact invariants for time dependent classical dynamical systems," *Int. J. Theor. Phys.* **37**, 1793–1856 (1998).
- ¹³R. S. Kaushal, "Quantum analogue of Ermakov systems and the phase of the quantum wave function," *Int. J. Theor. Phys.* **40** 835–847 (2001).
- ¹⁴V. I. Man'ko, "Quantum invariants and state system generating algebra," in *Group Theoretical Methods in Physics: Proceedings of the Yurmana Seminar* (VNU Science, 1985), Vol. II.