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Tsukamoto's Theorem in Characteristic two

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ABSTRACT: In this paper it is proved that hermitian forms over quaternion division algebras over local fields of characteristic two are classified by their dimension and discriminant.

Key Words: Hermitian form, quaternion algebra, local field.

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1. Introduction

The Tsukamoto's theorem classifies skew-hermitian forms over quaternion division algebras over local fields of characteristic different from two. It was generalized by Becher and Mahmoudi to a quaternion division algebra over a Kaplansky field, (see §6 of [2]). In this article we consider non-singular (or regular) hermitian forms over a quaternion division algebra over a local field of characteristic two and we show that Tsukamoto's classification is also valid in this case. We see that these forms correspond to skew-hermitian forms over quaternion division algebras over fields of characteristic different from two. The main theorem (see Theorem 3.1) is very similar to theorem 3.6 of Chapter 10 of [7] and the Theorem 3 of [8]. However we can see in the following corollary that the structures of these forms are independent of the characteristic.

In order to state our results we need some notation. Throughout this paper F will always denotes a field of characteristic two, and $\dot{F}$ its multiplicative group of nonzero elements.

We denote by $\mathcal{Q}$ a *quaternion algebra* over a field F. There always exists an F-basis $\{1, i, j, k\}$ of $\mathcal{Q}$ with multiplication given by $ij + ji = i,\ j^2 + j = b \in F,\ i^2 = a \in \dot{F},\ ji = k$, (see Chapter 8, Section 11 of [7]). Every basis $\{1, i, j, k\}$ satisfying

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the above relations is called *standard basis* of the quaternion algebra. In this case, we also denote $\mathcal{Q}$ by $\big[(b,a)/F\big)$.

The *standard involution* $\sigma : \mathcal{Q} \to \mathcal{Q}$ is given by $\sigma(x) = \alpha + \beta + \beta j + \gamma i + \delta k$, for all $x = \alpha + \beta j + \gamma i + \delta k \in \mathcal{Q}$. The element $x\sigma(x)$ belongs to F, and is called the *norm* of x and is denoted by $N(x)$. Considering the standard basis $\{1, i, j, k\}$ of the F-vector space $\mathcal{Q}$, the norm $N : \mathcal{Q} \to F$ is the quadratic form denoted by $[1, b] \perp \langle a\rangle[1, b]$, that is $N(x) = \alpha^2 + \alpha\beta + b\beta^2 + a(\gamma^2 + \gamma\delta + b\delta^2)$. In general, the two-dimensional quadratic forms $c\alpha^2 + d\beta^2$ and $ec\alpha^2 + e\alpha\beta + ed\beta^2$ over F will be denoted by $[c] \perp [d]$ and $\langle e\rangle[c, d]$, respectively. For a quadratic form $q : V \to F$ (V an F-vector space), $D_F q = \{q(u) \in F,\ u \in V \setminus \{0\}\}$ denotes the subset of elements of F represented by q. For instance, $D_F([1] \perp [a]) \subset D_F([1] \perp \langle a\rangle[1, b]) \subset D_F N$. It is known that $\mathcal{Q}$ is a division algebra if and only if $0 \notin D_F N$. The quadratic form q is *universal* if q represents all $\alpha \in \dot{F}$. We refer to [1] for general facts about quadratic forms in characteristic two.

The element $x \in \mathcal{Q}$ is said to be symmetric if $\sigma(x) = x$, and we denote by $Sym(\mathcal{Q})$ the subset of all symmetric elements of $\mathcal{Q}$. It is easy to see that $Sym(\mathcal{Q}) = F + Fi + Fk = \{x \in \mathcal{Q} \mid x^2 = N(x) \in F\}$. The quadratic form $N_{Sym(\mathcal{Q})} : Sym(\mathcal{Q}) \to F$ is denoted by $[1] \perp \langle a\rangle[1, b]$. For each $x = \alpha + \gamma i + \delta k \in Sym(\mathcal{Q})$ we have $N(x) = \alpha^2 + a(\gamma^2 + \gamma\delta + b\delta^2)$.

2. Preliminaries

Let $\mathcal{Q}$ be a quaternion division algebra over a field F. An *hermitian form* on a finite dimensional $\mathcal{Q}$-right vector space V is a map $h : V \times V \to \mathcal{Q}$ which satisfies the following conditions:

$$h(u+v, w) = h(u, w) + h(v, w), \quad h(u, v+w) = h(u, v) + h(u, w),$$
$$h(u\alpha, v\beta) = \sigma(\alpha)h(u, v)\beta, \quad \text{and} \quad \sigma(h(u, v)) = h(v, u),$$

for all $u,\ v,\ w \in V$ and all $\alpha,\ \beta \in \mathcal{Q}$.

We will refer to h as being an *hermitian form* over $\mathcal{Q}$ and V as its *underlying vector space*. The pair (V, h) is called an *hermitian space*. The $\mathcal{Q}$-dimension of V is said to be the *dimension* of h over $\mathcal{Q}$; $dim_{\mathcal{Q}} h$, and also the *dimension* of the hermitian space (V, h) over $\mathcal{Q}$.

The hermitian form h over $\mathcal{Q}$ (or hermitian space (V, h) is said to be *regular* or *nondegenerate* if, $h(u, v) = 0$, for every $v \in V$, then $u = 0$, that is, if for any $u \in V \setminus \{0\}$, the associated $\mathcal{Q}$-linear form $V \to \mathcal{Q},\ v \to h(u, v)$ is nontrivial. Otherwise, h or (V, h) is said to be *singular* or *degenerate*.

We say that an hermitian form h, or hermitian space (V, h) is *isotropic* if there exists a vector $u \in V \setminus \{0\}$ such that $h(u, u) = 0$, and h or (V, h) is *anisotropic* in otherwise.

We say that the hermitian form h *represents* the element $z \in \mathcal{Q}$ if there exists $u \in V \setminus \{0\}$ such that $h(u,u) = z$. Denote by Dh the subset of elements of $\mathcal{Q}$ represented by h. Thus $0 \in Dh$ if and only if h is isotropic. Of course, $Dh \subset Sym(\mathcal{Q})$ and we say that h is *universal* if h represents all $z \in Sym(\mathcal{Q})$.

An *isometry* between two hermitian spaces (V_1, h_1) and (V_2, h_2), or between h_1 and h_2 is an isomorphism of $\mathcal{Q}$-vector spaces $\tau : V_1 \to V_2$, such that $h_1(u,v) = h_2(\tau(u), \tau(v))$, for every $u,\ v \in V_1$. In this case we say that (V_1, h_1) and $V_2, h_2)$, or h_1 and h_2 are *isometric* and write $(V_1, h_1) \simeq (V_2, h_2)$, or $h_1 \simeq h_2$ to indicate this.

Given two hermitian spaces (V_1, h_1) and (V_2, h_2) over $\mathcal{Q}$ the *orthogonal sum* of h_1 and h_2, denoted by $h = h_1 \perp h_2$, is the hermitian form over $\mathcal{Q}$ with underlying vector space $V = V_1 \oplus V_2$ defined by $h(u,v) = h_1(u_1, v_1) + h_2(u_2, v_2)$, for every $u = (u_1, u_2),\ v = (v_1, v_2) \in V$. The hermitian space (V, h) is denoted by $(V_1, h_1) \perp (V_2, h_2)$. In particular, if $V_1,\ V_2$ are subspaces of V such that $V = V_1 \oplus V_2$ and $h(u,v) = 0$ for every $u \in V_1,\ v \in V_2$ then $(V,h) \simeq (V_1, h_{V_1}) \perp (V_2, h_{V_2})$.

An hermitian form h with underlying vector space V is said to be *diagonalizable* if there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of V such that $h(u,v) = \sum_{i=1}^{n} \sigma(x_i) a_i y_i$, (with $a_i \in Sym(\mathcal{Q})$) for all $u = \sum_{i=1}^{n} e_i x_i$ and $v = \sum_{i=1}^{n} e_i y_i \in V$. We denote h by $\langle a_1, a_2 \ldots, a_n \rangle$, or also by $n\langle a \rangle$, if $a_i = a$ for all $i = 1, 2, \ldots, n$. It follows that h is regular if and only if $a_i \neq 0$, for all $a_i \in \mathcal{Q}$.

Two elements $a,\ b \in Sym(\mathcal{Q})$ are *congruent* if there exists $c \in \mathcal{Q}$ such that $b = \sigma(c) a c$, which is equivalent to saying that $\langle a \rangle \simeq \langle b \rangle$ over $\mathcal{Q}$.

For an element $a \in \dot{F}$ we define the *scaled hermitian form* ah by $(ah)(u,v) = a.h(u,v)$, for all $u,\ v$ belonging to underlying vector space of h. In particular $a\langle a_1, a_2, \ldots, a_n \rangle = \langle aa_1, aa_2, \ldots, aa_n \rangle$. Two hermitian forms h and h_1 over $\mathcal{Q}$ are said to be *similar* if $h_1 \simeq ah$, for some $a \in \dot{F}$.

The *Grothendieck group* and the *Witt group* of the regular hermitian forms over $\mathcal{Q}$ are denoted, respectively, by $\widehat{W}(\mathcal{Q})$ and by $W(\mathcal{Q})$. A regular hermitian space (V, h) such that there exists a decomposition $V = N \oplus P$ with $N = N^{\perp}$, that is, $h = \begin{pmatrix} 0 & \alpha \\ \sigma(\alpha) & \beta \end{pmatrix}$ is called *metabolic hermitian space*. We denote the two-dimensional metabolic hermitian space $h = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ by $\mathbb{M}(a)$. The following lemma is due to knebusch and can be seen in ([4], Chapter I, Proposition 3.7.6) or ([7], Chapter 7, Lemma 3.7).

Lemma 2.1. *Let (V,h) be a metabolic hermitian space. Then $(V,h) \perp (V,-h) \simeq \mathbb{H}(N) \perp (V,-h)$ and thus $[V,h] = [\mathbb{H}(N)]$ in $\widehat{W}(\mathcal{Q})$, where $\mathbb{H}(N)$ is an hyperbolic space for some subspace N of V and $[V,h]$ is the isometry class of (V,h). In particular $[V,h]$ is zero in $W(\mathcal{Q})$.*

Proposition 2.2. (Chapter I; 6.1.1 and 6.1.4 of [4]) *Let* (V,h) *be a regular hermitian space over* $\mathcal{Q}$. *There exists an orthogonal decomposition* $(V,h) \simeq (V',h_{an}) \perp \mathbb{M}(a_1) \perp \cdots \perp \mathbb{M}(a_r)$, *with* h_{an} *anisotropic or zero and* $r \geq 0$. *Furthermore,* (V',h_{an}) *is uniquely determined up to isometry by* (V,h). *In particular,* (V,h), *(or* h*) is isotropic if and only if* $r \geq 1$.

We write $h \simeq h_{an} \perp h_{\mathbb{M}}$, where $h_{\mathbb{M}}$ is a metabolic hermitian space.

As in [8] and also [7] the *discriminant* of an hermitian form h (or hermitian space (V,h)) over $\mathcal{Q}$ will be denoted by $disc(h)$ and it is defined as follows: Let $\{e_1, e_2, \ldots, e_n\}$ be an $\mathcal{Q}$-basis of V. Denoting by Nrd the *reduced norm* from $M_n(\mathcal{Q})^{\cdot}$ to $\dot{F}$, we put $disc(h) = (-1)^n Nrd\big((h(e_i,e_j))\big) mod \dot{F}^2$. It is known that $disc(h)$ is independent of the choice of the basis of V and is also independent of the choice of the splitting field of $\mathcal{Q}$,(see Chapter 8, Lemma 5.7 of [7], 16.1 of [5] or §22 of [3]). In particular, given the quaternion algebra $\big[(b,a)/F\big)$, if we take the algebraic closure $\overline{F}$ of F, we have an F-algebra homomorphism $\varphi : \big[(b,a)/F\big) \to M_2(\overline{F})$ given by $\varphi(i) = i_0$ and $\varphi(j) = j_0$, where $i_0 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ and $j_0 = \begin{pmatrix} \beta & 0 \\ 0 & \beta+1 \end{pmatrix}$ in $M_2(\overline{F})$ (the algebra of 2×2 matrices over $\overline{F}$) and α, β are elements of $\overline{F}$ such that $\alpha^2 = a$, $\beta^2 + \beta + b = 0$. It follows that $Nrd(\langle z \rangle) = ([1,b] \perp \langle a \rangle [1,b])(x_1,x_2,x_3,x_4) = N(z)$, where $z = x_1 + x_2 j + x_3 i + x_4 k \in \mathcal{Q}$ and $x_1, x_2, x_3, x_4 \in F$.

Since reduced norm is multiplicative ([3], §22(7) and §20, Theorem 1 and §22, Theorem 1) or ([5], §16.5, Corollary b), it follows that $disc(h_1 \perp h_2) = disc(h_1).disc(h_2)$. Now, if $\tau : (V_1,h_1) \overset{\sim}{\to} (V_2,h_2)$ is an isometry and B_1, B_2 are F-basis of V_1 and V_2 respectively, take $(h_1)_{B_1}$, $(h_2)_{B_2}$, $(\alpha_{ij}) = (\tau)_{B_1 B_2}$ the matrices of h_1, h_2 and τ, with respect to the given basis. Then $(h_1)_{B_1} = \big(\sigma(\alpha_{ij})\big)^t (h_2)_{B_2}(\alpha_{ij})$, where $\sigma : \mathcal{Q} \to \mathcal{Q}$, is the standard involution. From Lemma 5 of ([3], §22) we get

$$Nrd\big((h_1)_{B_1}\big) = \sigma\big(Nrd\big((\alpha_{ij})^t\big)\big) Nrd\big((h_2)_{B_2}\big) Nrd\big((\alpha_{ij})\big) = Nrd\big((h_2)_{B_2}\big).$$

in $\dot{F}/\dot{F}^2$, since $Nrd\big((\alpha_{ij})^t\big) \in F$. Thus Nrd and $disc$ does not depend of the isometry class of (V_1,h_1). Furthermore, as hyperbolic space and metabolic hermitian space has *Dieudonné determinant* $1.\dot{F}^2$, (see [3] §19 Example 1, §20 Definitions 1 and 3) the Proposition 5 of [8] holds for any characteristic:

Proposition 2.3. *The mapping* $h \to disc(h)$ *induces an homomorphism from the Witt group* $W(\mathcal{Q})$ *into* $\dot{F}/\dot{F}^2$.

The mapping $W(\mathcal{Q}) \to \dot{F}/\dot{F}^2$ will also be denoted by *disc*.

Proposition 2.4. *Two one-dimensional hermitian forms over* $\mathcal{Q}$ *are similar if and only if their discriminants are the same in* $\dot{F}/\dot{F}^2$.

Proof: It is exactly the same as ([2], Proposition 4.2). $\square$

The following lemma in characteristic different from two is due to Scharlau ([7], Chapter 10, Lemma 3.4).

Lemma 2.5. *Let $\lambda \in Sym(\mathcal{Q})\backslash\{0\}$ and $c \in \dot{F}$. If $\lambda \notin F$, then the hermitian forms $\langle\lambda\rangle$ and $\langle c\lambda\rangle$ are isometric over $\mathcal{Q}$ if and only if c is represented over F by the quadratic form $[1] \perp [a]$. If $\lambda \in F$, then the hermitian form $\langle\lambda\rangle$ and $\langle c\lambda\rangle$ are isometric over $\mathcal{Q}$ if and only if c is a norm in F.*

Proof: We have $\langle\lambda\rangle \simeq \langle c\lambda\rangle$ over $\mathcal{Q}$ if and only if

$$\sigma(x)\lambda x = c\lambda, \qquad (*)$$

for some $x \in \mathcal{Q}$. Thus, if $\lambda \in F$, then $(*)$ holds if and only if $\sigma(x)x = c$, that is, c is a norm in F. If $\lambda \notin F$, then there exist $j' \in \mathcal{Q}$, $s \in F$ such that $j'^2 + j' + s = 0$, $j'\lambda + \lambda j' = \lambda$ (If $\lambda = \alpha + \beta i + \gamma k$, then β or γ is nonzero. Take $j' = j + \frac{\alpha}{a\beta}i$, if β is nonzero, or $j' = j + \frac{\alpha}{a\gamma}i$ if β is zero). Thus, $\big[(N(j'), N(\lambda))/F\big)$ is a quaternion algebra contained in $\mathcal{Q}$. It follows that $\mathcal{Q} = \big[(N(j'), N(\lambda))/F\big)$ and therefore we can suppose $\lambda = i$. Replacing in $(*)$ we get $\sigma(x)ix = ci$, for some $x \in \mathcal{Q}$, or equivalently $N(x)ix = cxi$. Writting $x = y + zj$, with $y,\ z \in Fi$. Thus $N(x)(iy + izj) = c\big(iy + z(i + ij)\big)$, (because $yi = iy$ and $ji = i + ij$). Equivalently, $N(x)iy + N(x)izj = ci(y + z) + cizj$, and so

$$\begin{cases} N(x)y = c(y+z) \\ N(x)z = cz. \end{cases}$$

If $z \neq 0$, then $N(x) = c$ and $cz = 0$. Thus $z = 0$, absurd. It follows that $z = 0$, $x = y$ and $c = N(y) \in D_F([1] \perp [a])$. $\square$

Remark 2.6. *If we consider $x = y + iz$, $y, z \in F + Fj$, we may conclude that $y, z \in F$. Thus, once again $N(x) \in D_F([1] \perp [a])$.*

3. Main Results

The field F in question is local field of characteristic two, that is, $F = K((t))$ (the field of Laurent's power series of K), where K is a finite field of characteristic two. Every element $f \in F$ is of the form $f = t^m(1 + a_1t + a_2t^2 + \cdots)$, $a_i \in K$, $m \in \mathbb{Z}$. Since $K = K^2$, f can be written in the form $f = g^2 + th^2$, for some $g,\ h \in F$. Thus $\{1, t\}$ is a basis for the F^2-vector space F and the quadratic form $[1] \perp [t]$ is universal over F. The unique quaternion division algebra over F, up to isomorphism, is $\mathcal{Q} = \big[(b, t)/F\big)$, for some $b \in F$ and their norm form is $N = [1, b] \perp \langle t\rangle[1, b]$ up to isometry, (see, for instance, ([1], Chapter II, Proposition 1.19 and [6], Lemma 1.7)

Theorem 3.1. *Let $F = K((t))$ be a local field of characteristic two and $\mathcal{Q} = \big[(b,t)/F\big)$ be the unique quaternion division algebra over F, up to isomorphism. Then*

(a) For any dimension ≥ 1 there are regular hermitian forms of any discriminant.

(b) A two-dimensional regular hermitian form over $\mathcal{Q}$ is isotropic if and only if has trivial discriminant.

(c) Any regular hermitian form h with $dim_{\mathcal{Q}} \geq 2$, is the form $h \simeq \langle z\rangle \perp h_{\mathbb{M}}$ if $dim_{\mathcal{Q}}h$ is odd and $h \simeq h_{an} \perp h_{\mathbb{M}}$ if $dim_{\mathcal{Q}}h$ is even, for some metabolic hermitian space $h_{\mathbb{M}}$ and $h_{an} = 0$ or $\langle 1, z\rangle$ for some $z \in Sym(\mathcal{Q})$.

(d) Let h_1 and h_2 be regular hermitian forms of equal dimension over $\mathcal{Q}$. Then $disc(h_1) = disc(h_2)$ if and only if $(h_1)_{an} \simeq (h_2)_{an}$.

Proof: (a) Since $D_F([1] \perp [t]) = \dot{F}$ and $D_F([1] \perp [t]) \subset D_F([1] \perp \langle t\rangle[1,b])$, for any $\alpha \in \dot{F}$ there exists $z_0 \in Sym(\mathcal{Q})$ such that $N(z_0) = \alpha$. Thus the hermitian forms $\langle z_0\rangle$ and $\langle 1, \dots, z_0\rangle$ have discriminant α.

Now, we show (d) for 1-dimensional forms. Let $z_1,\ z_2 \in Sym(\mathcal{Q})$ and assume that hermitian forms $\langle z_1\rangle$ and $\langle z_2\rangle$ over $\mathcal{Q}$ have the same discriminant. According to Proposition (2.4), $\langle z_1\rangle \simeq \langle cz_2\rangle$ for some $c \in \dot{F}$. Since $\dot{F} = D_F([1] \perp [t])$ and $D_F([1] \perp [t] \subset D_F N$, by Lemma (2.5) we obtain $\langle cz_2\rangle \simeq \langle z_2\rangle$ and so $\langle z_1\rangle \simeq \langle z_2\rangle$.

(b) Let $z_1, z_2 \in Sym(\mathcal{Q})$ be such that the form $\langle z_1, z_2\rangle$ has discriminant 1. Then $Nrd(\langle z_1\rangle)$ and $Nrd(\langle z_2\rangle)$ represent the same element in $\dot{F}/\dot{F}^2$. This means that $\langle z_1\rangle \simeq \langle z_2\rangle$ by what we showed above. It follows that $\langle z_1, z_2\rangle$ is isotropic.

Conversely, if h is an 2-dimensional regular hermitian form over $\mathcal{Q}$ and h is isotropic then there is a basis $B = \{u, v\}$ such that $h(u,u) = 0,\ h(u,v) = h(v,u) = 1$. Thus $h \simeq \mathbb{M}(h(v,v))$ and $disc(h) = 1$.

(c) First we give Tsukamoto's argument to show that every 3-dimensional regular hermitian form over $\mathcal{Q}$ is isotropic. Suppose that h is anisotropic. Since h can be diagonalized ([4], Chapter I, Lemma 6.2.1) we may assume that $h = \langle z_1, z_2, z_2\rangle$, with $z_1, z_2, z_3 \in Sym(\mathcal{Q})\backslash\{0\}$. From (a) there exists $z_0 \in Sym(\mathcal{Q})$ such that $Nrd(\langle z_0\rangle) = disc(h)$. As $Sym(\mathcal{Q})$ has F-dimension 3, there exist $c_0, c_1, c_2, c_3 \in F$, not all zero, such that $c_0z_0 + c_1z_1 + c_2z_2 + c_3z_3 = 0$. For $c_i \neq 0$, the Proposition (2.4) implies that $\langle c_iz_i\rangle$ and $\langle z_i\rangle$ are similar, that is, $c_iz_i = \sigma(d_i)z_id_i$, for some $d_i \in \dot{F},\ i = 0, 1, 2, 3$. If we take $d_i = 0$ for $c_i = 0$, we obtain $\sum_{i=0}^{3} \sigma(d_i)z_id_i = 0$ and therefore $\langle z_0\rangle \perp h$ is isotropic. From Proposition (2.2) we have $\langle z_0\rangle \perp h \simeq h_1 \perp \mathbb{M}(a)$, for some $a \in Sym(\mathcal{Q})$. Since $disc(\mathbb{M}(a)) = 1$ it follows that $disc(h_1) = disc(\langle z_0\rangle \perp h) = 1$. From (b) and Lemma (2.1) we get $\langle z\rangle \perp h = 0$ in $W(\mathcal{Q})$, that is, $h = \langle z_0\rangle$ in $W(\mathcal{Q})$. As $dim_{\mathcal{Q}}h = 3,\ h$ is isotropic, absurd. This concludes the first part. From Proposition (2.2) and the first part every regular hermitian form h with $dim_{\mathcal{Q}} h \geq 2$ is the form $h \simeq \langle z \rangle \perp h_{\mathbb{M}}$, for some $z \in Sym(\mathcal{Q})$, if $dim_{\mathcal{Q}} h$ is odd, and $h \simeq h_{an} \perp h_{\mathbb{M}}$, for some metabolic hermitian space $h_{\mathbb{M}}$ and $h_{an} = 0$ or $h_{an} \simeq \langle z_1, z_2 \rangle$, if $dim_{\mathcal{Q}} h$ is even.

If $h_{an} \simeq \langle z_1, z_2 \rangle$, by (a) $disc(h_{an}) = disc(\langle z_0 \rangle)$, for some $z_0 \in Sym(\mathcal{Q})$. Then $\langle 1, z_0 \rangle \perp h_{an}$ is isotropic from the first part. We write $\langle 1, z_0 \rangle \perp h_{an} \simeq h_1 \perp \mathbb{M}(a)$, for some two-dimensional regular hermitian form h_1 and $a \in Sym(\mathcal{Q})$. As before $disc(h_1) = 1$ and by part (b) and Proposition (2.2) we obtain $\langle 1, z_0 \rangle \perp h_{an} = 0$ in $W(\mathcal{Q})$. Thus $h_{an} \simeq \langle 1, z_0 \rangle$.

(d) Let h_1 and h_2 be regular hermitian forms of equal dimension over $\mathcal{Q}$. Then $h_1 \simeq \langle z_1 \rangle \perp (h_1)_{\mathbb{M}}$, $h_2 \simeq \langle z_2 \rangle \perp (h_2)_{\mathbb{M}}$ if $dim_{\mathcal{Q}} h_i$ is odd, $i = 1, 2$, or $h_1 \simeq (h_1)_{an} \perp (h_1)_{\mathbb{M}}$, $h_2 \simeq (h_2)_{an} \perp (h_2)_{\mathbb{M}}$, if $dim_{\mathcal{Q}} h_i$ is even, $i = 1, 2$. Suppose $dim_{\mathcal{Q}} h_i = 1$, $i = 1, 2$. Since $disc(h_{\mathbb{M}}) = 1$, from Proposition (2.4) $disc(h_1) = disc(h_2)$ if and only if $\langle z_1 \rangle \simeq \langle z_2 \rangle$. If $dim_{\mathcal{Q}} h_i$ is even for $i = 1, 2$ and $(h_1)_{an} \simeq \langle 1, z_1 \rangle$, $(h_2)_{an} \simeq \langle 1, z_2 \rangle$ then $disc(h_1) = disc(h_2)$ implies that $disc(\langle z_1 \rangle) = disc(\langle z_2 \rangle)$. From Proposition (2.4) $\langle z_1 \rangle \simeq \langle z_2 \rangle$, and so $(h_1)_{an} \simeq (h_2)_{an}$. Clearly $(h_1)_{an} \simeq (h_2)_{an}$ and $dim_{\mathcal{Q}} h_1 = dim_{\mathcal{Q}} h_2$ implies that $disc(h_1) = disc(h_2)$. Finally, if there is the case $(h_1)_{an} = 0$ and $(h_2)_{an} \simeq \langle 1, z_2 \rangle$ anisotropic, then $disc(h_1) \neq disc(h_2)$ and so $(h_1)_{an}$ is not isometric to $\langle 1, z_2 \rangle$. $\square$

Corollary 3.2. *Let $F = K((t))$ be a local field of characteristic two and $\mathcal{Q}$ the unique nonsplit quaternion algebra over F. Then $\widehat{W}(\mathcal{Q}) \tilde{\to} \mathbb{Z} \oplus \dot{F}/\dot{F}^2$, and $W(\mathcal{Q}) \tilde{\to} \mathbb{Z}/2\mathbb{Z} \oplus \dot{F}/\dot{F}^2$.*

Proof: We denote the elements of $\widehat{W}(\mathcal{Q})$ by $h_1 - h_2$ as into ([7], page 239) and we define $\varphi(h_1 - h_2) = (dim_{\mathcal{Q}} h_1 - dim_{\mathcal{Q}} h_2, disc(h_1).disc(h_2)^{-1})$. Then φ is clearly a group homomorphism. If $\varphi(h_1 - h_2) = (0, 1)$, then $dim_{\mathcal{Q}} h_1 = dim_{\mathcal{Q}} h_2$ and $disc(h_1) = disc(h_2)$. From Theorem (3.1, (d)) $(h_1)_{an} \simeq (h_2)_{an}$ and $dim_{\mathcal{Q}}\big((h_1)_{\mathbb{M}}\big) = dim_{\mathcal{Q}}\big((h_2)_{\mathbb{M}}\big)$. From Lemma (2.1) $(h_1)_{\mathbb{M}} \simeq (h_2)_{\mathbb{M}}$ and so $h_1 - h_2 = \big((h_1)_{an} - (h_2)_{an}\big) + \big((h_1)_{\mathbb{M}} - (h_2)_{\mathbb{M}}\big) = 0$ in $\widehat{W}(\mathcal{Q})$. Therefore φ is injective. For every $(m, a) \in \mathbb{Z} \oplus \dot{F}/\dot{F}^2$ take $z_0 \in Sym(\mathcal{Q})$ such that $disc(\langle z_0 \rangle) = a$ and $h = \langle 1, 1, \ldots, z_0 \rangle$, with $dim_{\mathcal{Q}} h = |m| + 1$. Then $\varphi(h - \langle 1 \rangle) = (m, a)$, if $m \geq 0$ and $\varphi(\langle 1 \rangle - h) = (m, a)$, if $m < 0$. This conclude that φ is an isomorphism and induces an isomorphism $\overline{\varphi} : W(\mathcal{Q}) \to \mathbb{Z}/2\mathbb{Z} \oplus \dot{F}/\dot{F}^2$, given by $\overline{\varphi}(h) = \big(dim_{\mathcal{Q}} h(mod 2), disc(h)\big)$. The Lemma (2.1) and the Proposition (2.2) imply that the elements of the Witt group are determined by isometric classes of anisotropic regular hermitian forms. Thus $W(\mathcal{Q}) = \{\langle z_0 \rangle, \langle 1, z_0 \rangle \mid z_0 \in Sym(\mathcal{Q})\}$. $\square$

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