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Comparing Tsai-Wu and Tsai-Hill Failure Criteria for High-Pressure Vessel Design under Uncertainty

Henrique Cordeiro Novais
Samuel da Silva
UNESP - Universidade Estadual Paulista, Ilha Solteira, SP, Brasil
henrique.novais@unesp.br, samuel.silva13@unesp.br

Mariana Pimenta Alves
Carlos Alberto Cimini Junior
UFMG - Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brasil
maripimenta8@gmail.com, carlos.cimini@gmail.com

Abstract. *This study compares two widely used failure theories for characterizing composite materials: the Tsai-Wu and Tsai-Hill failure criteria. The paper's main objective is to evaluate the performance of both criteria in a high-pressure vessel design project, taking into account uncertainties in material properties at the lamina level and in the internal pressure applied to the vessel. The analysis of histograms of failure index fluctuations reveals that, contrary to a purely deterministic approach, each failure criterion can be more or less conservative when there is variability in the system due to different factors. Additionally, a global sensitivity analysis using Sobol indices was conducted to determine the contributions of individual inputs to the output uncertainty. This probabilistic approach emphasizes the importance of considering uncertainty analysis to design structures that are both safer and more cost-effective.*

Keywords: *Composite materials, Failure criteria, High-pressure vessels, Uncertainty analysis, Sobol Indices*

1. INTRODUCTION

Composite materials are made by combining two or more distinct phases at a macroscopic scale. This unique combination results in a material that exhibits superior performance compared to its constituent materials when used separately (Daniel and Ishai, 2006). As the demand for high-performance materials continues to grow, composite materials have become an essential area of research and development, with new composite materials and manufacturing processes being developed. Due to their mechanical properties, e.g., high strength, low weight, and durability, composite materials have become increasingly popular (Tita, 1999). In particular, reinforced composites, such as carbon fiber-reinforced polymers (CFRP), have demonstrated exceptional mechanical properties that make them an adequate alternative in numerous applications, including aerospace, transportation, and construction. An illustration of this can be seen in pressure vessel applications, which have been extensively utilized to contain compressed natural gas in vehicles. Furthermore, analogous storage concepts are currently being explored for hydrogen gas storage (SOMERDAY and MARCHI, 2008).

Aiming to design safer and more economical structures, it is essential to consider uncertainty analysis in engineering projects. By incorporating uncertainty analysis, engineers can identify potential risk factors and determine the impact of these uncertainties on the overall design. This information can then be used to develop more robust and reliable structures to withstand unforeseen events and minimize the risk of failure (Li *et al.*, 2014). Two widely used failure theories for characterizing composite materials are the Tsai-Wu and Tsai-Hill failure criteria (Daniel and Ishai, 2006). In a purely deterministic analysis, there is space for discussion and comparison between failure criteria to determine which one is more or less conservative. However, Yanik *et al.* (2018) demonstrated that when considering the impact of uncertainties, the notion of one failure criterion being more conservative than the other, such as Tresca and Von Mises for metallic materials, becomes questionable.

Design involving composites can encounter various factors that contribute to the presence of uncertainties, such as the manufacturing process, microstructure of materials, mechanical testing variations, environmental factors, and operational conditions. Therefore, to effectively address real-world issues related to composite structures, combine deterministic failure analysis and structural design with non-deterministic factors (Azizian and Almeida, 2022).

In this paper, we utilized Monte Carlo (MC) and Polynomial Chaos Expansion (PCE) propagation methods to estimate uncertainties. Additionally, we conducted a global sensitivity analysis using Sobol Indices to identify critical input material properties and their interactions. Through these analyses, we aimed to compare the Tsai-Wu and Tsai-Hill failure criteria for a high-pressure vessel project, considering the impact of uncertainties on material properties. A graphical abstract of the research structure developed in this work is shown in Fig. 1.

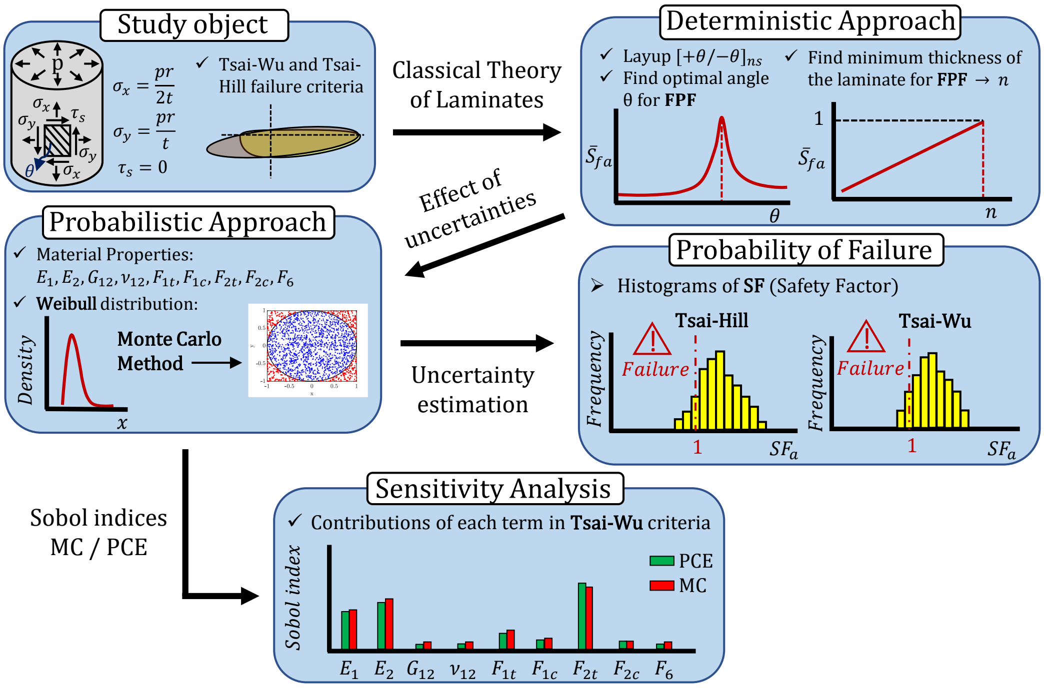

Figure 1. Graphical abstract representing the research developed in this work.

2. DETERMINISTIC APPROACH

The pressure vessel was assumed to be thin-walled, and we limited our analysis to the cylindrical section. However, our model did not include a localized bending correction, resulting in a purely membrane bi-axial stress state. This assumption allowed us to focus on the key factors affecting the vessel's structural integrity in this design section. An illustration of the vessel can be seen in Fig. 2, in addition to the state of stress in the overall system (off-axis).

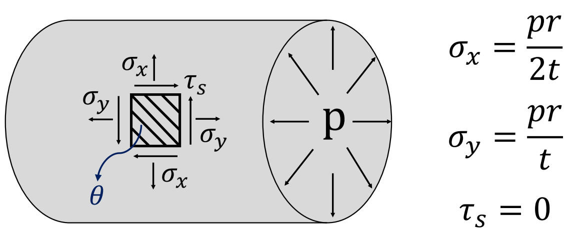

Figure 2. Schematic illustration depicting the pressure vessel and the stress state.

The composite material used in the simulations was carbon fiber with an epoxy resin matrix (CFRP T300/976), with a layer thickness of t_{ply} = 0.125 mm. Table 1 provides an overview of its material properties. The pressure vessel, with a diameter of D = 1 m, was subjected to an internal pressure of p = 10 MPa using a symmetric layup described as $[+\theta/-\theta]_{ns}$, where n represents the number of repetitions. The stress-strain relations were obtained using the Classical Laminate Theory Formulations (Daniel and Ishai, 2006). The internal pressure p was assumed to be the maximum value the vessel could withstand under critical conditions, well above its normal operating conditions.

To determine the ideal θ angle for the First Ply Failure condition (FPF) we used the Tsai-Wu failure criterion (Eq. (1)) to analyze the safety factor (SF) values for different θ angles. This enabled us to identify the optimal angle that would ensure the highest level of safety for the system, as shown in Fig. 3. Similar results were found by Onder *et al.* (2009), when investigating the burst failure load of composite pressure vessels using anti-symmetric laminated shell by using analytical and finite element simulations. Vignoli and Savi (2018) also found 55° as optimum angle in its multiscale failure analysis of cylindrical COPVs.

Table 1. Properties of the composite material CFRP T300/976.

Property	E_1	E_2	G_{12}	ν_{12}	F_{1t}	F_{1c}	F_{2t}	F_{2c}	F_6
λ	156 GPa	9.09 GPa	6.96 GPa	0.228	1520 MPa	1590 MPa	44.5 MPa	252 MPa	35.6 MPa
k	34	34	34	25	33	23	32	26	28

where $E_i(i = 1, 2)$ are the Young's modulus of the composite plies in the fiber direction and the direction transversely to the fiber direction, respectively, G_{12} is the shear modulus of the composite plies, F_{1t}, F_{2t}, F_{1c} and F_{2c} are the composite ply tensile and compressive strengths, respectively, F_6 is the composite ply shear strength, and λ and k are the scale and shape parameters of the Weibull distribution, respectively.

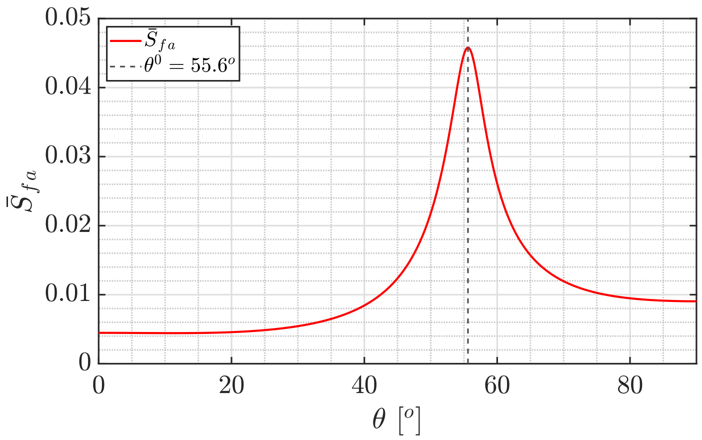

Figure 3. Safety factor as function of θ angles.

In order to determine the minimum thickness t of the laminate for the FPF condition, we calculated the minimum safety factor value for the laminate as a function of the number of layers repetitions n for both criteria. Figure 4 displays the results obtained.

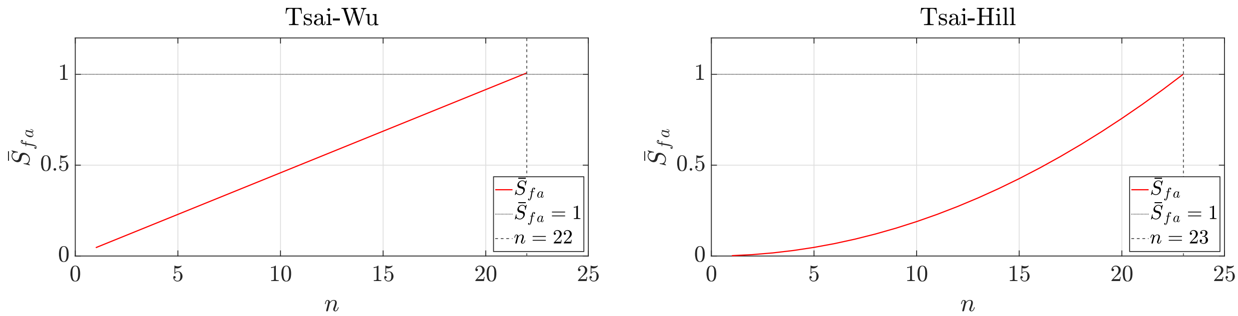

Figure 4. Safety factors as function of number of repetitions n for both criteria.

Using a deterministic approach, we determined that a layup of $[+55.6°/-55.6°]_{22s}$ and $[+55.6°/-55.6°]_{23s}$, composed of the CFRP T300/976 composite material, would be capable of withstanding the conditions to which the pressure vessel is subjected according to Tsai-Wu and Tsai-Hill failure criterion, respectively, which equations can be found in Eq. (1) and Eq. (2).

$$f_1\sigma_1 + f_2\sigma_2 + f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_6^2 + 2f_{12}\sigma_1\sigma_2 = 1 \quad (1)$$

where $f_1 = \frac{1}{F_{1t}} - \frac{1}{F_{1c}}$, $f_2 = \frac{1}{F_{2t}} - \frac{1}{F_{2c}}$, $f_{11} = \frac{1}{F_{1c}F_{1t}}$, $f_{22} = \frac{1}{F_{2c}F_{2t}}$, $f_{66} = \frac{1}{F_6^2}$, $f_{12} = -\frac{1}{2}\sqrt{f_{11}f_{22}}$.

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} + \frac{\tau_6^2}{F_6^2} - \frac{\sigma_1\sigma_2}{F_1^2} = 1 \quad (2)$$

where $F_1 = F_{1t}$ or F_{1c} and $F_2 = F_{2t}$ or F_{2c} depending on whether σ_1 and σ_2 are in a tension or compression state, respectively.

3. PROBABILISTIC APPROACH

In this work, variability in system parameters was assumed by considering the lamina material's mechanical properties as random variables, as shown in Tab. 1. The Weibull distribution function was used to describe each property, which is commonly used in composite material studies, as seen in previous literature (Azizian and Almeida, 2022), (Li *et al.*, 2014), (Vardhan *et al.*, 2019), (Wencheng, 2011). Equation 3 presents the probability density function of a Weibull random variable.

$$f(x;k,\lambda) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)e^{-(x/\lambda)^k}, & x \geq 0, \\ 0, & x < 0 \end{cases} \tag{3}$$

where k > 0 is the shape parameter and $\lambda > 0$ is the scale parameter of the Weibull distribution.

To visualize the histograms of each failure criterion (Tsai-Wu and Tsai-Hill) under material uncertainty, the Monte Carlo Method was implemented to evaluate the output. This allowed us to generate many random samples for the input variables (material properties) and assess the resulting output distribution. It should be noted that depending on the complexity of the system being studied, this method can be computationally expensive due to its slow convergence rate (Norenberg *et al.*, 2022). However, as we only deal with algebraic expressions in all simulations, Monte Carlo performed well in this case, converging with 5000 samples. Figure 5 displays the histograms of the Safety Factor for both failure criteria.

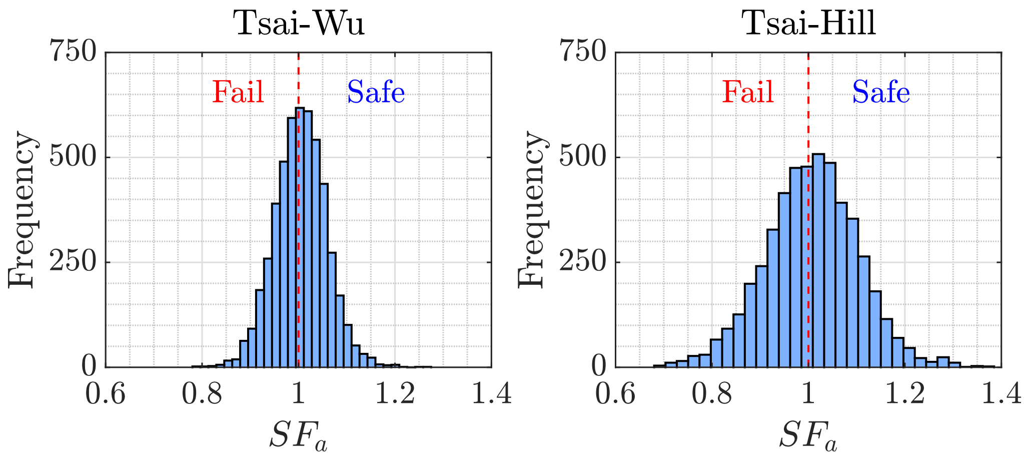

Figure 5. Safety factor histograms under material uncertainty estimated by Monte Carlo method for n = 23 repetitions.

Considering the effect of uncertainties, with the same layup obtained from the deterministic approach for each criteria, we found a 46.9% and 53.4% failure probability for Tsai-Wu and Tsai-Hill criteria, respectively. By increasing the number of layer repetitions, we found that the minimum value of n which provides a 0% failure probability is $n = 28$, which is common to both criteria, as shown in Fig. 6.

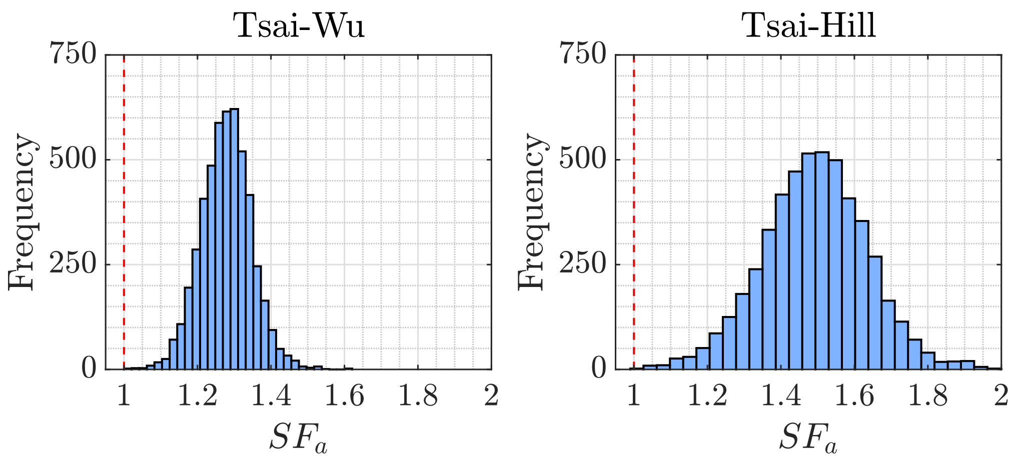

Figure 6. Safety factor histograms under material uncertainty estimated by Monte Carlo method for n = 28 repetitions.

Despite the same number of repetitions being common, the histograms highlights that Tsai-Wu failure criterion has a lower dispersion compared to Tsai-Hill. This can be observed in Fig. 7, where the probability density functions (PDFs) for both criteria are illustrated.

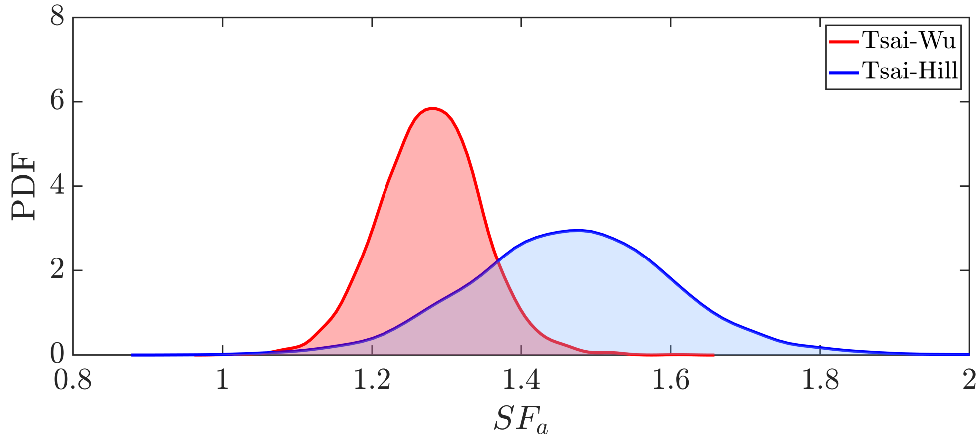

Figure 7. Probability density function for both criteria.

The results suggest that both the Tsai-Wu and Tsa-Hill failure criteria demonstrate a significant level of similarity for the specific application of the pressure vessel illustrated in Fig. 2. However, it is essential to note that the similarity between the Tsai-Wu and Tsa-Hill failure criteria is highly dependent on the material properties, more specifically, related to the anchoring points of each failure criterion as a function of different materials. A failure envelope study can provide a comprehensive understanding of this relationship. Figure 8 presents the failure envelope in the first quadrant for both criteria for the CFRP T300/976 material (with a 95% confidence interval). Under an uncertainty analysis, we can conclude that both failure criteria exhibit remarkably similar behavior in the first quadrant, corresponding to the vessel's tension-tension stress state. Consequently, this similarity explains the identical composite layout obtained from the Tsai-Wu and Tsa-Hill criteria.

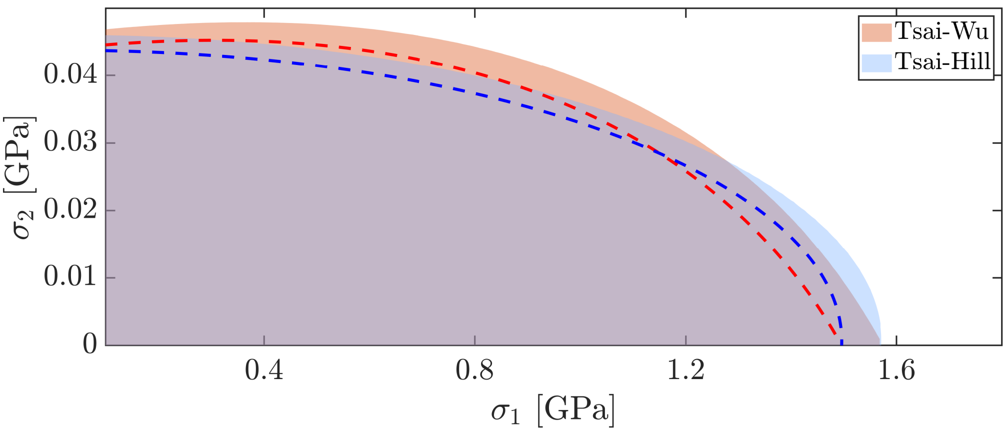

Figure 8. Envelope of failure criteria in the first quadrant for CFRP T300/976 material. The dashed lines are the mean envelopes for Tsai-Wu (red) and Tsai-Hill (blue) failure criteria.

Extending the analysis to a different composite material, T800-carbon fiber (Hwang *et al.*, 2003), it's possible to visualize a significant difference between both failure criteria, as shown in Fig. 9, which consequently leads to different layups for Tsai-Wu and Tsai-Hill. Therefore, we can conclude that selecting an appropriate failure criterion will also depend on the specific composite material being used. The choice of a failure criterion introduces several approximations, which may have varying levels of significance depending on the loading conditions under consideration. When represented graphically, different failure criteria result in different envelope shapes. The significance of these distinctions will vary based on their position within the design space, making them more or less noticeable.

4. SOBOL SENSITIVITY ANALYSIS

Sensitivity Analysis (SA) has become increasingly prominent in engineering in recent years, as it provides a valuable tool for understanding how input parameters affect the response of a given system. By systematically varying the input

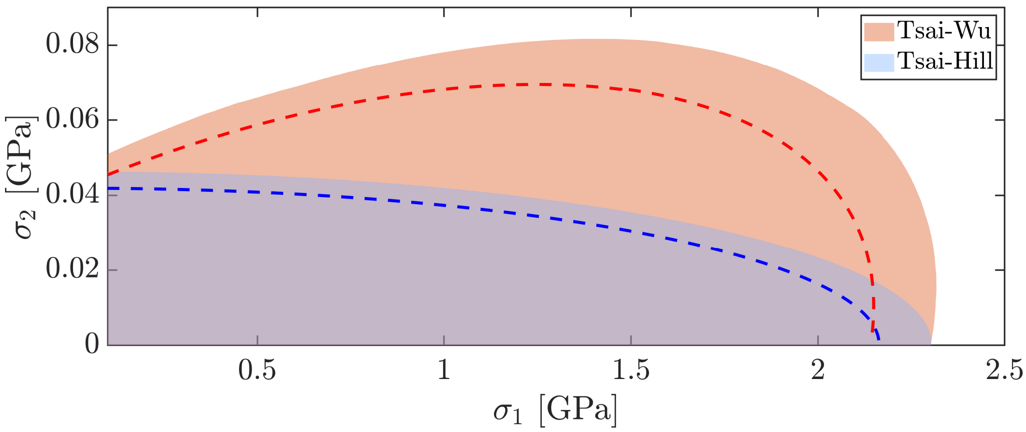

Figure 9. Envelope of failure criteria in the first quadrant for T800-carbon fiber material. The dashed lines are the mean envelopes for Tsai-Wu (red) and Tsai-Hill (blue) failure criteria.

parameters and analyzing their impact on the system output, SA enables us to identify and prioritize the most influential factors in the design process (Abbiati *et al.*, 2021).

There are two types into which SA techniques can be classified: local and global. The first analyzes how the variation in one specific parameter affects the system's output while the others remain unchanged. The global one thoroughly examines the impact of simultaneously changing the parameters across the multidimensional input domain (Cacuci, 2003).

In this work, we intend to analyze the contributions of each term (composite material properties) for the total value of the Safety Factor, more specifically for the Tsai-Wu criterion. The technique implemented for this purpose is called Sobol indices, a Global Sensitivity Analysis (GSA) widely used in many applications found in the literature.

Sobol (1993) was the first to introduce the Sobol indices, which are related to a variance decomposition technique that expresses the response output as the combination of each input variable's individual and collective contributions. An interesting output can be described in general terms as shown by the following equation (Tosin *et al.*, 2020):

$$\mathcal{Y} = \mathcal{M}(\mathbf{X}), \quad \mathbf{X} = [X_1, X_2, X_3, ...X_N] \tag{4}$$

where $\mathbf{X}$ is an input vector with N independent parameters that, from the mathematical operator $\mathcal{M}$, return the scalar output $\mathcal{Y}$, in this case, the value of the Safety Factor. From the Hoeffding-Sobol decomposition (Sudret, 2008), Eq. (4) can be rewritten in terms of sums of different dimensions:

$$\mathcal{Y} = \mathcal{M}_0 + \sum_{i=1}^{N} \mathcal{M}_i \left(X_i\right) + \sum_{i<j} \mathcal{M}_{ij} \left(X_i, X_j\right) + \ldots + \mathcal{M}_{1\ldots N} \left(X_1 \ldots X_N\right), \tag{5}$$

where $\mathcal{M}_0$ is a constant associated to the expected value of the output, and the terms $\mathcal{M}_i$ and $\mathcal{M}_{ij}$ are associated to the expected value conditioned on parameter i and ij. The Sobol indices take into account the contribution of the individual and combination of parameters to the total variance of the output. For example, the first-order indices measure the independent impact of each input variable on the overall variance:

$$S_i = \frac{Var\left[\mathcal{M}_i\left(X_i\right)\right]}{Var[\mathcal{M}(\mathbf{X})]}, \quad i = 1, 2, 3, ...N \tag{6}$$

Similarly, we have the second-order indices that consider the impact of interactions among X_i and X_j:

$$S_{ij} = \frac{Var\left[\mathcal{M}_{ij}\left(X_i, X_j\right)\right]}{Var[\mathcal{M}(\mathbf{X})]}, \quad 1 \leq i < j \leq N \tag{7}$$

By continuing this process, we can obtain Sobol's indices of all orders, resulting in a final index that denotes the combined contribution of interactions among all variables.

The toolbox developed by Marelli and Sudret (2014) was utilized to obtain Sobol indices. In addition to the Monte Carlo Method, Sobol indices were computed using Polynomial Chaos Expansion (PCE). PCE is a surrogate model representing an approximate model of the original system's stochastic model. As mentioned previously, despite the Monte Carlo Method's satisfactory performance in this study's simulations, computing the Sobol's indices with PCE significantly improved, reducing the simulation time from 40 seconds to just 2 seconds. Therefore, it provides a faster and more computationally efficient alternative for performing the same process (Tosin *et al.*, 2020). Figure 10 displays the first-order Sobol indices for evaluating the Safety Factor from the Tsai-Wu criterion.

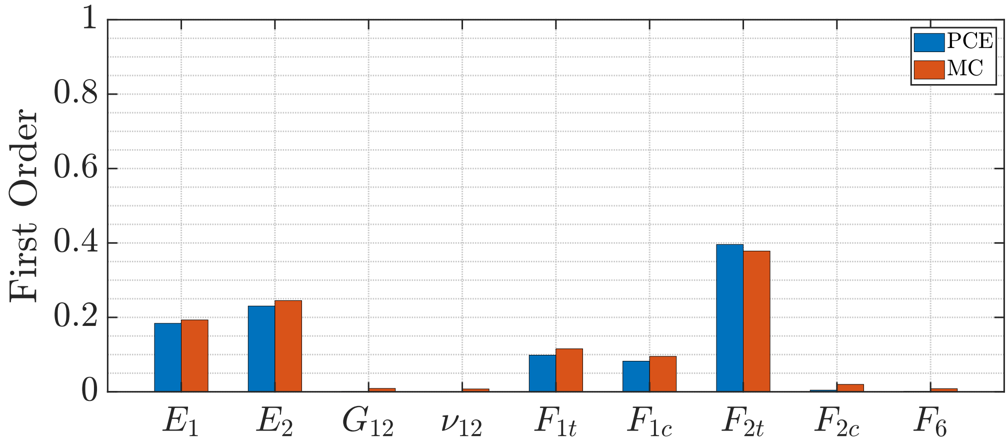

Figure 10. First order Sobol' indices.

The close agreement between the values obtained using the Monte Carlo method and PCE indicates the latter's accuracy. Based on its values, we can analyze that three parameters, E_1, E_2, and F_{2t}, dominate over others, meaning that these parameters significantly influence the safety factor. This highlights the importance of treating these parameters as stochastic variables, as minor changes in their values can significantly impact the output, in this case, the value of SF, an essential project parameter. Conversely, parameters with low sensitivity (such as G_{12} and ν_{12}) allow for knowledge of the exact value and its fine control to be negligible. The second-order Sobol's indices obtained were insignificant (less than 0.1%), and the same is expected for higher orders. This implies that the interaction effects among the input variables are insignificant compared to the individual effects.

5. FINAL REMARKS

Our study emphasizes the importance of considering uncertainties in failure analysis when designing composite structures. The comparison of the Tsai-Wu and Tsai-Hill failure criteria for a high-pressure vessel project highlights the necessity of incorporating uncertainty analysis to ensure the design's safety and reliability. Incorporating Monte Carlo and PCE propagation methods and Sobol Indices can help identify critical input material properties and their interactions and provide insight into the parameters that significantly impact the design.

Our research highlights the importance of incorporating uncertainty analysis and selecting appropriate failure criteria for designing safe and reliable composite structures. Further studies can explore the impact of additional factors, such as uncertainties typically associated with manufacturing (e.g., variation in thickness, winding angles, general dimensions, fiber volume fraction, etc.), as well as progressive damage analyses, to enhance the accuracy and reliability of the design process.

6. ACKNOWLEDGEMENTS

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