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Dynamical algebra of quasi-exactly-solvable potentials

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We show that by using second-order differential operators as a realization of the $so(2,1)$ Lie algebra, we can extend the class of quasi-exactly-solvable potentials with dynamical symmetries. As an example, we dynamically generate a potential of tenth power, which has been treated in the literature using other approaches, and discuss its relation with other potentials of lowest orders. The question of solvability is also studied.

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In the last decade, the problem of obtaining exact solutions for quantum anharmonic potentials has attracted increasing interest. As far as we know, Flessas [1] was the first person to obtain directly from the Schrödinger equation solutions for such potentials. Many other authors have been working with this problem [2–20], enlarging the class of quasi-exactly-solvable potentials.

This search has been growing in two different ways. The first one starts from the expression of the potential and uses some ansatz for the wave function to obtain a set of equations that leads to exact solutions, provided that certain relations between the parameters of the potential hold [1–6]. Independently, other authors, particularly the Soviet ones, have also been working on such types of problems, whose solutions are closely related to dynamical groups. An excellent review of this approach appeared in a recent work by Shiffman [7].

In the second approach, Shiffman [7] and Turbiner [8], in particular, searched for exact solutions by symmetry arguments. In fact, they showed that the Hamiltonian of these quasi-exactly-solvable potentials can be written in terms of symmetry generators, demonstrating the existence of a dynamical algebra in these potentials. An important characteristic of the works in this approach is that they use a constructive approach, in the sense that they start from the solution (the ansatz of the wave function) and obtain the quasi-exact potentials whose param-

eters are written in terms of the parameters of the wave function. This enables us to see directly the relations between both parameters. In fact, this constructive approach was also used in the recent paper of Taylor and Leach [10], in which they work with two-dimensional potentials, without mentioning any symmetry. More recently, these quasi-exactly-solvable potentials have also been related to supersymmetric quantum mechanics [11–14].

If we compare these two lines of study of this problem, we observe that with the first method one obtains few eigenstates for a given potential [1–4] as a consequence of a poor choice of the ansatz for the wave functions, and with the second method one obtains more eigenstates, but only for a limited number of anharmonic potentials. This can easily be seen by inspection of the Table 1, appearing in the work of Turbiner [8], where we note that there is no potential with a power higher than the sextic one. The fundamental problem of this second approach is that they work with the first-order generators:

$$J^+ = z^2 \frac{d}{dz} - 2jz, \quad J^0 = z \frac{d}{dz} - j, \quad J^- = \frac{d}{dz}, \quad (1)$$

which satisfy the $sl(2)$ algebra

$$[J^+, J^-] = -2J^0, \quad [J^0, J^\pm] = \pm J^\pm, \quad (2)$$

where $j(j+1)$ is the eigenvalue of the Casimir operator. The principal characteristic of the anharmonic potentials discussed by Turbiner is that their Hamiltonians are at most bilinear products of the generators J^i . This is also currently used in other algebraic methods [21]. This characteristic, together with the fact that all generators J^i have at least one first-order derivative term, led to a limitation on the highest power of the anharmonic potential that can be handled with this technique [8].

In this paper we show that by working with an algebra isomorphic to $sl(2)$, but using second-order generators and Hamiltonians written as multilinear combinations of these generators, the class of quasi-exact potentials that present a dynamical algebra is considerably broadened. These generators are [21–24]

$$T_1 = \alpha_2 x^{2-j} \left[\frac{d}{dx} \right]^2 + \alpha_1 x^{1-j} \left[\frac{d}{dx} \right] + \alpha_0 x^{-j}, \quad (3a)$$

$$T_2 = -\frac{i}{j} x \left[\frac{d}{dx} \right] - i\sigma, \quad (3b)$$

$$T_3 = \lambda x^j, \quad (3c)$$

which satisfy the $so(2,1)$ algebra

$$[T_1, T_2] = -iT_1, \quad [T_2, T_3] = -iT_3, \quad [T_1, T_3] = -iT_2, \quad (4)$$

when their parameters are related by

$$\sigma = \frac{1}{2j} \left[\frac{\alpha_1}{\alpha_2} + j - 1 \right], \quad \lambda = -(2\alpha_2 j^2)^{-1}, \quad (5)$$

and α_i and j can be adjusted to a desired form of the Hamiltonian (α_2 and $j \neq 0$). Note that the commutation relations (4) and (2) are equivalent because of the isomorphism between the $so(2,1)$ and $sl(2)$ algebras.

Let us now construct a Hamiltonian using a multilinear combination of the generators (3), which in general can be written as

$$H = \sum_{k=0}^K a_k (T_3)^k T_1 + \sum_{l=0}^L b_l (T_3)^l T_2 + \sum_{m=0}^M c_m (T_3)^m, \quad (6)$$

which involves arbitrary x powers and naturally generalizes the results of Shiffman [7] and Turbiner [8]. It is important to note that since the generator T_3 , Eq. (3c), is of zero derivative order, we can eliminate the restriction on the highest order of the polynomial potentials and so construct potentials with arbitrary powers.

Before writing the polynomial potentials with Hamiltonian (6), we remember that this method consists of using a constructive approach, where a “gauge transformation” is performed in the wave function [7]

$$\Psi(x) = \hat{\Psi}(x) e^{-a(x)}, \quad (7)$$

where $\hat{\Psi}(x)$ and $a(x)$ are polynomials in x . The “gauged wave function” $\hat{\Psi}(x)$ satisfies an eigenvalue equation

$$H_G \hat{\Psi}(x) = E \hat{\Psi}(x), \quad (8)$$

where the “gauged Hamiltonian” H_G can be written as

$$H_G = -\frac{1}{2} \frac{d^2}{dx^2} + A(x) \frac{d}{dx} + \Delta V(x), \quad (9)$$

where

$$\Delta V(x) = V(x) + \frac{1}{2} A'(x) - \frac{1}{2} A^2(x) \quad (10)$$

and $a(x) = \int^x A(x') dx'$.

As we want to construct arbitrary polynomial potentials using the Hamiltonian (6), we must identify it with the gauged Hamiltonian (9). It is also clear from Eqs. (8)–(10) that the ansatz (7) and especially the polynomial $a(x)$ determine the desired potential $V(x)$, so we must take care with the “gauged potential” $\Delta V(x)$. Once we choose $a(x)$ as a polynomial of maximum power n , then the highest power of $\Delta V(x)$ will be $2(n-1)$, as can be seen from (10). If this is not the case, one cannot factorize the wave function $\Psi(x)$ as an exponential times a pure polynomial. Note, however, that n is arbitrary, so one can construct potentials with arbitrary powers as stated in Eq. (6).

Let us now discuss a potential of tenth power, which has been recently treated in the literature using other approaches [15], as an example of the generalization given by the representation of second-order operators (3) and their multilinear combination (6). To do this we choose $j=2$ in the generators (3) and rewrite the Hamiltonian (6) as

$$H_G = T_1 + \left[b_0 + b_1 T_3 + b_2 (T_3)^2 \right] T_2 + c_0 + c_1 T_3 + c_2 (T_3)^2, \quad (11)$$

so that the potential $V(x)$ is given by

$$\begin{aligned} V(x) = & \frac{1}{2} \beta^2 x^{10} + \beta \gamma x^8 + \left(\frac{1}{2} \gamma^2 + \alpha \beta \right) x^6 \\ & + \left[\epsilon + \left(g - \frac{5}{2} \right) \beta + \alpha \gamma \right] x^4 \\ & + \left[\delta + \left(g + \frac{3}{2} \right) \gamma + \frac{1}{2} \alpha^2 \right] x^2 \\ & + \frac{1}{2} g(g+1) x^{-2} + \left(g - \frac{1}{2} \right) \alpha, \end{aligned} \quad (12)$$

when we identify $A(x)$ and $\Delta V(x)$, appearing in Eq. (10), as

$$A(x) = \beta x^5 + \gamma x^3 + \alpha x + g x^{-1} \quad (13a)$$

and

$$\Delta V(x) = \epsilon x^4 + \delta x^2. \quad (13b)$$

To establish the correspondence between the gauged Hamiltonian (11) and the potential (12) we can substitute Eqs. (13a) and (13b) in Eq. (9), implying that the coefficients are related by

$$\begin{aligned} b_0 &= 2i\alpha, \quad b_1 = 8i\gamma, \quad b_2 = 32i\beta, \quad c_0 = \alpha(2g-1)/2, \\ c_1 &= 4\delta - 2\gamma(1-2g), \quad c_2 = 16\epsilon - 8\beta(1-2g), \end{aligned}$$

since the coefficients appearing in the generators (3) are

$$\alpha_0 = 0, \quad \alpha_1 = g, \quad \alpha_2 = -\frac{1}{2}, \quad \sigma = (1-2g)/4, \quad \lambda = 4.$$

Apart from the intrinsic interest in solving the potential of tenth power, it will also permit us to illustrate this algebraic technique related to the generators T_i ($i=1,2,3$) of $so(2,1)$ Lie algebra in the context of quasi-exactly-solvable potentials and discuss possible relations between quasi-exactly and exactly solvable potentials. This will be done using the fact that the sextic potential

$$V(x) = \frac{1}{2}\gamma^2 x^6 + \alpha\gamma x^4 + [\delta + (g - \frac{3}{2})\gamma + \frac{1}{2}\alpha^2]x^2 + \frac{1}{2}g(g+1)x^{-2} + (g - \frac{1}{2})\alpha \quad (14)$$

is a particular case of potential (12) ($\beta=\epsilon=0$), as well as

$$a_{n+2} = \frac{(E - n\alpha)a_n - [(n-2)\gamma + \delta]a_{n-2} - [(n-4)\beta + \epsilon]a_{n-4}}{(n+2) \left[\frac{n+1}{2} - g \right]}, \quad (17)$$

where E is the energy eigenvalue of the gauged Hamiltonian which coincides with the eigenvalues of the canonical Hamiltonian. Furthermore, one can verify, from Eqs. (8) and (9), that odd wave functions exists only when $g=0$ and there is no restriction on g for even wave functions.

In order to ensure finiteness, and consequently normalizability of the wave function $\Psi(x)$, we must as usual break off the series, Eq. (16), at a certain $n=N$, so that we impose

$$a_{N+2} = a_{N+4} = a_{N+6} = 0. \quad (18)$$

Substituting (18) in (17), we find

$$(E - N\alpha)a_N - [(N-2)\gamma + \delta]a_{N-2} - [(N-4)\beta + \epsilon]a_{N-4} = 0, \quad (19a)$$

$$(N\gamma + \delta)a_N + [(N-2)\beta + \epsilon]a_{N-2} = 0, \quad (19b)$$

$$(N\beta + \epsilon)a_N = 0. \quad (19c)$$

These equations determine the spectrum of the problem and give relations between the energy E and the parameters $\alpha, \beta, \gamma, \delta, \epsilon$, and g , without any additional restrictions on a_n 's. So when we solve Eq. (19a) for the energy E , we get in general only one energy level for a given set of parameters fixed up by Eqs. (19b) and (19c). When we have vanishing parameters that imply vanishing terms in Eqs. (19a)–(19c), this will diminish the restrictions on the possible energy levels and the relations between the parameters, so that more energy eigenvalues may be permitted. This is what happens in the sextic potential (14), where $\beta=\epsilon=0$, so Eq. (19a) reduces to

the harmonic oscillator with a “centrifugal barrier”

$$V(x) = \frac{1}{2}\alpha^2 x^2 + \frac{1}{2}g(g+1)x^{-2} + (g - \frac{1}{2})\alpha, \quad (15)$$

when we take $\beta=\epsilon=\gamma=\delta=0$.

Assuming that the eigenfunction $\hat{\Psi}(x)$ of the gauged Hamiltonian, Eq. (9), to be of the form

$$\hat{\Psi}(x) = \sum_n a_n x^n, \quad (16)$$

substituting it in Eq. (8) and collecting the terms of the same power of x , we obtain the recurrence relation

$$(E - N\alpha)a_N + 2\gamma a_{N-2} = 0, \quad (20)$$

Eq. (19b) reduces to $N\gamma = -\delta$, which relates the potential parameters and the number of exact energy levels, and Eq. (19c) becomes an identity. The polynomial equation (20) for the energy E gives us $1+N/2$ levels when N is even (arbitrary g) and $(1+N)/2$ when N is odd ($g=0$). In the case of the harmonic oscillator plus the centrifugal barrier (15), where $\beta=\epsilon=\gamma=\delta=0$, Eqs. (19b) and (19c) are trivially satisfied and Eq. (19a) becomes

$$E = N\alpha, \quad (21)$$

which means that this solution is exact, since we have an arbitrary number of energy levels without any relation between the potential parameters and the number of exact levels.

At this point we can compare quasi-exactly and exactly solvable potentials in the light of the examples discussed above. We observe that the multilinear combination (6) of generators T_i leads to a potential with arbitrary powers, which has in general only one exact energy level, for a given set of parameters. This is the case of the tenth-power potential (12) with the gauged Hamiltonian (11). If we restrict the discussion to a bilinear combination of the generators, choosing $b_2=0$ in (11), this will generate the case of the sextic potential (14), which has various energy levels per potential. In the more restricted case of a linear combination of the generators, which can be obtained imposing $b_1=b_2=c_2=0$, we find the harmonic oscillator (15), which has an arbitrary number of exact levels without any restriction on the potential parameters. These facts seem to suggest that there is a relation between the number of exact levels for a given potential and the power of the gauged Hamiltonian in the generators.

As a final remark, we note that the technique presented here may also be applied to polynomial potentials with odd powers, choosing, for example, $j = 1$ in Eq. (3), and might be extended, using a point canonical transforma-

tion in the generators (3) [24], to treat nonpolynomial potentials involving, for example circular or hyperbolic functions [8] too. These aspects are presently under investigation and will be reported in due course.

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