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General method for reducing the two-body Dirac equation

A. P. Galeão and P. Leal Ferreira

Instituto de Física Teórica da Universidade Estadual Paulista, Rua Pamplona, 145-01405 São Paulo, SP-Brasil

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A semirelativistic two-body Dirac equation with an enlarged set of phenomenological potentials, including Breit-type terms, is investigated for the general case of unequal masses. Solutions corresponding to definite total angular momentum and parity are shown to fall into two classes, each one being obtained by solving a system of four coupled first-order radial differential equations. The reduction of each of these systems to a pair of coupled Schrödinger-type equations is also discussed.

I. INTRODUCTION

It appears natural to start this introduction with the Bethe-Salpeter equation,¹ which may be derived in a relativistically covariant approach based on quantum field theory. However, this equation is not always used in applications due to its extreme complexity. Furthermore, the Bethe-Salpeter equation is not free of inherent difficulties. This is the case with respect to the existence of quantum numbers associated to excitations of the relative time, which are not born out by experiments. However, it should be said that this difficulty has been circumvented in more recent approaches based on Dirac's relativistic constraint mechanics and supersymmetry.²

In an alternative approach, one can write an equal-time two-body equation, which is a natural extension of the one-body Dirac equation and, while not fully covariant, exhibits several desirable features such as invariance under spatial rotations and reflections, a correct nonrelativistic limit and reduction to the one-body equation when one of the masses goes to infinity. Such an equation was first set up by Breit in 1929 for the two electrons in orthohelium.³ Other applications, involving more general interactions, were made some time afterward to the deuteron⁴ and to a model for pions regarded as bound states of a nucleon and an antinucleon.^{5,6} More recently, this equation has been applied both to positronium⁷⁻⁹ and to mesons regarded as quark-antiquark systems.¹⁰⁻¹³ The phenomenological character of the two-body Dirac equation should be emphasized, especially for heuristic applications beyond the purely electrodynamic ones (helium-like atoms and positronium), namely in the nuclear and mesonic cases referred to above.

In the present work, we present an explicit derivation of all classes of solutions to the two-body Dirac equation, with an enlarged set of interactions and corresponding to a total angular momentum J and definite parity of the system, applicable to particle-particle and particle-antiparticle cases. By an enlarged set of interactions we mean that, besides the five Lorentz tensorial interactions originally introduced in Refs. 4-6, we have included a

Coulomb-type interaction,⁹ a Breit interaction, and two additional terms, which are generalized scalar terms considered recently by Childers.¹¹ This set of interactions allows the necessary flexibility to attack the different types of problems mentioned above.

This paper is organized as follows. Section II is devoted to our Breit-type equation in the unequal masses case. It contains a general interaction consisting of nine phenomenological terms, as described above. The original 16-component spinor $\Psi(\mathbf{r})$ is more conveniently transformed into a new spinor $\Phi(\mathbf{r})$, whose properties are discussed in detail. This new spinor, written as a 4×4 matrix, is decomposed in terms of the 16 Dirac matrices [see Eq. (2.9)], and the corresponding components are shown to satisfy the set of equations, Eqs. (2.10a)-(2.10h), which were first obtained by Moseley and Rosen⁶ for the more restrictive case of ordinary tensor interactions.

Section III deals with the separation of radial and angular variables. It is shown how two independent sets of coupled first-order radial equations are obtained [see Eqs. (3.4a)-(3.4d) and (3.5a)-(3.5d)]. In Sec. IV, we perform the reduction of the first-order systems obtained in the previous section into sets of two coupled second-order equations, which, in turn, can be put in Schrödinger form by eliminating the first-order terms on their left-hand sides.

Section V deals with the classification of the solutions obtained. For clarity, the $J \neq 0$ and $J = 0$ cases are discussed separately, as well as the case of equal masses. Finally, Sec. VI contains a summary of our work and our main conclusions.

II. THE TWO-BODY DIRAC EQUATION

The two-body Dirac equation for two spin- $\frac{1}{2}$ particles of masses m_1 and m_2 has the following form in natural units ($\hbar = c = 1$):

$$\{-i\alpha_1 \cdot \nabla_1 + m_1 \beta_1 - i\alpha_2 \cdot \nabla_2 + m_2 \beta_2 + H_{12}\} \Psi = E \Psi, \quad (2.1)$$

where α_1, β_1 and α_2, β_2 are the Dirac matrices for each of the two particles, H_{12} represents their interaction, E is their total energy, and the wave function Ψ is a 16-component spinor. For the Dirac matrices, we are following the conventions of Ref. 14. In the center-of-mass frame, Eq. (2.1) takes the simpler form

$$\{-i(\alpha_1 - \alpha_2) \cdot \nabla + \mu(\beta_1 - \beta_2) + m(\beta_1 + \beta_2) + H_{12}\} \Psi = E\Psi, \quad (2.2)$$

where

$$\mu = \frac{1}{2}(m_1 - m_2), \quad (2.3)$$

$$m = \frac{1}{2}(m_1 + m_2), \quad (2.4)$$

and the derivatives are with respect to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. We shall consider an interaction of the general form

$$H_{12} = - \sum_{n=1}^9 \Omega_n(r) \omega_n, \quad (2.5)$$

where $\Omega_n(r)$ are arbitrary shape functions and ω_n are the following spin-dependent operators:

$$\omega_1 = \beta_1 \beta_2 \quad (\text{scalar}), \quad (2.6a)$$

$$\omega_2 = \frac{1}{2}(1 - \alpha_1 \cdot \alpha_2) \quad (\text{vector}), \quad (2.6b)$$

$$\omega_3 = \frac{1}{2}\beta_1 \beta_2 (\boldsymbol{\Sigma}_1 \cdot \boldsymbol{\Sigma}_2 + \alpha_1 \cdot \alpha_2) \quad (\text{tensor}), \quad (2.6c)$$

$$\omega_4 = \frac{1}{2}(\boldsymbol{\Sigma}_1 \cdot \boldsymbol{\Sigma}_2 - \Gamma_1 \Gamma_2) \quad (\text{pseudovector}), \quad (2.6d)$$

$$\omega_5 = \beta_1 \beta_2 \Gamma_1 \Gamma_2 \quad (\text{pseudoscalar}), \quad (2.6e)$$

$$\omega_6 = 1 \quad (\text{Coulomb type}), \quad (2.6f)$$

$$\omega_7 = \frac{(\alpha_1 \cdot \mathbf{r})(\alpha_2 \cdot \mathbf{r})}{r^2} \quad (\text{Breit}), \quad (2.6g)$$

$$\omega_8 = \beta_1 \beta_2 \frac{(\alpha_1 \cdot \mathbf{r})(\alpha_2 \cdot \mathbf{r})}{r^2} \quad (\text{Childers}), \quad (2.6h)$$

$$\omega_9 = \beta_1 \beta_2 \alpha_1 \cdot \alpha_2 \quad (\text{Childers}), \quad (2.6i)$$

with $\boldsymbol{\Sigma} = (1/2i)\boldsymbol{\alpha} \times \boldsymbol{\alpha}$ and $\boldsymbol{\Gamma} = (1/6i)\boldsymbol{\alpha} \times \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$. Among those, only the first five operators, given in Eqs. (2.6a)–(2.6e), are invariant under Lorentz transformations. However, even if one restricted H_{12} to such terms, Eq. (2.1) would still not be strictly covariant, unless the interaction were of the contact type, a case that, however, is unable to sustain bound states of finite energy.^{4–6} One is, therefore, forced to adopt a phenomenological point of view and consider interactions of less restrictive forms

like Eq. (2.5), taken to hold only in the center-of-mass frame. For the sake of generality, we have included here the four extra terms with $n=6$ to 9, corresponding, respectively, to an instantaneous Coulomb-type interaction,⁹ $V(r)$, to part of Breit's retardation interaction,³ viz.,

$$-\frac{1}{2}V(r) \left[\alpha_1 \cdot \alpha_2 + \frac{(\alpha_1 \cdot \mathbf{r})(\alpha_2 \cdot \mathbf{r})}{r^2} \right],$$

and to the two terms of a generalization thereof for scalar potentials proposed by Childers.¹¹ Such retardation terms have usually been ignored in applications or, at most, treated as first-order perturbations. We present here a way to treat them, *a priori*, on the same footing as the remaining terms in the interaction.

To solve Eq. (2.2), we start by writing the 16-component spinor $\Psi(\mathbf{r})$ in matrix form as follows:

$$\Psi(\mathbf{r}) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{bmatrix}, \quad (2.7)$$

with the first index referring to particle 1 and the second to particle 2. In this way, α_1, β_1 can be simply replaced by the standard Dirac matrices α, β , while α_2, β_2 must be replaced by the corresponding transposed matrices, and made to act on the right. Furthermore, it is convenient to change to the new spinor $\Phi(\mathbf{r})$, related to the original one by

$$\Psi(\mathbf{r}) = S_2 \Phi(\mathbf{r}), \quad (2.8)$$

where $S_2 = \frac{1}{2}(C\Gamma\beta)_2$ and $C_2 \equiv -i(\beta\alpha_y)_2$ is the charge conjugation operator for particle 2. [The reason for the factor $\frac{1}{2}$ is that the components to be introduced in Eq. (2.9) agree exactly with the ones defined in Ref. 6, Eq. (4)].

Introducing Eq. (2.8) into Eq. (2.2), with $\Phi(\mathbf{r})$ written in the form

$$\Phi(\mathbf{r}) = I(\mathbf{r}) + \mathbf{A}(\mathbf{r}) \cdot \beta \boldsymbol{\alpha} - A_0(\mathbf{r})\beta + \mathbf{G}(\mathbf{r}) \cdot \boldsymbol{\Sigma} + \mathbf{F}(\mathbf{r}) \cdot \boldsymbol{\alpha} + J(\mathbf{r})\Gamma + \mathbf{U}(\mathbf{r}) \cdot \beta \boldsymbol{\alpha} \Gamma - U_0(\mathbf{r})\beta \Gamma, \quad (2.9)$$

one gets the following set of equations:

$$[E + V_1(r)]I + 2i\nabla \cdot \mathbf{F} + 2mA_0 = 0, \quad (2.10a)$$

$$\left[E + V_2(r) + V_9(r) \left(\frac{\mathbf{r}}{r} \times \right)^2 \right] \mathbf{A} + 2\nabla \times \mathbf{U} - 2\mu \mathbf{F} = 0, \quad (2.10b)$$

$$[E + V_3(r)]A_0 + 2mI = 0, \quad (2.10c)$$

$$\left[E + V_4(r) - V_{10}(r) \left(\frac{\mathbf{r}}{r} \times \right)^2 \right] \mathbf{G} + 2i \nabla J - 2m \mathbf{U} = 0, \quad (2.10d)$$

$$\left[E + V_5(r) - V_9(r) \left(\frac{\mathbf{r}}{r} \times \right)^2 \right] \mathbf{F} + 2i \nabla I - 2\mu \mathbf{A} = 0, \quad (2.10e)$$

$$[E + V_6(r)] J + 2i \nabla \cdot \mathbf{G} + 2\mu U_0 = 0, \quad (2.10f)$$

$$\left[E + V_7(r) + V_{10}(r) \left(\frac{\mathbf{r}}{r} \times \right)^2 \right] \mathbf{U} + 2 \nabla \times \mathbf{A} - 2m \mathbf{G} = 0, \quad (2.10g)$$

$$[E + V_8(r)] U_0 + 2\mu J = 0, \quad (2.10h)$$

first obtained by Moseley and Rosen⁶ for a more restricted class of interactions. In Eqs. (2.10), we have made use of the compact notation

$$\left(\frac{\mathbf{r}}{r} \times \right)^2 \mathbf{V} \equiv \frac{\mathbf{r}}{r} \times \left(\frac{\mathbf{r}}{r} \times \mathbf{V} \right), \quad (2.11)$$

valid for any three-vector $\mathbf{V}$, and introduced the following potentials:

$$V_1 = \Omega_1 + 2\Omega_2 - 3\Omega_3 - 2\Omega_4 + \Omega_5 + \Omega_6 - \Omega_7 - \Omega_8 - 3\Omega_9, \quad (2.12a)$$

$$V_2 = -\Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 - \Omega_8 + \Omega_9, \quad (2.12b)$$

$$V_3 = \Omega_1 - \Omega_2 - \Omega_4 - \Omega_5 + \Omega_6 + \Omega_7 + \Omega_8 + 3\Omega_9, \quad (2.12c)$$

$$V_4 = \Omega_1 + \Omega_3 + \Omega_5 + \Omega_6 - \Omega_7 - \Omega_8 + \Omega_9, \quad (2.12d)$$

$$V_5 = -\Omega_1 - \Omega_3 - \Omega_5 + \Omega_6 - \Omega_7 + \Omega_8 - \Omega_9, \quad (2.12e)$$

$$V_6 = -\Omega_1 + 2\Omega_2 + 3\Omega_3 - 2\Omega_4 - \Omega_5 + \Omega_6 - \Omega_7 + \Omega_8 + 3\Omega_9, \quad (2.12f)$$

$$V_7 = \Omega_1 + \Omega_2 + \Omega_4 - \Omega_5 + \Omega_6 + \Omega_7 + \Omega_8 - \Omega_9, \quad (2.12g)$$

$$V_8 = -\Omega_1 - \Omega_2 - \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 - \Omega_8 - 3\Omega_9, \quad (2.12h)$$

$$V_9 = 2\Omega_7 - 2\Omega_8, \quad (2.12i)$$

$$V_{10} = 2\Omega_7 + 2\Omega_8. \quad (2.12j)$$

III. RADIAL EQUATIONS

To solve Eqs. (2.10), we notice that Eqs. (2.10c) and (2.10h) can be used to eliminate $I(\mathbf{r})$ and $J(\mathbf{r})$ by writing

$$I(\mathbf{r}) = -(1/2m) [E + V_3(r)] A_0(\mathbf{r}), \quad (3.1a)$$

$$J(\mathbf{r}) = -(1/2\mu) [E + V_8(r)] U_0(\mathbf{r}). \quad (3.1b)$$

Furthermore, Eq. (2.2) is rotationally invariant, which allows us to look for solutions with well-defined total angular momentum J . We, therefore, write for the remaining components

$$A_0(\mathbf{r}) = a_J(r) Y_{JM}(\theta, \varphi), \quad (3.2a)$$

$$U_0(\mathbf{r}) = u_J(r) Y_{JM}(\theta, \varphi), \quad (3.2b)$$

and, for the three-vectors $\mathbf{V} \equiv \mathbf{A}, \mathbf{F}, \mathbf{G}$, and $\mathbf{U}$,

$$\mathbf{V}(\mathbf{r}) \equiv [v_J^{(1)}(r) \mathbf{L} + v_J^{(2)}(r) \mathbf{r} \nabla + v_J^{(3)}(r) \times (\mathbf{r}/r)] Y_{JM}(\theta, \varphi), \quad (3.2c)$$

with $v \equiv a, f, g$, and u , respectively, and $\mathbf{L} = -i\mathbf{r} \times \nabla$. Introducing the above expressions into Eqs. (2.10), making use of linear independence, and introducing the auxiliary functions

$$w_1(r) = ir^2 f_J^{(3)}(r), \quad (3.3a)$$

$$w_2(r) = \frac{E + V_3(r)}{2m} a_J(r), \quad (3.3b)$$

$$w_3(r) = ru_J^{(1)}(r), \quad (3.3c)$$

$$w_4(r) = ira_J^{(2)}(r), \quad (3.3d)$$

$$w_5(r) = r^2 g_J^{(3)}(r), \quad (3.3e)$$

$$w_6(r) = i \frac{E + V_8(r)}{2\mu} u_J(r), \quad (3.3f)$$

$$w_7(r) = ira_J^{(1)}(r), \quad (3.3g)$$

$$w_8(r) = ru_J^{(2)}(r), \quad (3.3h)$$

we get two independent sets of four coupled first-order differential equations each, namely

$$\frac{dw_1}{dr} = C_{12}w_2 + C_{14}w_4, \quad (3.4a)$$

$$\frac{dw_2}{dr} = C_{21}w_1 + C_{23}w_3, \quad (3.4b)$$

$$\frac{dw_3}{dr} = C_{32}w_2 + C_{34}w_4 \quad (J \neq 0), \quad (3.4c)$$

$$\frac{dw_4}{dr} = C_{41}w_1 + C_{43}w_3 \quad (J \neq 0), \quad (3.4d)$$

and

$$\frac{dw_5}{dr} = C_{56}w_6 + C_{58}w_8, \quad (3.5a)$$

$$\frac{dw_6}{dr} = C_{65}w_5 + C_{67}w_7, \quad (3.5b)$$

$$\frac{dw_7}{dr} = C_{76}w_6 + C_{78}w_8 \quad (J \neq 0), \quad (3.5c)$$

$$\frac{dw_8}{dr} = C_{85}w_5 + C_{87}w_7 \quad (J \neq 0), \quad (3.5d)$$

whose coefficients have the following explicit expressions:

$$C_{12} = -\frac{2J(J+1)}{E+V_5+V_9} + \frac{[(E+V_1)(E+V_3)-(2m)^2]r^2}{2(E+V_3)}, \quad (3.6a)$$

$$C_{14} = \frac{2\mu J(J+1)}{E+V_5+V_9}, \quad (3.6b)$$

$$C_{21} = -\frac{(E+V_2)(E+V_5)-(2\mu)^2}{2(E+V_2)r^2}, \quad (3.6c)$$

$$C_{23} = \frac{2\mu J(J+1)}{(E+V_2)r^2}, \quad (3.6d)$$

$$C_{32} = \frac{2\mu}{E+V_5+V_9}, \quad (3.6e)$$

$$C_{34} = \frac{(E+V_2-V_9)(E+V_5+V_9)-(2\mu)^2}{2(E+V_5+V_9)}, \quad (3.6f)$$

$$C_{41} = \frac{2\mu}{(E+V_2)r^2}, \quad (3.6g)$$

$$C_{43} = \frac{2J(J+1)}{(E+V_2)r^2} - \frac{(E+V_4+V_{10})(E+V_7-V_{10})-(2m)^2}{2(E+V_4+V_{10})}, \quad (3.6h)$$

$$C_{56} = \frac{2J(J+1)}{E+V_4+V_{10}} - \frac{[(E+V_6)(E+V_8)-(2\mu)^2]r^2}{2(E+V_8)}, \quad (3.6i)$$

$$C_{58} = \frac{2mJ(J+1)}{E+V_4+V_{10}}, \quad (3.6j)$$

$$C_{65} = \frac{(E+V_4)(E+V_7)-(2m)^2}{2(E+V_7)r^2}, \quad (3.6k)$$

$$C_{67} = \frac{2mJ(J+1)}{(E+V_7)r^2}, \quad (3.6l)$$

$$C_{76} = \frac{2m}{E+V_4+V_{10}}, \quad (3.6m)$$

$$C_{78} = -\frac{(E+V_4+V_{10})(E+V_7-V_{10})-(2m)^2}{2(E+V_4+V_{10})}, \quad (3.6n)$$

$$C_{85} = \frac{2m}{(E+V_7)r^2}, \quad (3.6o)$$

$$C_{87} = -\frac{2J(J+1)}{(E+V_7)r^2} + \frac{(E+V_2-V_9)(E+V_5+V_9)-(2\mu)^2}{2(E+V_5+V_9)}. \quad (3.6p)$$

The remaining equations give purely algebraic relations between the radial components of $\Phi(\mathbf{r})$ and can be re-written in terms of the auxiliary radial functions as follows:

$$a_J^{(3)} = -\frac{2i}{(E+V_2)r^2} [J(J+1)w_3 + \mu w_1], \quad (3.7a)$$

$$u_J^{(3)} = \frac{2}{(E+V_7)r^2} [-J(J+1)w_7 + mw_5], \quad (3.7b)$$

and, for $J \neq 0$,

$$g_J^{(1)} = \frac{2m}{(E+V_4+V_{10})r} w_3, \quad (3.7c)$$

$$g_J^{(2)} = \frac{2}{(E+V_4+V_{10})r} (w_6 + mw_8), \quad (3.7d)$$

$$f_J^{(1)} = -\frac{2i\mu}{(E+V_5+V_9)r} w_7, \quad (3.7e)$$

$$f_j^{(2)} = -\frac{2i}{(E + V_5 + V_9)r}(-w_2 + \mu w_4). \quad (3.7f)$$

We have thus succeeded in reducing the problem of solving the two-body Dirac equation (2.2) to that of solving a quadruple of first-order radial differential equations, namely Eqs. (3.4) or Eqs. (3.5), depending on the type of eigenfunction one is looking for, as discussed in Sec. V.

IV. SCHRÖDINGER-TYPE EQUATIONS

A. Case $J \neq 0$

To continue, it is best to deal with the states of zero angular momentum separately. We first take the case with $J \neq 0$. It is, then, possible to reduce Eqs. (3.4) to two coupled second-order equations in any two of the four functions, the best choice being w_2 and w_3 due to the structure of the coefficients. We get

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{C_{21}} \frac{dw_2}{dr} \right) + \left(\frac{C_{14}C_{32}}{C_{34}} - C_{12} \right) w_2 \\ = \left(\frac{C_{23}}{C_{21}} + \frac{C_{14}}{C_{34}} \right) \frac{dw_3}{dr} + \left(\frac{C_{23}}{C_{21}} \right)' w_3, \end{aligned} \quad (4.1a)$$

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{C_{34}} \frac{dw_3}{dr} \right) + \left(\frac{C_{23}C_{41}}{C_{21}} - C_{43} \right) w_3 \\ = \left(\frac{C_{41}}{C_{21}} + \frac{C_{32}}{C_{34}} \right) \frac{dw_2}{dr} + \left(\frac{C_{32}}{C_{34}} \right)' w_2, \end{aligned} \quad (4.1b)$$

where the prime, here and in what follows, indicates differentiation with respect to r . Similarly, we can replace Eqs. (3.5) by

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{C_{65}} \frac{dw_6}{dr} \right) + \left(\frac{C_{58}C_{76}}{C_{78}} - C_{56} \right) w_6 \\ = \left(\frac{C_{67}}{C_{65}} + \frac{C_{58}}{C_{78}} \right) \frac{dw_7}{dr} + \left(\frac{C_{67}}{C_{65}} \right)' w_7, \end{aligned} \quad (4.2a)$$

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{C_{78}} \frac{dw_7}{dr} \right) + \left(\frac{C_{67}C_{85}}{C_{65}} - C_{87} \right) w_7 \\ = \left(\frac{C_{85}}{C_{65}} + \frac{C_{76}}{C_{78}} \right) \frac{dw_6}{dr} + \left(\frac{C_{76}}{C_{78}} \right)' w_6. \end{aligned} \quad (4.2b)$$

The above equations can be put in Schrödinger form by eliminating the first-order terms on their left-hand sides. To achieve this for Eqs. (4.1), we change to the new functions

$$W_2 = \frac{1}{\sqrt{-C_{21}}} w_2, \quad (4.3a)$$

$$W_3 = \frac{1}{\sqrt{C_{34}}} w_3, \quad (4.3b)$$

getting

$$\begin{aligned} \frac{d^2}{dr^2} W_2 + \left[C_{21} \left(\frac{C_{14}C_{32}}{C_{34}} - C_{12} \right) + \frac{1}{2} \left(\frac{C_{21}''}{C_{21}} \right) - \frac{3}{4} \left(\frac{C_{21}'}{C_{21}} \right)^2 \right] W_2 \\ = -\sqrt{-C_{21}C_{34}} \left[\left(\frac{C_{23}}{C_{21}} + \frac{C_{14}}{C_{34}} \right) \frac{dW_3}{dr} + \left[\frac{1}{2} \left(\frac{C_{23}}{C_{21}} + \frac{C_{14}}{C_{34}} \right) \right. \right. \\ \left. \left. \times \left(\frac{C_{34}'}{C_{34}} \right) + \left(\frac{C_{23}}{C_{21}} \right)' \right] W_3 \right], \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \frac{d^2}{dr^2} W_3 + \left[C_{34} \left(\frac{C_{23}C_{41}}{C_{21}} - C_{43} \right) + \frac{1}{2} \left(\frac{C_{34}''}{C_{34}} \right) - \frac{3}{4} \left(\frac{C_{34}'}{C_{34}} \right)^2 \right] W_3 \\ = \sqrt{-C_{21}C_{34}} \left[\left(\frac{C_{41}}{C_{21}} + \frac{C_{32}}{C_{34}} \right) \frac{dW_2}{dr} + \left[\frac{1}{2} \left(\frac{C_{41}}{C_{21}} + \frac{C_{32}}{C_{34}} \right) \left(\frac{C_{21}'}{C_{21}} \right) \right. \right. \\ \left. \left. + \left(\frac{C_{32}}{C_{34}} \right)' \right] W_2 \right]. \end{aligned} \quad (4.4b)$$

For Eqs. (4.2), the appropriate transformation is

$$W_6 = \frac{1}{\sqrt{C_{65}}} w_6, \quad (4.5a)$$

$$W_7 = \frac{1}{\sqrt{-C_{78}}} w_7, \quad (4.5b)$$

leading to

$$\begin{aligned} \frac{d^2}{dr^2} W_6 + \left[C_{65} \left(\frac{C_{58}C_{76}}{C_{78}} - C_{56} \right) + \frac{1}{2} \left(\frac{C_{65}''}{C_{65}} \right) - \frac{3}{4} \left(\frac{C_{65}'}{C_{65}} \right)^2 \right] W_6 \\ = \sqrt{-C_{65}C_{78}} \left[\left(\frac{C_{67}}{C_{65}} + \frac{C_{58}}{C_{78}} \right) \frac{dW_7}{dr} + \left[\frac{1}{2} \left(\frac{C_{67}}{C_{65}} + \frac{C_{58}}{C_{78}} \right) \left(\frac{C_{78}'}{C_{78}} \right) \right. \right. \\ \left. \left. + \left(\frac{C_{67}}{C_{65}} \right)' \right] W_7 \right], \end{aligned} \quad (4.6a)$$

$$\begin{aligned} \frac{d^2}{dr^2} W_7 + \left[C_{78} \left(\frac{C_{67}C_{85}}{C_{65}} - C_{87} \right) + \frac{1}{2} \left(\frac{C_{78}''}{C_{78}} \right) - \frac{3}{4} \left(\frac{C_{78}'}{C_{78}} \right)^2 \right] W_7 \\ = -\sqrt{-C_{65}C_{78}} \left[\left(\frac{C_{85}}{C_{65}} + \frac{C_{76}}{C_{78}} \right) \frac{dW_6}{dr} + \left[\frac{1}{2} \left(\frac{C_{85}}{C_{65}} + \frac{C_{76}}{C_{78}} \right) \right. \right. \\ \left. \left. \times \left(\frac{C_{78}'}{C_{78}} \right) + \left(\frac{C_{76}}{C_{78}} \right)' \right] W_6 \right]. \end{aligned} \quad (4.6b)$$

For the most general case we are considering in this paper, the coefficients in Eqs. (4.4) and (4.6) are rather

complicated expressions that can be obtained from Eq. (3.6). However, if, following the usual practice, one does not include in the interaction the Breit and Childers–Breit terms, i.e., those with $n=7$ and 8 in Eq. (2.5), then, as is clear from Eqs. (2.12i) and (2.12j)

$$V_9 = V_{10} = 0, \quad (4.7)$$

resulting in considerable simplification of the above equations, the most important being the disappearance of the derivative couplings, by which we mean that the coefficients of the first-order terms on their right-hand sides vanish.

B. Case $J=0$

In this case a great simplification takes place since Eqs. (3.4) and (3.5) become, respectively,

$$\frac{dw_1}{dr} = C_{12}w_2, \quad (4.8a)$$

$$\frac{dw_2}{dr} = C_{21}w_1, \quad (4.8b)$$

and

$$\frac{dw_5}{dr} = C_{56}w_6, \quad (4.9a)$$

$$\frac{dw_6}{dr} = C_{65}w_5. \quad (4.9b)$$

Each of these systems of coupled equations can be reduced to a single second-order equation as follows:

$$\frac{d}{dr} \left(\frac{1}{C_{21}} \frac{dw_2}{dr} \right) - C_{12}w_2 = 0, \quad (4.10)$$

$$\frac{d}{dr} \left(\frac{1}{C_{65}} \frac{dw_6}{dr} \right) - C_{56}w_6 = 0. \quad (4.11)$$

Those can, in turn, be put in Schrödinger form by the same transformations given in Eqs. (4.3a) and (4.5a). One gets

$$\frac{d^2 W_2}{dr^2} + \left\{ \frac{1}{2} \left(\frac{C_{21}''}{C_{21}} \right) - \frac{3}{4} \left(\frac{C_{21}'}{C_{21}} \right)^2 - C_{12}C_{21} \right\} W_2 = 0, \quad (4.12)$$

$$\frac{d^2 W_6}{dr^2} + \left\{ \frac{1}{2} \left(\frac{C_{65}''}{C_{65}} \right) - \frac{3}{4} \left(\frac{C_{65}'}{C_{65}} \right)^2 - C_{56}C_{65} \right\} W_6 = 0. \quad (4.13)$$

V. CLASSIFICATION OF SOLUTIONS

A. Case $J \neq 0$

Since we have obtained two independent sets of radial equations, namely, Eqs. (3.4) and (3.5), it is natural to classify the solutions of the two-body Dirac equation into those of class I, with $w_5 = w_6 = w_7 = w_8 = 0$, and those of class II, with $w_1 = w_2 = w_3 = w_4 = 0$. Making use of Eqs. (3.3) and (3.7) we can write, for the corresponding transformed spinors, as given by Eq. (2.9), the following expressions:

$$\begin{aligned} \Phi_{JM}^{(I)}(\mathbf{r}) = & \left[-w_2 + \beta\alpha \cdot \left[-\frac{i}{r} w_4 r \nabla - \frac{2i}{(E+V_2)r^2} (J(J+1)w_3 + \mu w_1) \frac{\mathbf{r}}{r} \right] - \beta \left[\frac{2m}{(E+V_3)} w_2 \right] + \Sigma \cdot \left[\frac{2m}{(E+V_4+V_{10})r} w_3 \mathbf{L} \right] \right. \\ & \left. + \alpha \cdot \left[-\frac{2i}{(E+V_5+V_9)r} (-w_2 + \mu w_4) r \nabla - \frac{i}{r^2} w_1 \frac{\mathbf{r}}{r} \right] + \beta\alpha \Gamma \cdot \left[\frac{1}{r} w_3 \mathbf{L} \right] \right] Y_{JM}(\theta, \varphi) \end{aligned} \quad (5.1)$$

and

$$\begin{aligned} \Phi_{JM}^{(II)}(\mathbf{r}) = & \left[\beta\alpha \cdot \left[-\frac{i}{r} w_7 \mathbf{L} \right] + \Sigma \cdot \left[\frac{2}{(E+V_4+V_{10})r} (w_6 + m w_8) r \nabla + \frac{1}{r^2} w_5 \frac{\mathbf{r}}{r} \right] + \alpha \cdot \left[-\frac{2i\mu}{(E+V_5+V_9)r} w_7 \mathbf{L} \right] + \Gamma [i w_6] \right. \\ & \left. + \beta\alpha \Gamma \cdot \left[\frac{1}{r} w_8 r \nabla + \frac{2}{(E+V_7)r^2} (-J(J+1)w_7 + m w_5) \frac{\mathbf{r}}{r} \right] + \beta \Gamma \cdot \left[\frac{2i\mu}{E+V_8} w_6 \right] \right] Y_{JM}(\theta, \varphi). \end{aligned} \quad (5.2)$$

The parity operator can be written as $\Pi = \eta_1 \eta_2 \beta_1 \beta_2 \Pi_r$, where η_1 and η_2 stand for the intrinsic parities of particles 1 and 2 and Π_r changes $\mathbf{r}$ into $-\mathbf{r}$. We get, then, from Eq. (2.8),

$$\Pi \Psi(\mathbf{r}) = \Pi S_2 \Phi(\mathbf{r}) = \eta_1 \eta_2 \beta \Phi(-\mathbf{r}) \beta S^T. \quad (5.3)$$

From this and Eqs. (5.1) and (5.2) one can easily see that the states of class I have parity $\eta_1 \eta_2 (-)^J$ and those of class II have parity $\eta_1 \eta_2 (-)^{J+1}$.

B. Case $J=0$

In this case, many of the components in Eqs. (5.1) and (5.2) are identically zero, and one is left with

$$\begin{aligned} \Phi_{J=0}^{(I)}(\mathbf{r}) = & \left[-w_2 + \beta \alpha \cdot \left[-\frac{2i\mu}{(E+V_2)r^2} w_1 \frac{\mathbf{r}}{r} \right] \right. \\ & - \beta \left[\frac{2m}{(E+V_3)} w_2 \right] \\ & \left. + \alpha \cdot \left[-\frac{i}{r^2} w_1 \frac{\mathbf{r}}{r} \right] \right] \frac{1}{\sqrt{4\pi}}, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \Phi_{J=0}^{(II)}(\mathbf{r}) = & \left[\Sigma \cdot \left[\frac{1}{r^2} w_3 \frac{\mathbf{r}}{r} \right] + \Gamma[iw_6] \right. \\ & + \beta \alpha \Gamma \cdot \left[\frac{2m}{(E+V_7)r^2} w_5 \frac{\mathbf{r}}{r} \right] \\ & \left. + \beta \Gamma \left[\frac{2i\mu}{E+V_8} w_6 \right] \right] \frac{1}{\sqrt{4\pi}}. \end{aligned} \quad (5.5)$$

C. Equal masses

In many applications one deals with two particles of equal masses, i.e., $\mu=0$ in our notation. It can be easily seen that the coefficients on the right-hand sides of Eqs. (4.4a) and (4.4b) are proportional to μ and, therefore, cancel in the present case, leading to two uncoupled equations. This allows us to further divide class I into states of type A, with $W_3=0$, and states of type B, with $W_2=0$.

Considering states of type A first, Eqs. (4.3b) and (3.4c) show that, for $J \neq 0$, one has $w_3=w_4=0$, and Eq. (5.1) gives, for the corresponding spinor,

$$\begin{aligned} \Phi_{JM}^{(A)}(\mathbf{r}) = & \left[-w_2 - \beta \left[\frac{2m}{(E+V_3)} w_2 \right] \right. \\ & + \alpha \cdot \left[-\frac{2i}{(E+V_5+V_9)r} w_2 r \nabla \right. \\ & \left. \left. - \frac{i}{r^2} w_1 \frac{\mathbf{r}}{r} \right] \right] Y_{JM}(\theta, \varphi). \end{aligned} \quad (5.6)$$

On the other hand, for $J=0$, Eq. (5.4) gives

$$\begin{aligned} \Phi_{J=0}^{(A)}(\mathbf{r}) = & \left[-w_2 - \beta \left[\frac{2m}{(E+V_3)} w_2 \right] \right. \\ & \left. + \alpha \cdot \left[-\frac{i}{r^2} w_1 \frac{\mathbf{r}}{r} \right] \right] \frac{1}{\sqrt{4\pi}}, \end{aligned} \quad (5.7)$$

which can be seen as a particular case of Eq. (5.6).

Coming to states of type B, Eqs. (4.3a) and (3.4b) show that $w_2=w_1=0$, this remaining true for all values of J . Then, Eq. (5.1) gives, for $J \neq 0$,

$$\begin{aligned} \Phi_{JM}^{(B)}(\mathbf{r}) = & \left[\beta \alpha \cdot \left[-\frac{i}{r} w_4 r \nabla - \frac{2iJ(J+1)}{(E+V_2)r^2} w_3 \frac{\mathbf{r}}{r} \right] \right. \\ & + \Sigma \cdot \left[\frac{2m}{(E+V_4+V_{10})r} w_3 \mathbf{L} \right] \\ & \left. + \beta \alpha \Gamma \cdot \left[\frac{1}{r} w_3 \mathbf{L} \right] \right] Y_{JM}(\theta, \varphi), \end{aligned} \quad (5.8)$$

while Eq. (5.4) shows that no state of type B exists for $J=0$. It is interesting to notice that this conclusion would also follow by setting $J=0$ in Eq. (5.8).

If we now examine Eqs. (4.6a) and (4.6b) with the help of Eqs. (3.6), we see that the coefficients on their right-hand sides are proportional to m and not to μ , and the equations remain coupled. Consequently, no further simplification occurs in the equal-masses case for solutions of class II, and we get a single type of state, which, following the original classification of Moseley and Rosen,⁶ we shall call type C. The corresponding spinor is given, according to Eqs. (5.2) and (5.5), by

$$\begin{aligned} \Phi_{JM}^{(C)}(\mathbf{r}) = & \left[\beta \alpha \cdot \left[-\frac{i}{r} w_7 \mathbf{L} \right] + \Sigma \cdot \left[\frac{2}{(E+V_4+V_{10})r} \right. \right. \\ & \left. \left. \times (w_6 + m w_8) r \nabla + \frac{1}{r^2} w_5 \frac{\mathbf{r}}{r} \right] + \Gamma[iw_6] \right. \\ & + \beta \alpha \Gamma \cdot \left[\frac{1}{r} w_8 r \nabla + \frac{2}{(E+V_7)r^2} \right. \\ & \left. \left. \times (-J(J+1)w_7 + m w_5) \frac{\mathbf{r}}{r} \right] \right] Y_{JM}(\theta, \varphi), \end{aligned} \quad (5.9)$$

which, for $J=0$, reduces to

$$\Phi_{j=0}^{(C)}(\mathbf{r}) = \left[\boldsymbol{\Sigma} \cdot \left[\frac{1}{r^2} w_5 \frac{\mathbf{r}}{r} \right] + \Gamma [i w_6] \right. \\ \left. + \beta \alpha \Gamma \cdot \left[\frac{2m}{(E + V_7) r^2} w_5 \frac{\mathbf{r}}{r} \right] \right] \frac{1}{\sqrt{4\pi}}. \quad (5.10)$$

VI. SUMMARY AND CONCLUSIONS

Here we have presented a general discussion of the main properties of the two-body Dirac equation for a rather general class of phenomenological interactions, which includes, besides the usual five tensorial terms, four extra ones that are of importance in particular instances, and allow the inclusion of retardation effects.¹⁵ We have shown that the states with well-defined angular momentum and parity fall into two classes, corresponding, in the general case, to the solutions of two distinct sets of two coupled Schrödinger-type radial equations. These results are in close correspondence with those obtained quite recently by Cheung and Li,¹⁶ who, however, restrict their treatment to the usual five tensorial interactions only. We have also shown how those two classes give rise, in the particular case of equal masses, to the three types of solutions originally introduced by Moseley and Rosen,⁶ based on Kemmer's⁴ classification scheme.

Before closing, we wish to point out that the presence of Breit or Childers terms in the interaction may require a special treatment of the radial equations, such as the analytical asymptotic⁸ or the semiperturbative⁷ methods elaborated on by Królikowski in particular instances.

As an example of the last considerations, we calculate the energy levels of the parapositronium. Here, the interaction is of the form

$$H_{12} = V - \frac{1}{2} \left[\boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2 + \frac{(\boldsymbol{\alpha}_1 \cdot \mathbf{r})(\boldsymbol{\alpha}_2 \cdot \mathbf{r})}{r^2} \right] V', \quad (6.1)$$

with $V = V' = -\alpha/r$, where α is the fine-structure constant.

We have to do with class I, type A solutions. Particularizing the radial equations of Sec. IV for this case, Eq. (4.4a), and following closely the prescription of Ref. 7, we obtain the familiar result¹⁷

$$E = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{4n^4} \left(\frac{11}{16} - \frac{n}{J + \frac{1}{2}} \right) + O(\alpha^6). \quad (6.2)$$

Notice that according to that prescription,⁷ V and V' , though equal, are not treated in the radial equation obtained on the same footing. Thus, in the spirit of treating Breit terms as a lowest-order perturbation only,^{7,8,11,18} expansions up to the first order in V' were performed, to obtain Eq. (6.2).

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