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Cross-Ladder Diagrams Via Hierarchical Equations in the Light Front

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Abstract We calculate the correction to the propagator of two bosons up to the g^4 order in the "cross-ladder" approximation in the light front. The hierarchy of the coupled equations describe the propagator of two bosons in various cases, including the "cross-ladder" approximation. Our results show that the contribution of the pair is small and is limited to a range of bound state.

1 Introduction

The two body Green function in the light front includes the propagation of intermediate states with any number of particles. We start our discussion evaluating the second order correction to the coupling constant associated with the propagator. We define the matrix element for the interaction and so we obtain the correction to the Green function in the light front. Then we evaluate the correction to the Green function to the fourth order in the coupling constant, where we use the technique of factorizing the energy denominators, which is important to identify the global propagation of four bodies after the integration in the energies k^- [1].

2 Bosons Propagation Model in the Light-Front

In this section we consider the two body Green function in the "ladder" approximation for the dynamics defined in the light front. Within this treatment, we are not going to deal with perturbative corrections that can be decomposed into one body problem. Our interest is to define in the light front the interaction between two bodies mediated by the interchange of a particle and obtain the correction to the two body Green function originated in this interaction.

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For this purpose, we use a bosonic model for which the interaction Lagrangian is defined as:

$$\mathcal{L}_I = g\phi_1^*\phi_1\sigma + g\phi_2^*\phi_2\sigma, \tag{1}$$

where the bosons ϕ_1 and ϕ_2 have equal mass m and the intermediate boson, σ, has the mass m_σ, and the coupling constant is g.

The propagation of a free particle with spin zero in four dimensional space-time is represented by the covariant Feynman propagator

$$S(x^\mu) = \int i \frac{d^4k}{(2\pi)^4} \frac{e^{-i(k^0x^0 - \mathbf{k}\cdot\mathbf{x})}}{k^2 - m^2 + i\varepsilon}, \tag{2}$$

where the coordinate x^0 represents the time and k^0 the energy.

The light-cone coordinates are given by $x^\pm = \frac{1}{\sqrt{2}}(ct \pm z)$ and $\mathbf{x}_\perp = x^1\hat{\imath} + x^2\hat{\jmath}$, such that for Lorentz transformations along z direction this implies $\mathbf{x}'_\perp = \mathbf{x}_\perp$. Our light-front metric $(+, -, 1, 2)$ is therefore $g^{+-} = -g^{11} = -g^{22} = 1$. Scalar product of two vectors is then $a \cdot b = a^+b^- + a^-b^+ - \mathbf{a}_\perp \cdot \mathbf{b}_\perp$.

We make the projection of the propagator for a boson in time associated to the null plane rewriting the coordinates in terms of time coordinate x^+ and the position coordinates (x^- and $\mathbf{x}_\perp$). With these, the momenta are given by k^-, k^+ and $\mathbf{k}_\perp$, and the Jacobian of the transformation $(k^0, \mathbf{k}) \to (k^-, k^+, \mathbf{k}_\perp)$ is equal to 1 and k^+, $\mathbf{k}_\perp$ are momentum operators. Evaluating the Fourier transform, we obtain

$$\widetilde{S}(k^-) = \frac{i}{2k^+\left(k^- - \frac{\mathbf{k}_\perp^2 + m^2}{2k^+} + \frac{i\varepsilon}{2k^+}\right)}, \tag{3}$$

The Green funtion in the light front $G(x^+)$ acting in the Fock space is defined as the probability amplitude of the transition from the initial state in the Fock space $|i\rangle$ to the final state $|f\rangle$. In the case of a free boson, the Green function for the propagation of a particle is defined by the operator

$$G_0^{(p,a)}(k^-) = \frac{\theta(\pm k^+)}{k^- - k^-_{\text{on}} \pm i\epsilon}; \tag{4}$$

where $k^-_{\text{on}} = \frac{\mathbf{k}_\perp^2 + m^2}{2k^+} > 0$ is the energy of the particle propagation, $G_0^{(p)}$, and $k^-_{\text{on}} < 0$ the antiparticle propagation, $G_0^{(a)}$.

We can see that the difference between the Green functions in Eqs. (4) and (3) for the propagator in the light front is the absence of the imaginary (complex) number i and of the factor of phase space $2k^+$ which appears in the denominator of Eq. (3).

3 Hierarchical Equations

In principle and generally, the Green function in the light front for a system of two bodies could be obtained from the solution for the covariant Bethe–Salpeter equation which has all the two body irreducible diagrams in the kernel and self-energy corrections to the intermediate propagators of the ϕ_1 and ϕ_2 bosons [2]. However, this is easier said than done. In practice, what we can do as a first approximation is to consider the two boson Green function in the light front, without including self-energy corrections to the intermediate bosons, that is, closed loops for the bosons ϕ_1 and ϕ_2 but with crossed diagrams, as a solution to the following set of hierarchical equations:

$$
\begin{aligned}
G^{(2)}(K^-) &= G_0^{(2)}(K^-) + G_0^{(2)}(K^-)V(G^{(3)})(K^-)VG^{(2)}(K^-) \, , \\
G^{(3)}(K^-) &= G_0^{(3)}(K^-) + G_0^{(3)}(K^-)V(G^{(4)})(K^-)VG^{(3)}(K^-) \, , \\
G^{(4)}(K^-) &= G_0^{(4)}(K^-) + \Delta G_\times^{(4)}(K^-) + G_0^{(4)}(K^-)V(G^{(5)})(K^-)VG^{(4)}(K^-) \, , \\
&\vdots \quad \vdots \\
G^{(N)}(K^-) &= G_0^{(N)}(K^-) + G_0^{(N)}(K^-)V(G^{(N+1)})(K^-)VG^{(N)}(K^-), \\
&\vdots \quad \vdots
\end{aligned} \tag{5}
$$

where $\Delta G_\times^{(4)}(K^-) = G_\times^{(4)}(K^-) + G_\times^{(4)\text{ant}}(K^-)$ is the inclusion of the "cross ladder" diagram and where the vertex V is given explicitly in the matrix elements of the interaction Hamiltonian that creates or destroys a quantum of the intermediate boson; for example in the matrix elements

$$
\begin{aligned}
< qk_\sigma|V|k > &= 2\delta(q + k_\sigma - k)\frac{g}{\sqrt{q^+k_\sigma^+k^+}}\theta(k_\sigma^+) \\
< q|V|k_\sigma k > &= 2\delta(k + k_\sigma - q)\frac{g}{\sqrt{q^+k_\sigma^+k^+}}\theta(k_\sigma^+) \, ,
\end{aligned} \tag{6}
$$

where $k_\sigma^+ = k^+ - q^+$ and $\mathbf{k}_{\sigma\perp} = \mathbf{k}_\perp - \mathbf{q}_\perp$. As we want to have a correction for the two boson propagators to the fourth order in the coupling constant, we have eigth propagators in the "cross-ladder" diagram, two of which are propagators of the interacting bosons, with momenta identified as k_σ and $k_{\sigma'}$. For the moment, we shall identify the other six propagators with momenta k_i, where $i = 1, 2, 3, 4, 5, 6$.

The set of equations above, Eq. (5), include, in particular, the "ladder" approximation for the covariant Bethe–Salpeter equation. The hierarchical equations, Eq. (5), correspond to a truncation in the Fock space in the light front, so that only states with two particles ϕ_1 and ϕ_2 without restriction in the number of σ bosons are permitted in the intermediate states. The free propagation of these states is represented by the Green function $G_0^{(N)}(K^-)$, where the number of σ bosons is $N-2$. The iterated equations in Eq. (5) do not, however, include the totality of crossed "ladder" diagrams. For example, the intermediate propagation in the light front of a state of one ϕ_1 boson, two ϕ_2 bosons and one ϕ_2 antiboson (four body Fock state) are not included in the proposed hierarchical equations. In order to get the two body propagator in the light front time in the "ladder" approximation, we shall restrict ourselves to the hierarchy of equations Eq. (5).

4 Cross Ladder Contribution

The construction of the integral equation for the vertex follows the same procedure used for the ladder diagram [3], and the corresponding Green's function for the crossed diagram is

$$
\begin{aligned}
|\Psi_B > = G_0^{(2)}(K_B^-)V\{G_0^{(3)}(K_B^-) &+ G_0^{(3)}(K_B^-)V[G^{(4)}(K_B^-) \\
&+ G_\times^{(4)}(K_B^-) + G_\times^{(4)\text{ant}}(K_B^-)]VG_0^{(3)}(K_B^-)\}V\,|\Psi_B\rangle \, .
\end{aligned} \tag{7}
$$

Note that we have an explicit non vanishing contribution coming from the antiparticle sector of the cross ladder diagram, that is summed into the kernel of the equation. The vertex for the bound state satisfies the following homogeneous integral equation

$$
\begin{aligned}
\Gamma_B(\mathbf{q}_\perp, y) = \int \frac{dxd^2\mathbf{k}_\perp}{x(1-x)}\frac{\Gamma_B(\mathbf{k}_\perp, x)}{M_B^2 - M_0^2} \\
\times \Big[K^{(3)}(\mathbf{q}_\perp, y; \mathbf{k}_\perp, x) + K^{(4)}(\mathbf{q}_\perp, y; \mathbf{k}_\perp, x) \\
+ K_\times^{(4)}(\mathbf{q}_\perp, y; \mathbf{k}_\perp, x) + K_\times^{(4)\text{ant}}(\mathbf{q}_\perp, y; \mathbf{k}_\perp, x)\Big] \, ,
\end{aligned} \tag{8}
$$

where $K^{(3)}$, $K^{(4)}$, $K_{\times}^{(4)}$ and $K_{\times}^{(4)\mathrm{ant}}$ are respectively the three-bodies, four-bodies, "cross-ladder" and antiparticle kernels respectively, given by

$$\begin{aligned}
K^{(3)}(\mathbf{q}_\perp, \mathbf{y}; \mathbf{k}_\perp, \mathbf{x}) = {} & \frac{g^2}{16\pi^3}\frac{\theta(x-y)}{(x-y)} \\
& \times \frac{1}{\left(M_B^2 - \frac{q_\perp^2+m^2}{y} - \frac{k_\perp^2+m^2}{1-x} - \frac{(q-k)_\perp^2+m_\sigma^2}{x-y}\right)} \\
& + [k \leftrightarrow q], \qquad (9)\\
K^{(4)}(\mathbf{q}_\perp, \mathbf{y}; \mathbf{k}_\perp, \mathbf{x}) = {} & \left(\frac{g^2}{16\pi^3}\right)^2 \int \frac{d^2 p_\perp dz}{z\,(z-x)\,(y-z)\,(1-z)} \\
& \times \frac{\theta(y-z)\theta(z-x)}{\left(M_B^2 - \frac{k_\perp^2+m^2}{x} - \frac{p_\perp^2+m^2}{1-z} - \frac{(p-k)_\perp^2+m_\sigma^2}{z-x}\right)} \\
& \times \frac{1}{\left(M_B^2 - \frac{p_\perp^2+m^2}{z} - \frac{q_\perp^2+m^2}{1-y} - \frac{(q-p)_\perp^2+m_\sigma^2}{y-z}\right)} \\
& \times \frac{1}{\left(M_B^2 - \frac{k_\perp^2+m^2}{x} - \frac{q_\perp^2+m^2}{1-y} - \frac{(q-p)_\perp^2+m_\sigma^2}{y-z} + \frac{(p-k)_\perp^2+m_\sigma^2}{z-x}\right)} \\
& + \left[x \leftrightarrow y, \mathbf{k}_\perp \leftrightarrow \mathbf{q}_\perp\right], \qquad (10)\\
K_{\times}^{(4)} = {} & \left(\frac{g^2}{16\pi^3}\right)^2 \int \frac{d^2\mathbf{p}_\perp dz\theta(1-z)}{z\,(x-z)\,(1-z-x-y)} \left(K'_{\times} + K''_{\times}\right), \qquad (11)
\end{aligned}$$

with

$$\begin{aligned}
K'_{\times} = {} & \frac{\theta(z-y)\theta(x-z)}{(z-y)\left[M_B^2 - \frac{\mathbf{p}_\perp^2+m^2}{z} - \frac{\mathbf{k}_\perp^2+m^2}{1-x} - \frac{(\mathbf{k}-\mathbf{p})_\perp^2+m_\sigma^2}{x-z}\right]} \\
& \times \frac{1}{M_B^2 - \frac{(\mathbf{p}-\mathbf{q})_\perp^2+m_\sigma^2}{z-y} - \frac{(\mathbf{k}-\mathbf{p})_\perp^2+m_\sigma^2}{x-z} - \frac{\mathbf{k}_\perp^2+m^2}{1-x} - \frac{\mathbf{q}_\perp^2+m^2}{y}} \\
& \times \frac{1}{M_B^2 - \frac{(\mathbf{k}-\mathbf{p})_\perp^2+m_\sigma^2}{x-z} - \frac{(\mathbf{p}-\mathbf{k}-\mathbf{q})_\perp^2+m^2}{1-z-x-y} - \frac{\mathbf{q}_\perp^2+m_\sigma^2}{y}} \\
& + \left[(x \leftrightarrow y), (\mathbf{k}_\perp \leftrightarrow \mathbf{q}_\perp)\right], \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}
K''_{\times} = {} & \frac{\theta(y-z)\theta(x-z)}{(y-z)} \\
& \times \frac{1}{M_B^2 - \frac{(\mathbf{k}-\mathbf{p})_\perp^2+m_\sigma^2}{x-z} - \frac{(\mathbf{K}-\mathbf{p}-\mathbf{k}-\mathbf{q})_\perp^2+m^2}{1-z-x-y} - \frac{\mathbf{p}_\perp^2+m^2}{z} - \frac{(\mathbf{q}-\mathbf{p})_\perp^2+m_\sigma^2}{y-z}} \\
& \times \frac{1}{M_B^2 - \frac{(\mathbf{q}-\mathbf{p})_\perp^2+m_\sigma^2}{y-z} - \frac{(\mathbf{K}-\mathbf{p}-\mathbf{k}-\mathbf{q})_\perp^2+m^2}{1-z-x-y} - \frac{\mathbf{k}_\perp^2+m^2}{x}} \\
& \times \left\{ \frac{1}{M_B^2 - \frac{(\mathbf{k}-\mathbf{p})_\perp^2+m_\sigma^2}{x-z} - \frac{(\mathbf{K}-\mathbf{p}-\mathbf{k}-\mathbf{q})_\perp^2+m^2}{1-z-x-y} - \frac{\mathbf{q}_\perp^2+m_\sigma^2}{y}} \right.
\end{aligned}$$

$$
+ \frac{1}{M_B^2 - \frac{(\mathbf{q}-\mathbf{p})_\perp^2 + m_\sigma^2}{y-z} - \frac{\mathbf{q}_\perp^2 + m^2}{1-y} - \frac{\mathbf{p}_\perp^2 + m^2}{z}} \Bigg\}
$$

$$
+ \big[(x \leftrightarrow 1-x)\,;\, (y \leftrightarrow 1-y)\,;\, (z \leftrightarrow 1+z-x-y)\,; \big(\mathbf{p}_\perp \leftrightarrow (\mathbf{K}+\mathbf{p}-\mathbf{k}-\mathbf{q})_\perp\big) \big], \tag{13}
$$

and finally, the kernel for antiparticle:

$$
\begin{aligned}
K_\times^{(4)\mathrm{ant}} = \left(\frac{g^2}{16\pi^3}\right)^2 \int & \frac{d^2\mathbf{k}_\perp dz \theta(1+z)\theta(-z)}{|z|\,(y+|z|)\,(1-|z|-x-y)\,(x+|z|)} \\
& \times \frac{\theta(y+z)\theta(x+z)}{M^2 - \frac{\mathbf{q}_\perp^2+m^2}{y} - \frac{(\mathbf{K}+\mathbf{k}-\mathbf{p}-\mathbf{q})_\perp^2+m^2}{1-|z|-x-y} + \frac{\mathbf{k}_\perp^2+m^2}{|z|} - \frac{\mathbf{p}_\perp^2+m_\sigma^2}{x}} \\
& \times \frac{1}{M^2 - \frac{(\mathbf{p}-\mathbf{k})_\perp^2+m_\sigma^2}{x+|z|} - \frac{(\mathbf{K}+\mathbf{k}-\mathbf{p}-\mathbf{q})_\perp^2+m^2}{1-|z|-x-y} - \frac{\mathbf{p}_\perp^2+m_\sigma^2}{x}} \\
& \times \frac{1}{M^2 - \frac{(\mathbf{q}-\mathbf{k})_\perp^2+m_\sigma^2}{y+|z|} - \frac{(\mathbf{K}+\mathbf{k}-\mathbf{p}-\mathbf{q})_\perp^2+m^2}{1-|z|-x-y} - \frac{\mathbf{q}_\perp^2+m_\sigma^2}{y}}.
\end{aligned} \tag{14}
$$

Equation (8) is the vertex equation for the bound state with the kernel expanded up to the fourth order in g in the light-front in the "cross-ladder" approximation.

To illustrate the contribution of the antiparticle diagram for the bound state $0.0 \le M \le 2.0$ (in appropriate units) up to the fourth order in the coupling constant $0.0 \le g \le 20.0$ (in appropriate units) we have plotted

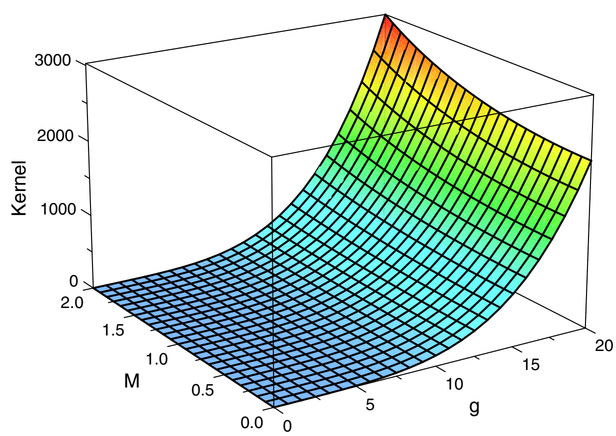

Fig. 1 Antiparticle kernel alone

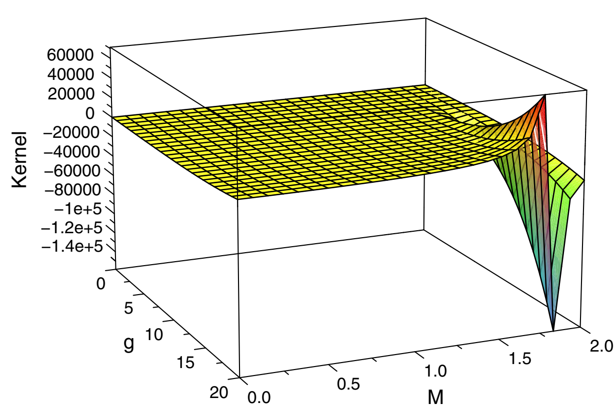

Fig. 2 Cross-ladder kernel alone

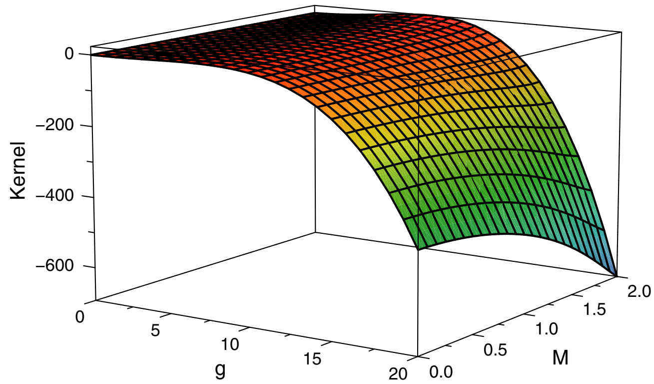

Fig. 3 Three and four bodies kernels taken (summed) together

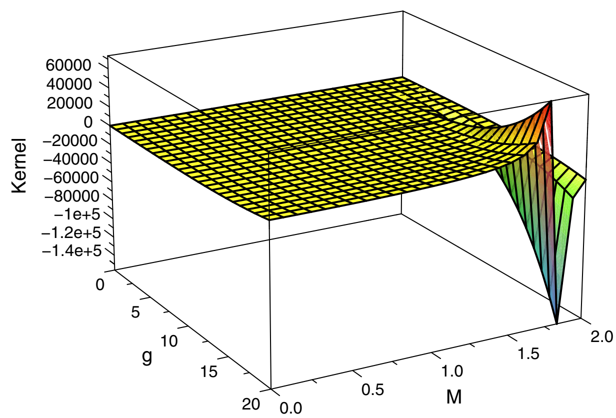

Fig. 4 Sum of all relevant kernels up to ladder approximation

some graphs for the relevant kernels and also parcial and total sum of them which may be seen in the following (Figs. 1, 2, 3, 4):

5 Conclusions

Although the solution to the Bethe–Salpeter equation is very hard to obtain, our results indicate that depending on the region of the bound state mass we may just consider some effective kernels. In our approach to obtain the Bethe–Salpeter equation to this order we followed the technique of hierarchical equations which has been previously applied to obtain the contributions to the second and fourth order in g for the ladder diagram. However, at this level, there is also a non-vanishing contribution coming from the pair-production in the crossed ladder diagram besides the usual cross ladder contribution. We may conclude that though there is an antiparticle contribution in the crossed diagrams, it remains almost imperceptible for low masses and small couplings as we may check in the figures.

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