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An algorithm based on negative probabilities for a separability criterion

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Abstract Here, we demonstrate that entangled states can be written as separable states [$\rho_{1\ldots N} = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$, 1 to N refering to the parts and p_i to the nonnegative probabilities], although for some of the coefficients, p_i assume negative values, while others are larger than 1 such to keep their sum equal to 1. We recognize this feature as a signature of non-separability or *pseudoseparability*. We systematize that kind of decomposition through an algorithm for the explicit separation of density matrices, and we apply it to illustrate the separation of some particular bipartite and tripartite states, including a multipartite $\bigotimes^N 2$ one-parameter Werner-like state. We also work out an arbitrary bipartite 2×2 state and show that in the particular case where this state reduces to an X-type density matrix, our algorithm leads to the separability conditions on the parameters, confirmed by the Peres-Horodecki partial transposition recipe. We finally propose a measure for quantifying the degree of entanglement based on these peculiar negative (and greater than one) probabilities.

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1 Introduction

The entanglement phenomenon has undertaken joint efforts from experimental and theoretical research since the proposition of Bell's theory in the mid-1960 [1]. With the goal to demonstrate the violation of Bell's inequalities, a major development in experimental techniques has been carried out to produce entangled photons, from the cascade atomic process [2] to the parametric down-conversion processes [3]. Massive entangled particles have also been produced through radiation–matter interaction in cavity QED [4] and trapped ions [5–7]. In the latter case, a controlled entanglement of 14 quantum bits has been recently generated enabling the implementation of the largest quantum register to date [8]. The violation of a form of Bell's inequality has also been verified with massive entangled particles within an ion trap [9]. Parallel to the experimental achievements, theoretical physics has been struggling with recently advanced striking features of entanglement, such as quantum correlations sudden death [10,11], quantum discord [12], and the derivation of separability criterion for density matrices [13,14].

A significant expansion of scope in the debate on quantum non-locality comes with the issue of separability of mixed density matrices, which is the focus of this work. Beyond the boundaries of testing non-locality of entangled pure states —guideline that defined the research on non-locality of the decade from 1970 to the 1990s— the introduction of the mixed density matrices brings the notion that while separable systems do satisfy Bell's inequality, the converse is not necessarily true [15–19]. Consequently, Bell's inequalities are not sufficiently sensitive tools for detecting inseparability of mixed states.

A simple necessary condition for separability of bipartite systems having dimensions 2×2 and 2×3, based on the partial transposition condition, was presented in Ref. [13]. Soon after it has been shown in Ref. [14] that for the dimensions considered, the necessary partial transposition condition is also a sufficient one. For higher dimensions, the necessary and sufficient condition for separability are derived from positive maps instead of the simple partial transposition technique, as also demonstrated in Ref. [14]. After all, the separability is an NP-hard problem [20–22]. In the wake of the criteria presented in Refs. [13,14], a number of alternative partial criteria for separability have surfaced which apply only for particular families of states [23–26]. More recently, separability criteria for different classes of multiparticle entanglement has been provided [27,28], which again are both necessary and sufficient for particular families of states, as for example N-qubit GHZ states [27]. We finally note that methods to characterize the set of separable states with arbitrary precision, the well-known complete criteria for separability detection, have also been formulated [29–32].

Taking a different approach to the separability problem, we consider the following point: A separable quantum state of a multipartite system can be written in the form $\rho_{1\ldots N} = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$ (1 to N standing for the parts), where the coefficients p_i are nonnegative and $\sum_i p_i = 1$, and therefore, they can be assumed being probabilities, once $\mathrm{Tr}_k \rho_i^{(k)} = 1$ and the marginal state for each subsystem $k = 1, \ldots, N$ is $\rho_k = \sum_i p_i \rho_i^{(k)}$. Here, we demonstrate that even non-separable or entangled states can be written in that very same form, although some of the p_i are forcefully negative while others are larger than 1. This feature we call *pseudoseparability*, and it constitutes a signature of a non-separable system. The negative coefficients p_i were called *negative probabilities* by Richard Feynman [33]. Regarding this *pseudoseparability*, we show —for a number of density matrices—how to realize it formally through an algorithm that consists of a sequence of procedures. The algorithm is based on the use of sets of selected state vectors, and the balancing of their expansion coefficients is the central ingredient for the implementation of a genuine probability or a *pseudoseparability*, so as to eliminate non-entangled states with negative probabilities. From the derivation of the negative probabilities, we propose a measure for quantifying the degree of entanglement of arbitrary multipartite states.

We must observe that our algorithm, although based on the density matrix formalism, presents some conceptual overlap with the discretized version of the Wigner function [35–37] and the quasiprobability-based criterion for classicality [38,39]. One could also recognize some overlap with Ref. [40], where the concept of negative probability is analyzed in the context of the micromaser which-path detector, and also with Ref. [41], where negative probabilities are used as a witness of non-classicality. Finally, we mention the cross-norm criterion for entanglement [42–44], whose idea is to write a density matrix ρ in a pseudoseparate decomposition $\rho = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$ and then define the cross-norm of ρ to be the smallest possible value of $\sum_i p_i$ in any such decomposition of ρ. It turns out that this norm equals 1 if and only if ρ is separable (if and only if $0 \leq p_i \leq 1$ for all i).

For the purpose of clarity, we start by presenting only a brief description of our algorithm, and then, to better illustrate its steps, we apply it to the case *(1)* of the single-parameter two-qubit $\left(\bigotimes^2 2\right)$ Werner state, because it allows a simple analytical manipulation. We next focus our attention on particular *(2)* bipartite 2×3 state, *(3)* tripartite $2 \times 2 \times 2$ state, and finally a *(4)* Werner-like multiqubit $\left(\bigotimes^N 2\right)$ state, confronting, again, our method with the partial transposition separability criterion. (The symbol $\bigotimes^N 2$ stands for a tensorial product $\rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$ of N 2×2 parts.) Thereafter, we handle the case *(5)* of an arbitrary bipartite state, finding that our algorithm leads to separability conditions which are analytically verified by the partial transposition method for the particular case of an X-type density matrix. We next demonstrate that an arbitrary multipartite state ρ can always be written in a pseudoseparable form. Finally, we propose a formula to measure the degree of entanglement based on the negative probabilities and we present our conclusions.

2 An overview of our separability algorithm

Although we really intend to clarify the whole idea of our algorithm by applying it to paradigmatic cases like Werner state, we here present an overview of the method, in order to facilitate the understanding of its application. We have a four-step algorithm which start with the decomposition of the desired N-partite state $\rho_{1\ldots N}$ into diagonal and non-diagonal components $\rho_{1\ldots N} = \rho_{1\ldots N}^{(D)} + \rho_{1\ldots N}^{(ND)}$ using a convenient basis of states for each particle composing the entanglement. Of course, a separable state is made only by the diagonal component. In the second step, the non-diagonal component $\rho_{1\ldots N}^{(ND)}$ must be transformed into a diagonal form with the help of another set of arbitrary superposition states, each again for a particle in the entanglement. The arbitrary expansion parameters $\{a_m\}$ of these superpositions are key ingredient of the algorithm and shall be determined from this and the next steps of the algorithm. We show that it is always possible to write $\rho_{1\ldots N}^{(ND)}$ as a sum of two diagonal forms Λ and Σ, the former (latter) containing only positive (negative) expansion coefficients or probabilities. In the third step, we add and subtract the separable operator Λ to $\rho_{1\ldots N}$, to obtain $\rho_{1\ldots N} = \left(\rho_{1\ldots N}^{(D)} + \Sigma - \Lambda\right) + 2\Lambda$. Thus, as far as the coefficients of Λ are already nonnegative, only the sum of terms in the parentheses must be analyzed so that we can derive a separability condition, i.e., the range of values of the parameters a_m that lead to a genuine separable state with positive probabilities. Finally, a fourth step is necessary since a pseudoseparable state with negative probabilities is not necessarily entangled. In fact, as to be demonstrate in Sect. 7, any state ρ can be written in a pseudoseparable form. To guarantee that the negative probabilities are a signature of entanglement, we have to introduce the fourth step, where a balance of the inequalities derived in the third step (made through the expansion coefficients $\{a_n\}$) ensures positive probabilities for separable states. The fourth step is an essential requirement to guarantee that any negative probability remaining after the balancing procedure is indeed a signature of entanglement.

3 The method applied to the 2 × 2 Werner state

To illustrate our algorithm, we consider the simple and ubiquitous two-qubit Werner state

$$\rho_{12} = x \left|\Psi_{12}^{-}\right\rangle\left\langle\Psi_{12}^{-}\right| + \frac{1-x}{4}\mathbb{I}, \tag{1}$$

which describes a bipartite 2×2 system in the Bell basis $\left\{\left|\Psi_{12}^{\pm}\right\rangle = (|1_1 0_2\rangle \pm |0_1 1_2\rangle)/\sqrt{2}, \left|\Phi_{12}^{\pm}\right\rangle = (|1_1 1_2\rangle \pm |0_1 0_2\rangle)/\sqrt{2}\right\}$, $\mathbb{I}$ being the unit operator. We next expose our four-step algorithm for separability:

i) Firt step: *Diagonal and non-diagonal decomposition of* ρ_{12}. Decompose the "two-particle" state ρ_{12} into its *diagonal* and *non-diagonal* components, $\rho_{12} = \rho_{12}^{(D)} + \rho_{12}^{(ND)}$, in the basis $\{|1_\ell\rangle, |0_\ell\rangle\}$, eigenstates of Pauli matrix σ_z^ℓ $(\ell = 1, 2)$,

$$\begin{aligned}\rho_{12}^{(D)} &= \frac{1-x}{4}\left[(|1_1\rangle\langle 1_1| + |0_1\rangle\langle 0_1|) \otimes (|1_2\rangle\langle 1_2| + |0_2\rangle\langle 0_2|)\right.\\ &\quad \left. + \frac{x}{2}|0_1\rangle\langle 0_1| \otimes |1_2\rangle\langle 1_2| + \frac{x}{2}|1_1\rangle\langle 1_1| \otimes |0_2\rangle\langle 0_2|\right], \end{aligned} \tag{2a}$$

$$\rho_{12}^{(ND)} = -\frac{x}{2}\left[|1_1\rangle\langle 0_1| \otimes |0_2\rangle\langle 1_2| + |0_1\rangle\langle 1_1| \otimes |1_2\rangle\langle 0_2|\right]. \tag{2b}$$

ii) Second step: *Transforming* $\rho_{12}^{(ND)}$ *into a diagonal form* $|\Phi\rangle\langle\Phi|$. Introduce $k = 1, \ldots, M$ arbitrary states for each particle $(\ell = 1, 2)$:

$$\left|\psi_k^{(\ell)}\right\rangle = \mathcal{N}^{(\ell)} \sum_{m=0}^{1} e^{i2\pi mk/M} a_m^{(\ell)} \left|m_\ell\right\rangle, \tag{3}$$

where $\mathcal{N}^{(\ell)} = \left(\left|a_0^{(\ell)}\right|^2 + \left|a_1^{(\ell)}\right|^2\right)^{-1/2}$. The expansion parameters $a_m^{(\ell)}$, so far arbitrary, shall be determined from this and the next steps of the algorithm. Moreover, we must have $M \geq M_{\min}$ where $M_{\min}$ stands for the minimum number of states $\left|\psi_k^{(\ell)}\right\rangle$ needed to write the interference terms ($\left|0_1\right\rangle \left\langle 1_1\right|$ and its Hermitian conjugate) into a diagonal form. In fact, the diagonal operator

$$\sum_{k=0}^{M-1} \left(\mathcal{N}^{(\ell)}\right)^{-2} e^{i2\pi k/M} \left|\psi_k^{(\ell)}\right\rangle\left\langle\psi_k^{(\ell)}\right|$$
$$= \sum_{m,n=0}^{1} \left[\sum_{k=0}^{M-1} e^{i2\pi(m-n+1)k/M}\right] a_m^{(\ell)} \left(a_n^{(\ell)}\right)^* \left|m_\ell\right\rangle \left\langle n_\ell\right|, \tag{4}$$

becomes proportional to the interference operator $\left|0_\ell\right\rangle \left\langle 1_\ell\right|$ if and only if the sum in the brackets, in the rhs of Eq. (4), equals $M\delta_{m,1-n}$, that happens whenever $M > M_{\min} = \max(m-n+1) = 2$. It is also straightforward to prove that, in the general case where D is the dimension of the arbitrary states $\left|\psi_k^{(\ell)}\right\rangle$ of a partite: $M_{\min} = 2D - 1$. For $M \geq 3$, a non-diagonal element can be expanded in terms of elementary projectors $\left|\psi_k^{(\ell)}\right\rangle\left\langle\psi_k^{(\ell)}\right|$,

$$\left|0_\ell\right\rangle \left\langle 1_\ell\right| = \frac{1}{M a_0^{(\ell)} \left(a_1^{(\ell)}\right)^*} \sum_{k=0}^{M-1} \left(\mathcal{N}^{(\ell)}\right)^{-2} e^{i2\pi k/M} \left|\psi_k^{(\ell)}\right\rangle\left\langle\psi_k^{(\ell)}\right|; \tag{5}$$

then, comparing and equaling this form with Eq. (2b), we set the following constraint for the coefficients $a_m^{(\ell)}$,

$$a_0^{(1)} a_1^{(2)} \left(a_1^{(1)} a_0^{(2)}\right)^* = x/2, \tag{6}$$

and we succeed to write, through the set of states $\left|\psi_k^{(\ell)}\right\rangle$, the term $\rho_{12}^{(ND)}$ into a sum of two diagonal forms,

$$\Lambda = \sum_{k,k'=0\left(\cos\left[\frac{2\pi}{M}(k-k')\right]<0\right)}^{M-1} \mathcal{P}_{kk'} \left|\psi_k^{(1)}\right\rangle\left\langle\psi_k^{(1)}\right| \otimes \left|\psi_{k'}^{(2)}\right\rangle\left\langle\psi_{k'}^{(2)}\right|, \tag{7a}$$

$$\Sigma = -\sum_{k,k'=0\left(\cos\left[\frac{2\pi}{M}(k-k')\right]\geq 0\right)}^{M-1} \mathcal{P}_{kk'} \left|\psi_k^{(1)}\right\rangle\left\langle\psi_k^{(1)}\right| \otimes \left|\psi_{k'}^{(2)}\right\rangle\left\langle\psi_{k'}^{(2)}\right|, \tag{7b}$$

involving the elementary projectors and the coefficients are $\mathcal{P}_{kk'} = 2\left(\mathcal{N}^{(1)}\mathcal{N}^{(2)}\right)^{-2}$ $\left|\cos\left[\frac{2\pi}{M}\left(k-k'\right)\right]\right|/M^2$. The sums in Eqs. (7a) and (7b) are done under different constraints, as explicitly specified in the underscript of the sum symbols, in order to separate terms having positive coefficients from those assuming negative values. Equation (7) is Werner state in a separablelike form. Before addressing the following step of the algorithm, which allows to distinguish between separable and entangled states, we observe that the pseudoseparable form is clearly not unique since it depends on the number of arbitrary states $\left|\psi_k^{(\ell)}\right\rangle$ introduced to write the interference terms into a diagonal form.

iii) Third step: *Conditions on the parameters*. We add and subtract the already separable operator Λ to ρ_{12}, such that $\rho_{12} = \left(\rho_{12}^{(D)} + \Sigma - \Lambda\right) + 2\Lambda$ (in the sense that $\left.\sum_{k,k'=0\left(\cos\left[\frac{2\pi}{M}(k-k')\right]<0\right)}^{M-1} + \sum_{k,k'=0\left(\cos\left[\frac{2\pi}{M}(k-k')\right]\geq 0\right)}^{M-1} = \sum_{k,k'=0}^{M-1}\right)$, thus only the sum of terms in the parentheses must be analyzed carefully (reminding that the coefficients of Λ, Eqs. (7a) and (3), are already nonnegative). Thus,

$$\begin{aligned}\rho_{12} = &\left[\frac{1-x}{4} - \xi\left|a_1^{(1)}a_1^{(2)}\right|^2\right]|1_1\rangle\langle 1_1|\otimes|1_2\rangle\langle 1_2| \\ &+\left[\frac{1-x}{4} - \xi\left|a_0^{(1)}a_0^{(2)}\right|^2\right]|0_1\rangle\langle 0_1|\otimes|0_2\rangle\langle 0_2| \\ &+\left[\frac{1+x}{4} - \xi\left|a_0^{(1)}a_1^{(2)}\right|^2\right]|0_1\rangle\langle 0_1|\otimes|1_2\rangle\langle 1_2| \\ &+\left[\frac{1+x}{4} - \xi\left|a_1^{(1)}a_0^{(2)}\right|^2\right]|1_1\rangle\langle 1_1|\otimes|0_2\rangle\langle 0_2| + 2\Lambda, \end{aligned} \tag{8}$$

where $\xi = 2\left(1-2^{1-j}\right)$ for $M = 2^j$ ($j \in \mathbb{N}$ and $j \geq 2$) and $\xi = 2$ otherwise. In order to determine the range of values of the parameters in Eq. (8) that lead to a genuine separable state ($p_n \geq 0$), the following four inequalities must be satisfied

$$1 - x \geq \begin{cases} 4\xi\left|a_1^{(1)}a_1^{(2)}\right|^2 \\ 4\xi\left|a_0^{(1)}a_0^{(2)}\right|^2 \end{cases}, \qquad 1 + x \geq \begin{cases} 4\xi\left|a_0^{(1)}a_1^{(2)}\right|^2 \\ 4\xi\left|a_1^{(1)}a_0^{(2)}\right|^2 \end{cases}, \tag{9}$$

plus the condition $\left|a_0^{(1)}a_1^{(1)}a_0^{(2)}a_1^{(2)}\right|^2 = x^2/4$ as established previously in Eq. (6).

In order to justify the fourth step, we observe that a pseudoseparable state with negative probabilities is not necessarily entangled. In fact, as a general result, we demonstrate in Sect. 7 that any state ρ can be written in a pseudoseparable form. To guarantee that the negative probabilities are a signature of entanglement, we have to introduce the fourth step, where a balance of the inequalities derived in the third step (made through the expansion coefficients $\{a_n\}$) ensures positive probabilities for separable states. Therefore, the fourth step is an essential requirement to guarantee that any negative probability remaining after the balancing procedure is indeed a signature of entanglement.

iv) Fourth step: *Balancing the expansion coefficients of the states* $\left|\psi_k^{(\ell)}\right\rangle$*: the separability criterion*. The final step is to adjust the coefficients $a_0^{(1)}$, $a_1^{(1)}$, $a_0^{(2)}$, $a_1^{(2)}$ in order to reduce the number of inequalities and, consequently, to maximize the range of values of the interpolator parameter x, which still ensures separability, i.e., $p_n \geq 0$ in Eq. (8). The parameter x becomes maximum for $\left|a_1^{(1)}a_1^{(2)}\right|^2 = \left|a_0^{(1)}a_0^{(2)}\right|^2$ which, together with condition $\left|a_0^{(1)}a_1^{(1)}a_0^{(2)}a_1^{(2)}\right|^2 = x^2/4$, allows us to choose, for simplicity, the solution $\left|a_0^{(1)}\right|^2 \kappa = \left|a_1^{(1)}\right|^2/\kappa' = \left|a_0^{(2)}\right|^2/\kappa = \left|a_1^{(2)}\right|^2 \kappa' = \sqrt{x/2}$, since $1/2 \leq \kappa\kappa' \leq 2$. It is worth noting that any other choice of these parameters will produce different basis states (3), and consequently different representations for the pseudoseparable state. Thence, the four inequalities in Eq. (9) reduce to two, $x \leq 1/\left(2\xi + 1\right)$, $x \leq 1/\left(2\xi - 1\right)$, for the separability criterion, or more concisely,

$$0 \leq x \leq \min\left(\frac{1}{2\xi + 1}, \frac{1}{2\xi - 1}\right) \leq 1, \tag{10}$$

showing that the greater value of x is attained for $\xi = 1$, and therefore, $j = 2$, $M = 4$, and $x \leq 1/3$. Consequently, the range of x for the separability of Werner state is the same as established by the Peres-Horodecki recipe, which is $[0, 1/3]$. Moreover, $M = 4$ is the necessary number of states $\left|\psi_k^{(\ell)}\right\rangle$ in the expansions (7a) and (7b) in order to establish the range of x that makes ρ_{12} genuinely separable.

4 Separability algorithm applied to particular states

4.1 A particular bipartite 2 × 3 state

Next, we use our method to treat the one-parameter particular bipartite 2×3 state, being an extension of Werner state (1),

$$\begin{aligned}\rho_{12} = \frac{x}{2}\left|\chi_{12}\right\rangle\left\langle\chi_{12}\right| + \frac{1-x}{6}\left(\left|0_1, 0_2\right\rangle\left\langle 0_1, 0_2\right| + \left|0_1, 1_2\right\rangle\left\langle 0_1, 1_2\right| + \left|0_1, 2_2\right\rangle\left\langle 0_1, 2_2\right|\right.\\ \left. + \left|1_1, 0_2\right\rangle\left\langle 1_1, 0_2\right| + \left|1_1, 1_2\right\rangle\left\langle 1_1, 1_2\right| + \left|1_1, 2_2\right\rangle\left\langle 1_1, 2_2\right|\right),\end{aligned} \tag{11}$$

where $\left|\chi_{12}\right\rangle = \left(\left|1, 0\right\rangle + \left|0, 2\right\rangle\right)/\sqrt{2}$, the second particle has its Hilbert space extended to three states $\left|0\right\rangle$, $\left|1\right\rangle$, and $\left|2\right\rangle$. This choice permits to short-circuit extended calculations and to go directly to the heart of the algorithm as we observe that non-diagonal terms, in the computational basis, are present only in $\left|\chi_{12}\right\rangle\left\langle\chi_{12}\right|$. Therefore, we can jump to the step ii), as we already know that the interference terms of $\left|\chi_{12}\right\rangle\left\langle\chi_{12}\right|$ may be rewritten into the desired factorized form by means of four superposition states for particle 1:

$$\left|\Psi_k^{(1)}\right\rangle = \mathcal{N}^{(1)} \sum_{m=0}^{1} e^{i2\pi mk/3} a_m^{(1)} \left|m_1\right\rangle . \tag{12}$$

Regarding particle 2, it is straightforward to verify that although we need at least five superposition states to reach a diagonal form for the interference terms, we must use eight superposition states ($k = 0, \ldots, 7$)

$$\left|\Psi_k^{(2)}\right\rangle = \mathcal{N}^{(2)} \sum_{m=0}^{2} e^{i\pi km/4} a_m^{(2)} \left|m_2\right\rangle , \tag{13}$$

in order to maximize the value of the interpolator parameter x, which ensures separability. Using the sets of states $\left\{\left|\Psi_k^{(1)}\right\rangle\right\}$ and $\left\{\left|\Psi_k^{(2)}\right\rangle\right\}$, we obtain

$$\left|0_1\right\rangle\left\langle 1_1\right| = \frac{1}{4a_0^{(1)}\left(a_1^{(1)}\right)^*} \sum_{k=0}^{3} \left(\mathcal{N}^{(1)}\right)^{-2} e^{i2\pi k/4} \left|\Psi_k^{(1)}\right\rangle\left\langle\Psi_k^{(1)}\right|, \tag{14a}$$

$$\left|0_2\right\rangle\left\langle 2_2\right| = \frac{1}{8a_0^{(2)}\left(a_2^{(2)}\right)^*} \sum_{k=0}^{7} \left(\mathcal{N}^{(2)}\right)^{-2} e^{i4\pi k/8} \left|\Psi_k^{(2)}\right\rangle\left\langle\Psi_k^{(2)}\right|, \tag{14b}$$

under the constraint

$$a_0^{(1)}\left(a_0^{(2)}\right)^*\left(a_1^{(1)}\right)^* a_2^{(2)} = \frac{x}{2}. \tag{15}$$

Going to the third step, we use Eqs. (14) to rewrite the interference term

$$\rho_{12}^{(ND)} = \frac{x}{2}\left[\left|1_1\right\rangle\left\langle 0_1\right| \otimes \left|0_2\right\rangle\left\langle 2_2\right| + \left|0_1\right\rangle\left\langle 1_1\right| \otimes \left|2_2\right\rangle\left\langle 0_2\right|\right] \tag{16}$$

in a diagonal form as a sum $\Lambda + \Sigma$, where

$$\Lambda = \frac{1}{16}\sum_{\ell=0}^{1}\sum_{\ell'=0}^{2}\left|a_\ell^{(1)}a_{\ell'}^{(2)}\right|^2 \sum_{n=0}^{3} \sum_{m=0\left(\cos\left[\frac{\pi}{2}(n-m)\right]\geq 0\right)}^{7} \left|\cos\left[\frac{\pi}{2}(n-m)\right]\right| \times \left|\Psi_n^{(1)}\right\rangle\left\langle\Psi_n^{(1)}\right| \otimes \left|\Psi_m^{(2)}\right\rangle\left\langle\Psi_m^{(2)}\right|, \tag{17}$$

$$\Sigma = -\frac{1}{16}\sum_{\ell=0}^{1}\sum_{\ell'=0}^{2}\left|a_\ell^{(1)}a_{\ell'}^{(2)}\right|^2 \sum_{n=0}^{3} \sum_{m=0\left(\cos\left[\frac{\pi}{2}(n-m)\right]< 0\right)}^{7} \left|\cos\left[\frac{\pi}{2}(n-m)\right]\right| \left|\Psi_n^{(1)}\right\rangle\left\langle\Psi_n^{(1)}\right| \otimes \left|\Psi_m^{(2)}\right\rangle\left\langle\Psi_m^{(2)}\right|. \tag{18}$$

Again, following the same procedure of the previous section, we add and subtract Λ to Σ and obtain

$$\rho_{12} = \sum_{\ell_1=0}^{1}\sum_{\ell_2=0}^{2}\left[\frac{1-x}{6} + \frac{x}{2}\left(\delta_{\ell_1,1}\delta_{\ell_2,0} + \delta_{\ell_1,0}\delta_{\ell_2,2}\right) - \left|a_{\ell_1}^{(1)}a_{\ell_2}^{(2)}\right|^2\right] \\ \times \left|\ell_1\right\rangle_1 \left\langle \ell_1\right| \otimes \left|\ell_2\right\rangle_2 \left\langle \ell_2\right| + 2\Lambda. \tag{19}$$

In order to establish the range of values that x takes for separable states, $p_n \geq 0$, we have to satisfy six inequalities,

$$1 - x \geq \begin{cases} 6\left|a_0^{(1)}a_0^{(2)}\right|^2 \\ 6\left|a_0^{(1)}a_1^{(2)}\right|^2 \\ 6\left|a_1^{(1)}a_1^{(2)}\right|^2 \\ 6\left|a_1^{(1)}a_2^{(2)}\right|^2 \end{cases}, \qquad 1 + 2x \geq \begin{cases} 6\left|a_0^{(1)}a_2^{(2)}\right|^2 \\ 6\left|a_1^{(1)}a_0^{(2)}\right|^2 \end{cases} \tag{20}$$

plus the constraint (15). Finally, in the fourth step, we balance the coefficients $a_0^{(1)}$, $a_1^{(1)}, a_0^{(2)}, a_1^{(2)}, a_2^{(2)}$ to maximize the value of the interpolator parameter x, by imposing the equalities $\left|a_0^{(1)}a_0^{(2)}\right|^2 = \left|a_1^{(1)}a_2^{(2)}\right|^2$ and $\left|a_0^{(1)}a_1^{(2)}\right|^2 = \left|a_1^{(1)}a_1^{(2)}\right|^2$ which, together with condition $\left|a_0^{(1)}a_0^{(2)}a_1^{(1)}a_2^{(2)}\right|^2 = x^2/4$, leads to $\left|a_0^{(1)}\right|^2 = \left|a_0^{(2)}\right|^2 = \left|a_1^{(1)}\right|^2 = \left|a_2^{(2)}\right|^2 = \sqrt{x/2}$ and $a_1^{(2)} = 0$. Under these conditions, the inequalities in Eq. (9) reduce to two inequalities, $(1-4x)/6 \geq 0$, $(1-x)/6 \geq 0$, from which we establish that the separability of state (11) occurs whenever $x \in [0, 1/4]$.

4.2 A particular tripartite 2 × 2 × 2 state

Let us now consider the particular tripartite state

$$\rho_{123} = \left|\Psi\right\rangle \mathbb{M} \left\langle\Psi\right|, \tag{21}$$

where we have defined the matrices

$$\left|\Psi\right\rangle = \left(\left|0,0,0\right\rangle\ \left|0,0,1\right\rangle\ \left|0,1,0\right\rangle\ \left|1,0,0\right\rangle\ \left|0,1,1\right\rangle\ \left|1,0,1\right\rangle\ \left|1,1,0\right\rangle\ \left|1,1,1\right\rangle\right), \tag{22}$$

$$\mathbb{M} = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 & 0 & 0 & \rho_{17} & \rho_{18} \\ \rho_{12}^* & \rho_{22} & 0 & 0 & 0 & 0 & \rho_{27} & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{34}^* & \rho_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_{66} & 0 & 0 \\ \rho_{17}^* & \rho_{27}^* & 0 & 0 & 0 & 0 & \rho_{77} & 0 \\ \rho_{18}^* & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{88} \end{pmatrix}. \tag{23}$$

Following the steps of the proposed algorithm and under the condition $\rho_{88} = \chi^2\rho_{66}\rho_{55}/\xi^2\sigma$, we are able to derive analytically the inequality

$$\max\{|\rho_{18}|, |\rho_{27}|\} \leq \min\left\{\sqrt{\chi\rho_{88}}, \sqrt{\chi\sigma}, \sqrt{\xi\rho_{66}}, \sqrt{\xi\rho_{55}}\right\}, \tag{24}$$

for the separability condition, where

$$\lambda = \max\{|\rho_{17}|, |\rho_{34}|\} \tag{25a}$$

$$\mu = \frac{\rho_{11} + \rho_2}{2} - \sqrt{\left(\frac{\rho_{11} - \rho_2}{2}\right)^2 + |\rho_{12}|^2} \tag{25b}$$

$$\sigma = \frac{\rho_{22} + \rho_{11}}{2} - \sqrt{\left(\frac{\rho_{22} - \rho_{11}}{2}\right)^2 + |\rho_{12}|^2}, \tag{25c}$$

$$\chi = \frac{\sigma + \rho_{77}}{2} - \lambda\sqrt{\left(\frac{\sigma - \rho_{77}}{2\lambda}\right)^2 + \frac{\mu}{\rho_{77}}}, \tag{25d}$$

$$\xi = \frac{\rho_{33} + \rho_{44}}{2} - \lambda\sqrt{\left(\frac{\rho_{33} - \rho_{44}}{2\lambda}\right)^2 + \frac{\rho_{33}}{\rho_{44}}}. \tag{25e}$$

Any violation of the inequality (24) implies that (21) is an entangled state. The inequality (24) demands an extremization procedure which may become an increasingly laborius task for states of higher dimensionality. We stress that we were able to deduce the optimal inequality (24) because of the condition $\rho_{88} = \chi^2\rho_{66}\rho_{55}/\xi^2\sigma$; otherwise, we would have to search for a numerical solution to the problem. The increase in the number of parties composing the state and/or their dimensions implies the increase in the difficulty for balancing the coefficients of the expansion of states $\left|\psi_k^{(\ell)}\right\rangle$.

4.3 A particular multipartite $\bigotimes^N 2$ state

We analyze the case of the single-parameter multipartite $\bigotimes^N 2$ state

$$\rho_{1\ldots N} = x\left|\Psi_{1\ldots N}\right\rangle\left\langle\Psi_{1\ldots N}\right| + \frac{1-x}{2^N}\left(\sum_{m_1=0}^{1}\cdots\sum_{m_N=0}^{1}\left|m_1,\ldots,m_N\right\rangle\left\langle m_1,\ldots,m_N\right|\right), \tag{26}$$

where $|\Psi_{1...N}\rangle = (|0_1,\ldots,0_N\rangle + |1_1,\ldots,1_N\rangle)/\sqrt{2}$, here again, the non-diagonal (in the computational basis) components are in the state $|\Psi_{1...N}\rangle$. Following the steps outlined in Sects. 2 and 3, we go directly to the third step of the algorithm, displaying the factorized form of state (26),

$$\rho_{12} = \sum_{m_1,m_2,\ldots,m_N=0}^{1}\left[\frac{1-x}{2^N}+\frac{x}{2}\left(\prod_{\ell=0}^{N}\delta_{m_\ell,0}+\prod_{\ell=0}^{N}\delta_{m_\ell,1}\right)-\left|\prod_{\ell=0}^{N}a_{m_\ell}^{(\ell)}\right|^2\right]\bigotimes_{\ell=1}^{N}|m_\ell\rangle\langle m_\ell|$$
$$+4\left(\prod_{n=1}^{N}\frac{\left(\mathcal{N}^{(n)}\right)^{-2}}{4}\right)\bigotimes_{\ell=1}^{N}\sum\nolimits_{m_\ell=0}^{\prime\,3}\left|\Psi_{m_\ell}^{(\ell)}\right\rangle\left\langle\Psi_{m_\ell}^{(\ell)}\right|, \tag{27}$$

where $j\in\mathbb{N}$, such that $4j\leq 3N$, and the prime on the sum symbol stands for the constraint $\sum_{\ell=1}^{N}m_\ell = 4j$. The determination of the range of values of x for the state $\rho_{1...N}$ be separable demands that the coefficients of the expansion be nonnegative. So 2^N inequalities are necessary,

$$\frac{1-x}{2^N}\geq\left\{\left|\left(\prod_{m=1}^{M}a_1^{(\ell_m)}\right)\prod_{\ell'=1}^{N}a_0^{(\ell')}\right|^2,\qquad \frac{1-x}{2^N}+\frac{x}{2}\geq\left\{\left|\prod_{\ell=1}^{N}a_0^{(\ell)}\right|^2,\left|\prod_{\ell=1}^{N}a_1^{(\ell)}\right|^2\right.\right. \tag{28}$$

for $\ell'\neq\ell_1\neq\ell_2\cdots\neq\ell_{N-1}=1,\ldots,N$ and $M=1,\ldots,N-1$. By balancing the expansion coefficients, we obtain the relation

$$\left|a_0^{(\ell')}\right|^2=\left|a_1^{(\ell')}\right|^2=\left(\frac{x}{2}\right)^{1/N}, \tag{29}$$

enabling us to reduce all the above inequalities in Eq. (28) to two simple ones, $(1-x)/2^N\geq x/2$ and $(1-x)/2^N\geq 0$, leading to the range $x\in\left[0,\left(1+2^{N-1}\right)^{-1}\right]$ for the separability of the multipartite state. For $N=2$, we recover the range of x values for the separability of the two-qubit Werner state and with the increase in the number of qubits N that interval diminish rapidly. If we assume the dynamical evolution map $x=\exp(-\gamma t)$, γ being a coherence decay rate, we note that as $x\in\left(\left(1+2^{N-1}\right)^{-1},1\right]$, then for a greater number N, the multipartite state will remain entangled for most part of the time of its evolution, thus making the state resisting its separability.

5 An arbitrary bipartite 2 × 2 state

Here, we apply the algorithm presented in Sect. 2 to an arbitrary bipartite 2×2 state

$$\rho_{12}=\sum_{m,n=1}^{4}p_{mn}\,|\phi_m\rangle\langle\phi_n|, \tag{30}$$

where $|\phi_1\rangle = |0_1 0_2\rangle$, $|\phi_2\rangle = |0_1 1_2\rangle$, $|\phi_3\rangle = |1_1 0_2\rangle$, $|\phi_4\rangle = |1_1 1_2\rangle$, and p_{mn} are the coefficients of the expansion on which separability condition will impose constraints. Following the steps established by the algorithm, the separability condition demands the inequality

$$\max\{|p_{14}|, |p_{23}|\} \leq \min\left\{\sqrt{a\tilde{b}}, \sqrt{\tilde{a}b}\right\}, \tag{31}$$

where we have defined

$$a = p_{11} + \sigma_{21}^{-}\left(1 - \delta_{p_{12},0}\right) + \left(\chi_1 - \sqrt{\chi_1^2 + |p_{13}|^2}\right)\left(1 - \delta_{p_{13},0}\right), \tag{32a}$$

$$b = p_{22} - \sigma_{21}^{-}\left(1 - \delta_{p_{12},0}\right) + \left(\chi_2 - \sqrt{\chi_2^2 + |p_{24}|^2}\right)\left(1 - \delta_{p_{24},0}\right), \tag{32b}$$

$$\tilde{a} = p_{33} + \sigma_{43}^{-}\left(1 - \delta_{p_{34},0}\right) - \left(\chi_1 + \sqrt{\chi_1^2 + |p_{13}|^2}\right)\left(1 - \delta_{p_{13},0}\right), \tag{32c}$$

$$\tilde{b} = p_{44} - \sigma_{43}^{+}\left(1 - \delta_{p_{34},0}\right) - \left(\chi_2 + \sqrt{\chi_2^2 + |p_{24}|^2}\right)\left(1 - \delta_{p_{24},0}\right), \tag{32d}$$

with

$$\sigma_1^{\pm} = \frac{p_{22} - p_{11}}{2} \pm \sqrt{\left(\frac{p_{22} - p_{11}}{2}\right)^2 + |p_{12}|^2}, \tag{33a}$$

$$\sigma_2^{\pm} = \frac{p_{44} - p_{33}}{2} \pm \sqrt{\left(\frac{p_{44} - p_{33}}{2}\right)^2 + |p_{34}|^2}, \tag{33b}$$

$$\chi_1 = \frac{p_{33} - p_{11} + \sigma_2^{-}\left(1 - \delta_{p_{34},0}\right) - \sigma_1^{-}\left(1 - \delta_{p_{12},0}\right)}{2}, \tag{33c}$$

$$\chi_2 = \frac{p_{44} - p_{22} - \sigma_2^{+}\left(1 - \delta_{p_{34},0}\right) + \sigma_1^{+}\left(1 - \delta_{p_{12},0}\right)}{2}. \tag{33d}$$

We note that by assigning numerical values to the coefficients p_{mn}, the inequality (31) can be numerically confirmed by the partial transposition criterion of Peres and Horodecki. For the particular case where $p_{12} = p_{13} = p_{24} = p_{34} = 0$, that characterizes an X-type density matrix, the separability conditions (31) simplify with $\sqrt{a\tilde{b}} = p_{11}p_{44}$ and $\sqrt{\tilde{a}b} = p_{22}p_{33}$, confirmed by the partial transposition criterion.

6 Any arbitrary state ρ can be written in a pseudoseparable form

An arbitrary multipartite state made of r subsystems, each having dimension D_i, $i = 1, \ldots, r$, can be written as

$$\rho_{1\ldots r} = \sum_{m_1,n_1=1}^{D_1} \cdots \sum_{m_r,n_r=1}^{D_r} C_{m_1,\ldots,m_r;n_1,\ldots,n_r} \left|m_1, \ldots, m_r\right\rangle \left\langle n_1, \ldots, n_r\right|. \tag{34}$$

Each operator $|m_j\rangle\langle n_j|$ of subsystem j can be written as a weighted sum of elementary projectors

$$\left|m_j\right\rangle\left\langle n_j\right| \equiv \left|m_j\right\rangle\left\langle m_j+\phi_j\right| = \left(N_{mj}\right)^2 \sum_{\ell=1}^{D_j} e^{i2\pi(\ell+1)\phi_j/D_j} \left|\Psi_j^{(\ell)}\right\rangle\left\langle\Psi_j^{(\ell)}\right|, \quad (35)$$

where we define the arbitrary vector as

$$\left|\Psi_j^{(\ell)}\right\rangle = \left(N_{m_j}\right)^{-1} e^{i2\pi(\ell+1)m_j/D_j} \left(a_j^{(m_j)}\left|m_j\right\rangle + e^{i2\pi(\ell+1)\phi_j/D_j} a_j^{(m_j+\phi_j)}\left|m_j+\phi_j\right\rangle\right), \quad (36)$$

with

$$\left(N_{m_j}\right)^2 = \left|a_j^{(m_j)}\right|^2 + \left|a_j^{(m_j+\phi_j)}\right|^2. \quad (37)$$

Substituting Eq. (35) in Eq. (34) for each subsystem, the arbitrary state ρ can now be written in the following pseudoseparable form

$$\rho = \sum_{\ell_1=1}^{D_1} \cdots \sum_{\ell_r=1}^{D_r} \tilde{C}_{\ell_1,\ldots,\ell_r} \prod_{j=1}^{r} \left|\Psi_j^{(\ell_j)}\right\rangle\left\langle\Psi_j^{(\ell_j)}\right|, \quad (38)$$

where

$$\tilde{C}_{\ell_1,\ldots,\ell_r} = \sum_{m_1,n_1=1}^{D_1} \cdots \sum_{m_r,n_r=1}^{D_r} C_{m_1,\ldots,m_r;n_1,\ldots,n_r} \prod_{j=1}^{r} \left(N_{m_j}\right)^2 e^{i2\pi(\ell_j+1)\phi_j/D_j}. \quad (39)$$

We therefore conclude that for finite-dimensional systems, there is a finite-time algorithm that allows one to write every state of N qudits in the pseudoseparable form. Essentially, we cannot guarantee that an arbitrary state is not entangled if one or more of its coefficients $\tilde{C}_{\ell_1,\ldots,\ell_r}$ are negative. Only after the implementation of the fourth step of the algorithm, that establishes the values of the parameters $a_j^{(m_j)}$, one can guarantee that if at least one of the coefficients $\tilde{C}_{\ell_1,\ldots,\ell_r}$ is negative, then ρ is an entangled state.

Regarding the scalability of the algorithm, for an arbitrary multipartite state of r subsystems of $D_r = 2$, the number of free parameters $a_j^{(m_j)}$ increases with r as $2^{r-1}(3^r-1)$ and so the difficulty to determine them. It would be worthwhile to analyze how costly is our algorithm in comparison with the complex methods mentioned in the Introduction where a formal separability criterion is derived (for bipartite systems of limited dimensions). In fact, the algorithms for the complete criteria detection have worst-case complexity, as discussed in [34].

7 A proposal for quantifying the degree of entanglement through the negative probabilities

Our algorithm for separability also enables us to propose a measure for the degree of entanglement. As far as we have derived the inequality following from our fourth step, given by the general expression

$$\max\{A_i\} \leq \min\{B_i\}, \tag{40}$$

(where the sets $\{A_i\}$ and $\{B_i\}$ stand for functions of the coefficients of the density matrix), we may compute the degree of entanglement through the relation

$$\mathcal{E} = \max\{\max\{A_i\} - \min\{B_i\}, 0\}, \tag{41}$$

which is clearly a monotonic function of the interpolator parameter $\max\{A_i\}$. As an example, we consider the particular multipartite $\bigotimes^N 2$ state (26) where the measure (41), apart from a normalization factor, becomes

$$\mathcal{E} = \max\left\{x - \left(1 + 2^{N-1}\right)^{-1}, 0\right\}. \tag{42}$$

For the case of a bipartite Werner state, this measure gives $E = \max\{x - 1/3, 0\}$, and as already known, the higher the value of x, the more entangled is the state.

8 Summary and conclusions

We have presented an algorithm for writing any arbitrary density matrix in the so-called *separable form* $\rho_{1\cdots N} = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)}$ where the coefficients p_i are nonnegative $\left(\text{and } \sum_i p_i = 1\right)$ only for separable states, whereas for entangled states, some p_i assume negative values, while others are greater than 1, i.e., we cannot recognize these weights as probabilities. We proposed and developed a four-step algorithm applying it to the emblematic case of the two-qubit Werner state, establishing the interval of values the parameter x takes for truly separable states. The *modus operandi* of our method, which we demonstrated to be applicable to any density matrix, generalizes the Peres-Horodecki separability recipe, proved to be valid for bipartite states of dimensions 2×2 and 2×3.

After presenting our method, we then prove its usefulness by applying it for a variety of states: a bipartite state of higher dimensions (2×3), a particular tripartite $2 \times 2 \times 2$ state, a particular multipartite $\bigotimes^N 2$ state, and an arbitrary bipartite 2×2 state. We verified that for a Werner-like multipartite state evolving in time, the greater the number of qubits N, the more time it will remain entangled during its evolution, thus making the state more resistant to separability.

From the derivation of the negative probabilities that allowed us to characterize entanglement, we also proposed a measure to quantify the degree of entanglement of arbitrary multipartite states. We finally stress that our algorithm, that demands an extremization procedure, may become an increasingly laborious task for states of higher dimensionality, characterizing the complexity associated with the separability problem in Hilbert space.

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