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Comparison of estimation methods for the Marshall–Olkin extended Lindley distribution

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The aim of this paper is to compare the parameters' estimations of the Marshall–Olkin extended Lindley distribution obtained by six estimation methods: maximum likelihood, ordinary least-squares, weighted least-squares, maximum product of spacings, Cramér–von Mises and Anderson–Darling. The bias, root mean-squared error, average absolute difference between the true and estimate distributions' functions and the maximum absolute difference between the true and estimate distributions' functions are used as comparison criteria. Although the maximum product of spacings method is not widely used, the simulation study concludes that it is highly competitive with the maximum likelihood method.

Keywords: estimation methods; Lindley distribution; Marshall–Olkin family; Monte Carlosimulations

AMS Subject Classification: F1.1; F4.3

1. Introduction

By various methods, new parameters can be introduced to expand families of distribution.[1] Marshall and Olkin [2] introduced a general method to obtain more flexible distributions by adding a new parameter to an existing one, called the Marshall–Olkin family of distributions. Starting with a baseline survival function $S_1(y \mid \theta)$, for a continuous random variable Y, the Marshall–Olkin family has survival function given by

$$S(y \mid \theta, \alpha) = \frac{\alpha S_1(y \mid \theta)}{1 - \bar{\alpha} S_1(y \mid \theta)}, \tag{1}$$

where $-\infty < y < +\infty$, $\theta = (\theta_1, \ldots, \theta_p)$ and $\bar{\alpha} = 1 - \alpha$. $\alpha > 0$ is called *tilt* parameter. Clearly, when $\alpha = 1$ we get the baseline survival function $S_1(y \mid \theta)$.

In general, the addition of a *tilt* parameter makes the resulting distribution richer and more flexible for modelling data. The study of the *tilt* parameter effect and the hazard rate function monotonicity was conducted in [2–4]. In [5], the *tilt* parameter was taken as a random variable.

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In what follows from Equation (1), the corresponding probability density function and the hazard rate function are written, respectively, as

$$f(y \mid \theta, \alpha) = \frac{\alpha f_1(y \mid \theta)}{[1 - \bar{\alpha} S_1(y \mid \theta)]^2}, \tag{2}$$

and

$$h(y \mid \theta, \alpha) = \frac{h_1(y \mid \theta)}{1 - \bar{\alpha} S_1(y \mid \theta)}, \tag{3}$$

where $f_1(y \mid \theta)$ and $h_1(y \mid \theta)$ are, respectively, the baseline probability density function and the baseline hazard rate function.

An interesting property of Marshall–Olkin family is the geometric-extreme stable property as follows. If Y_i, $i = 1, 2, \ldots$, is a sequence of independent and identically distributed random variables with survival function $S_1(y \mid \theta)$ and if N has a geometric distribution with probability mass function $P(N = n) = \alpha(1 - \alpha)^{n-1}$ taken values $\{1, 2, \ldots\}$, independent of Y_i, then the random variables $U = \min\{Y_1, \ldots, Y_N\}$ and $V = \max\{Y_1, \ldots, Y_N\}$ have survival function (1) with $0 < \alpha = p \leq 1$ and $\alpha = 1/p \geq 1$, respectively. Marshall and Olkin [2] have also noted that the method has a stability property, that is, if the method is applied twice, nothing new is obtained in the second time around.

Several papers have appeared in the last few years dealing with Marshall–Olkin extended family. A literature review showed that more than 20 distributions were used as baseline distributions.

Beta: [6];

Birnbaum–Saunders: [7];

Burr: [8,9];

Exponential: [2,4,8,10–23];

Exponentiated exponential: [24,25];

Exponentiated log-normal: [26];

Exponentiated Weibull: [27];

Extreme-value: [8];

Fréchet: [8,11,28,29];

Gamma: [30];

Kumaraswamy: [11];

Lindley: [31];

Linear failure-rate: [32];

Logistic: [8];

Log-logistic: [33];

Lomax: [34–36];

Log-normal: [37].

Makeham: [38];

Normal: [39,40].

Pareto: [4,8,11,41,42];

Power series: [11];

q-Weibull: [43];

Semi-Burr: [9];

Semi-Weibull: [44];

Student-*t*: [45].

Uniform: [46];

Weibull: [2–4,8,11,15,20,47–52].

By considering the survival function of one-parameter Lindley distribution, we have the Marshall–Olkin extended Lindley distribution (MOEL), named as Lindley-Geometric distribution by Zakerzadeh and Mahmoudi [53] with survival function:

$$S(y \mid \theta, \alpha) = \frac{\alpha(1+\theta y/(1+\theta))\,\mathrm{e}^{-\theta y}}{1-\bar{\alpha}(1+\theta y/(1+\theta))\,\mathrm{e}^{-\theta y}}, \tag{4}$$

where $0 < y < \infty$, $\theta > 0$, $\alpha > 0$ and $\bar{\alpha} = 1 - \alpha$. When $\alpha = 1$ we get the survival function of one-parameter Lindley distribution. The one-parameter Lindley distribution was introduced by Lindley [54] (see also [55]) as a distribution that can be useful to analyse lifetime data, especially in modelling stress-strength reliability applications. Ghitany et al. [56] studied the properties of the one-parameter Lindley distribution under a careful mathematical treatment. They also showed, in a numerical example, that the Lindley distribution gives better modelling than obtained using the exponential distribution. A two-parameter weighted Lindley distribution was proposed by Ghitany et al.[57] A generalized Lindley distribution, which includes as special cases the Lindley, exponential and gamma distributions, was proposed by Zakerzadeh and Dolati.[58] The one-parameter Lindley distribution in the competing risks scenario was considered in [59].

The probability density and hazard rate functions of MOEL distribution are, respectively:

$$f(y \mid \theta, \alpha) = \frac{\alpha\theta^2(1+y)\,\mathrm{e}^{-\theta y}}{(1+\theta)[1-\bar{\alpha}(1+\theta y/(1+\theta))\,\mathrm{e}^{-\theta y}]^2}, \tag{5}$$

$$h(y \mid \theta, \alpha) = \frac{\theta^2(1+y)}{(1+\theta+\theta y)[1-\bar{\alpha}(1+\theta y/(1+\theta))\,\mathrm{e}^{-\theta y}]}. \tag{6}$$

Note that $f(0|\theta,\alpha) = h(0|\theta,\alpha) = \theta^2/\alpha(1+\theta)$, $f(\infty|\theta,\alpha) = 0$ and $h(\infty|\theta,\alpha) = \theta$. The probability density function (5) is decreasing if $\alpha \le 2\theta^2/(\theta^2+1)$ and unimodal if $\alpha > 2\theta^2/(\theta^2+1)$. Figure 1 illustrates the probability density function of MOEL selected values of α and $\theta = 1$. It is clear that for values of α close to 1, the curve resembles the one-parameter Lindley distribution, while when $\alpha \longrightarrow \infty$ the curve tends to be symmetric.

In Figure 2, we have the hazard rate function graph of MOEL distribution for some values of α and $\theta = 1$. From Ghitany et al.,[31] $h(y \mid \theta, \alpha)$ has increasing, decreasing–increasing and increasing–decreasing–increasing behaviour, while the hazard rate function of the one-parameter Lindley distribution has only increasing behaviour.

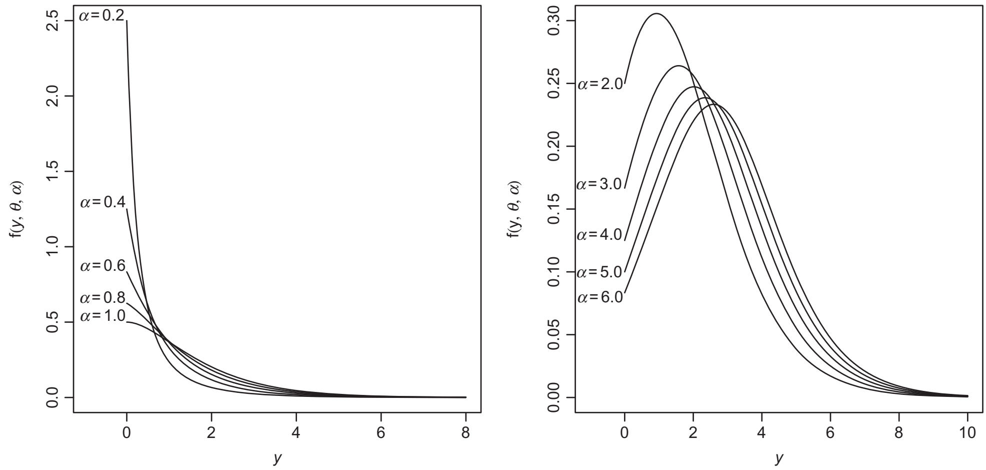

Figure 1. Density of MOEL distribution for selected values of α and $\theta = 1$.

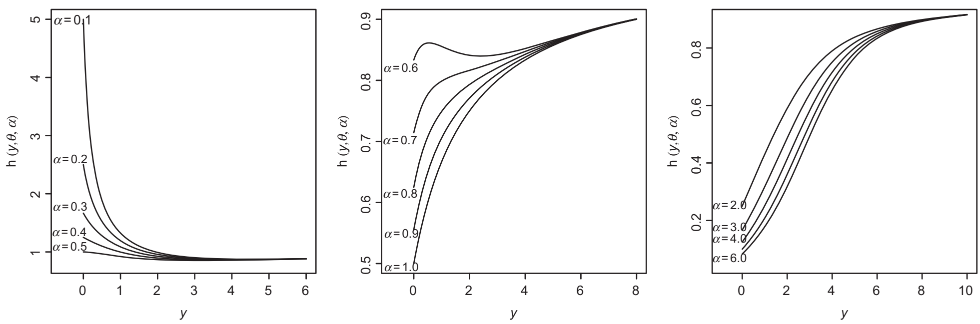

Figure 2. Hazard rate function of MOEL distribution for selected values of α and $\theta = 1$.

The quantile function of the MOEL distribution is given by

$$F^{-1}(u) = -1 - \frac{1}{\theta} - \frac{1}{\theta} W_{-1}\left(\frac{(\theta+1)}{e^{\theta+1}} \frac{(u-1)}{(1-\bar{\alpha}u)} \right),$$

where $0 < u < 1$ and $W_{-1}(\cdot)$ denotes the negative branch of the Lambert W function (i.e. the equation $W(z)\,e^{W(z)} = z$ solution) because $(1+\theta+\theta y) > 1$ and $(u-1)(\theta+1)\,e^{-\theta-1}/(1-\bar{\alpha}u) \in (-1/e, 0)$.[31,60]

Using the series expansion

$$(1-z)^{-w} = \sum_{j=0}^{\infty} \frac{\Gamma(w+j)}{\Gamma(w) j!} z^j, \tag{7}$$

where $|z| < 1$ and $w > 0$, the probability density function (5) can be written as

$$f(y \mid \theta, \alpha) = \frac{\theta^2}{(\theta+1)} \alpha (1+y)\, e^{-\theta y} \times \sum_{j=0}^{\infty} (j+1)(1-\alpha)^j \left(1 + \frac{\theta y}{\theta+1}\right)^j e^{-\theta y j}. \tag{8}$$

By considering Equation (8) and applying the binomial expression for $(1 + \theta y/(\theta + 1))^j$, the rth moment of Y is given by

$$E(Y^r) = \frac{\theta^2\alpha}{(\theta+1)} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \binom{j}{i}(j+1)(1-\alpha)^j \left(\frac{\theta}{\theta+1}\right)^i \times \frac{\Gamma(r+i+1)}{[\theta(j+1)]^{r+i+1}}\left(1+\frac{r+i+1}{\theta(j+1)}\right). \tag{9}$$

The moment generating function of the MOEL distribution is given by

$$M_Y(t) = \frac{\theta^2\alpha}{(\theta+1)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{t^k}{k!}\binom{j}{i}(j+1)(1-\alpha)^j \left(\frac{\theta}{\theta+1}\right)^i \times \frac{\Gamma(k+i+1)}{[\theta(j+1)]^{k+i+1}}\left(1+\frac{k+i+1}{\theta(j+1)}\right). \tag{10}$$

PROPOSITION 1.1 *The mean of the MOEL distribution is given by*

$$E(Y) = \frac{\theta^2\alpha}{(\theta+1)} \sum_{j=0}^{\infty}\sum_{i=0}^{j} \frac{(j+1)!}{(j-i)!}(1-\alpha)^j \left(\frac{\theta}{\theta+1}\right)^i \times \frac{(i+1)}{[\theta(j+1)]^{i+2}}\left(1+\frac{i+2}{\theta(j+1)}\right). \tag{11}$$

More statistical properties of the MOEL distribution are discussed in [31].

For any probability distribution, parameters estimation is always of fundamental importance although in general only the maximum likelihood estimation (MLE) method is considered. It is of interest to compare the MLE method with other estimation methods. In this paper, we consider five additional methods to estimate the parameters of MOEL distribution. These additional methods are the ordinary least-squares (OLS), weighted least-squares (WLS), maximum product of spacings (MPS), Cramér–von Mises (CM) and Anderson–Darling (AD). The main aim of this paper is to identify, for the MOEL distribution, the most efficient estimation method for different shape parameters' values and sample sizes.

In Section 2, we discuss the six estimation methods considered in this paper. The comparison of these methods in terms of bias, root mean-squared error, average absolute difference between the true and estimate distributions' functions and the maximum absolute difference between the true and estimate distributions' functions are presented in Section 3. Some concluding remarks in Section 4 finalize this paper.

2. Estimation methods

In this section, by considering the Marshall–Olkin model formulation, we describe six methods used to get the estimates for α and θ. For all methods, we consider the case where both α and θ are unknown. This is also considered in the simulation study presented in Section 3.

2.1. *Maximum likelihood*

Let $\mathbf{y} = (y_1, \ldots, y_n)$ be a random sample of size n from the Marshall–Olkin extended distribution with parameters α and $\theta = (\theta_1, \ldots, \theta_p)$. From Equation (2), the likelihood and log-likelihood functions are written, respectively, as follows:

$$L(\theta,\alpha \mid \mathbf{y}) = \prod_{i=1}^{n} f(y_i|\theta,\alpha) = \alpha^n \sum_{i=1}^{n} \frac{f_1(y_i|\theta)}{[1-\bar{\alpha}S_1(y_i|\theta)]^2}, \tag{12}$$

$$l(\theta,\alpha \mid \mathbf{y}) = n\log(\alpha) + \sum_{i=1}^{n} \log[f_1(y_i \mid \theta)] - 2\sum_{i=1}^{n} \log[1-\bar{\alpha}S_1(y_i|\theta)]. \tag{13}$$

The maximum likelihood estimates of θ and α, $\hat{\theta}_{\mathrm{MLE}}$ and $\hat{\alpha}_{\mathrm{MLE}}$, respectively, can be obtained numerically by maximizing the log-likelihood function (12). In this case, the log-likelihood function is maximized by solving numerically $(\partial/\partial\theta_j)l(\theta,\alpha \mid \mathbf{y}) = 0$ and $(\partial/\partial\alpha)l(\theta,\alpha \mid \mathbf{y}) = 0$ in θ and α, respectively, where

$$\frac{\partial}{\partial\theta_j} l(\theta,\alpha \mid \mathbf{y}) = \sum_{i=1}^{n} \frac{f'_{1j}(y_i \mid \theta)}{f_1(y_i \mid \theta)} - 2\bar{\alpha}\sum_{i=1}^{n} \frac{f_1(y_i \mid \theta)}{1-\bar{\alpha}S_1(y_i|\theta)}, \tag{14}$$

$$\frac{\partial}{\partial\alpha} l(\theta,\alpha \mid \mathbf{y}) = \frac{n}{\alpha} - 2\sum_{i=1}^{n} \frac{S_1(y_i \mid \theta)}{1-\bar{\alpha}S_1(y_i|\theta)}, \tag{15}$$

where $f'_{1j}(y_i \mid \theta) = (\partial/\partial\theta_j)f_1(y_i \mid \theta)$, $j = 1,\ldots,p$.

2.2. *Ordinary least-squares*

Let $y_{1:n} < y_{2:n} \cdots < y_{n:n}$ be the order statistics of a size n random sample from a distribution with cumulative distribution function $F(y)$. It is well known that

$$E[F(y_{i:n})] = \frac{i}{n+1} \quad \text{and} \quad \mathrm{Var}[F(y_{i:n})] = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{16}$$

For the Marshall–Olkin extended distribution, the least-square estimates $\hat{\theta}_{\mathrm{OLS}}$ and $\hat{\alpha}_{\mathrm{OLS}}$ of the parameters θ and α, respectively, are obtained by minimizing the function:

$$S(\theta,\alpha \mid \mathbf{y}) = \sum_{i=1}^{n} \left(F(y_{i:n} \mid \theta,\alpha) - \frac{i}{n+1}\right)^2. \tag{17}$$

These estimates can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n} \left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{i}{n+1}\right) \Delta_{1j}(y_{i:n}|\theta,\alpha) = 0, \tag{18}$$

$$\sum_{i=1}^{n} \left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{i}{n+1}\right) \Delta_2(y_{i:n}|\theta,\alpha) = 0, \tag{19}$$

where

$$\Delta_{1j}(y_{i:n}|\theta,\alpha) = \frac{(1-\bar{\alpha}S_1(y_{i:n}|\theta))F'_{1j} + \bar{\alpha}F_1(y_{i:n}|\theta)S'_{1j}}{[1-\bar{\alpha}S_1(y_{i:n}|\theta)]^2}, \tag{20}$$

$$\Delta_2(y_{i:n}|\theta,\alpha) = -\frac{S_1(y_{i:n}|\theta)F_1(y_{i:n}|\theta)}{[1-\bar{\alpha}S_1(y_{i:n}|\theta)]^2}, \tag{21}$$

$F'_{1j} = (\partial/\partial\theta_j)F_1(y_{i:n}|\theta)$ and $S'_{1j} = (\partial/\partial\theta_j)S_1(y_{i:n}|\theta)$, $j = 1,\ldots,p$.

2.3. *Weighted least-squares*

The weighted least-squares' estimates $\hat{\theta}_{\mathrm{WLS}}$ and $\hat{\alpha}_{\mathrm{WLS}}$ of the parameters θ and α, respectively, are obtained by minimizing the function:

$$W(\theta,\alpha \mid \mathbf{y}) = \sum_{i=1}^{n} w_i \left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{i}{n+1} \right)^2, \tag{22}$$

where the correction factor w_i is given by

$$w_i = \frac{1}{V[F(y_{(i:n)})]} = \frac{(n+1)^2(n+2)}{i(n-i+1)}. \tag{23}$$

These estimates can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n} \frac{1}{i(n-i+1)} \left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{i}{n+1} \right) \Delta_{1j}(y_{i:n}|\theta,\alpha) = 0, \tag{24}$$

$$\sum_{i=1}^{n} \frac{1}{i(n-i+1)} \left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{i}{n+1} \right) \Delta_{2}(y_{i:n}|\theta,\alpha) = 0, \tag{25}$$

where $\Delta_{1j}(y_{i:n}|\theta,\alpha)$ and $\Delta_2(y_{i:n}|\theta,\alpha)$ are given by Equations (20) and (21), respectively.

2.4. *Maximum product of spacings*

Cheng and Amin [61,62] introduced the maximum product of spacings (MPS) method as alternative to MLE in parameters estimation of continuous univariate distributions. Ranneby,[63] independently, developed the same method as an approximation to the Kullback–Leibler measure of information. In what follows, let $y_{1:n} < y_{2:n} < \cdots < y_{n:n}$ be an ordered random sample drawn from the Marshall–Olkin extended distribution. It defines the uniform spacings of the sample as the quantities: $D_1 = F(y_{1:n} \mid \theta,\alpha)$, $D_{n+1} = 1 - F(t_{n:n} \mid \theta,\alpha)$ and $D_i = F(t_{i:n} \mid \theta,\alpha) - F(t_{(i-1):n} \mid \theta,\alpha)$, $i = 2,\ldots,n$. Note that there are $(n+1)$ spacings of the first order.

Following,[62] the maximum product of spacings method consists in finding the values of θ and α which maximize the geometric mean of the spacings, the MPS statistics, given by

$$G(\theta,\alpha \mid \mathbf{y}) = \left(\prod_{i=1}^{n+1} D_i \right)^{1/(n+1)}, \tag{26}$$

or, equivalently, its logarithm $H = \log(G)$. By considering $0 = F(y_{0:n} \mid \theta,\alpha) < F(y_{1:n} \mid \theta,\alpha) < \cdots < F(y_{n:n} \mid \theta,\alpha) < F(y_{(n+1):n} \mid \theta,\alpha) = 1$, the quantity $H = \log(G)$ can be calculated as

$$H(\theta,\alpha \mid \mathbf{y}) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i). \tag{27}$$

The estimates for θ and α, $\hat{\theta}_{\mathrm{MPS}}$ and $\hat{\alpha}_{\mathrm{MPS}}$, can be found solving, respectively, in θ and α the nonlinear equations:

$$\frac{\partial}{\partial\theta_j} H(\theta,\alpha) = \sum_{i=1}^{n+1} \frac{1}{D_i} \Delta \left[\frac{\partial}{\partial\theta_j} F(y_{i:n}|\theta,\alpha) \right] = 0, \tag{28}$$

$$\frac{\partial}{\partial\alpha} H(\theta,\alpha) = \sum_{i=1}^{n+1} \frac{1}{D_i} \Delta \left[\frac{\partial}{\partial\alpha} F(y_{i:n}|\theta,\alpha) \right] = 0, \tag{29}$$

where Δ is the first-order difference operator.

Cheng and Amin [62] showed that maximizing H as a method of parameter estimation is as efficient as MLE estimation and the MPS estimators are consistent under more general conditions than the MLE estimators.

2.5. *Minimum distance methods*

In this subsection, we present two estimation methods for θ and α estimation based on the minimization of two well-known goodness-of-fit statistics. This class of statistics is based on the difference between the estimates of the cumulative distribution function and the empirical distribution functiono.[64,65]

2.5.1. *Cramér–von Mises*

The Cramér–von Mises estimates of the parameters $\hat{\theta}_{\mathrm{CM}}$ and $\hat{\alpha}_{\mathrm{CM}}$, respectively, are obtained by minimizing, with respect to θ and α, the function:

$$C(\theta,\alpha \mid \mathbf{y}) = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{2i-1}{2n}\right)^2. \tag{30}$$

These estimates can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n}\left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{2i-1}{2n}\right)\Delta_{1j}(y_{i:n}|\theta,\alpha) = 0, \tag{31}$$

$$\sum_{i=1}^{n}\left(\frac{F_1(y_{i:n}|\theta)}{1-\bar{\alpha}S_1(y_{i:n}|\theta)} - \frac{2i-1}{2n}\right)\Delta_2\,(y_{i:n}|\theta,\alpha) = 0, \tag{32}$$

where $\Delta_{1j}(y_{i:n}|\theta,\alpha)$ and $\Delta_2(y_{i:n}|\theta,\alpha)$ are given in Equations (20) and (21), respectively.

2.5.2. *Anderson–Darling*

The Anderson–Darling estimates of the parameters $\hat{\theta}_{AD}$ and $\hat{\alpha}_{AD}$, respectively, are obtained by minimizing, with respect to θ and α, the function:

$$A(\theta,\alpha \mid \mathbf{y}) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\{F(y_{i:n}|\theta,\alpha)[1-F(y_{n+1-i:n}|\theta,\alpha)]\}. \tag{33}$$

These estimates can also be obtained by solving the nonlinear equations:

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(y_{i:n}|\theta,\alpha)}{F(y_{i:n}|\theta,\alpha)} - \frac{\Delta_{1j}(y_{n+1-i:n}|\theta,\alpha)}{F(y_{n+1-i:n}|\theta,\alpha)}\right] = 0, \tag{34}$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(y_{i:n}|\theta,\alpha)}{F(y_{i:n}|\theta,\alpha)} - \frac{\Delta_2(y_{n+1-i:n}|\theta,\alpha)}{F(y_{n+1-i:n}|\theta,\alpha)}\right] = 0, \tag{35}$$

where $\Delta_{1j}(\cdot|\theta,\alpha)$ and $\Delta_2(\cdot|\theta,\alpha)$ are given in Equations (20) and (21), respectively.

These two methods together with other five were used in [64,65] for parameters estimation of the generalized Pareto distribution and three-parameter Weibull distribution, respectively.

3. Simulation studies

In this section we present results of some numerical experiments to compare the performance of the different estimation methods discussed in the previous section. We have taken sample sizes $n = 20, 50, 100$ and 200 and $\alpha = 0.5, 0.8, 1.5, 3.0$ and 5.0. The results were invariant with respect to θ, so we set $\theta = 1$.

For each of the 20 combinations, we have generated $B = 500,000$ pseudo-random samples from the Marshall–Olkin extended Lindley distribution. Since the MOEL distribution does not have an explicit inverse distribution function $F^{-1}(y \mid \theta, \alpha)$ to generate pseudo-random samples, we find the solution y of $F(y \mid \theta, \lambda) = U(0, 1)$ numerically.[66]

The estimates were obtained in *Ox* version 6.20,[67] using the *MaxBFGS* function. For each estimate, we compute the bias, the root mean-squared error, the average absolute difference between the true and estimate distributions' functions and the maximum absolute difference between the true and estimate distributions' functions, respectively, as

$$\operatorname{Bias}(\hat{\theta}) = \frac{1}{B}\sum_{i=1}^{B}\left(\hat{\theta}_i - \theta\right), \quad \operatorname{Bias}(\hat{\alpha}) = \frac{1}{B}\sum_{i=1}^{B}(\hat{\alpha}_i - \alpha), \tag{36}$$

$$\operatorname{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\theta}_i - \theta)^2}, \quad \operatorname{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{B}\sum_{i=1}^{B}(\hat{\alpha}_i - \alpha)^2}, \tag{37}$$

Table 1. Simulation results for $\theta = 1.0$ and $\alpha = 0.5$.

n	Qtd	MLE	MPS	OLS	WLS	CM	AD
20	Bias(θ)	0.2568^{5}	-0.0802^{3}	0.0264^{1}	0.0543^{2}	0.2784^{6}	0.1038^{4}
	RMSE(θ)	0.6385^{2}	0.5440^{1}	0.7550^{5}	0.7306^{4}	0.8798^{6}	0.6431^{3}
	Bias(α)	0.5174^{5}	0.1072^{1}	0.3501^{2}	0.3726^{4}	0.7217^{6}	0.3586^{3}
	RMSE(α)	1.3186^{3}	0.8030^{1}	1.4139^{4}	1.5403^{5}	1.9480^{6}	1.1921^{2}
	D_{abs}	0.0563^{3}	0.0551^{1}	0.0573^{5}	0.0567^{4}	0.0586^{6}	0.0561^{2}
	$D_{\max}$	0.0935^{3}	0.0879^{1}	0.0951^{5}	0.0937^{4}	0.1009^{6}	0.0922^{2}
	Total	21^{3}	8^{1}	22^{4}	23^{5}	36^{6}	16^{2}
50	Bias(θ)	0.0964^{6}	-0.0871^{4}	-0.0148^{1}	0.0155^{2}	0.0930^{5}	0.0275^{3}
	RMSE(θ)	0.3490^{2}	0.3379^{1}	0.4704^{5}	0.4182^{4}	0.4946^{6}	0.3875^{3}
	Bias(α)	0.1654^{5}	-0.0133^{1}	0.1022^{2}	0.1134^{4}	0.2151^{6}	0.1118^{3}
	RMSE(α)	0.4650^{2}	0.3546^{1}	0.5673^{5}	0.5265^{4}	0.6670^{6}	0.4760^{3}
	D_{abs}	0.0353^{2}	0.0350^{1}	0.0368^{5}	0.0360^{4}	0.0371^{6}	0.0357^{3}
	$D_{\max}$	0.0580^{2}	0.0566^{1}	0.0618^{5}	0.0598^{4}	0.0634^{6}	0.0590^{3}
	Total	19^{3}	9^{1}	23^{5}	22^{4}	35^{6}	18^{2}
100	Bias(θ)	0.0468^{5}	-0.0693^{6}	-0.0150^{3}	0.0107^{1}	0.0410^{4}	0.0121^{2}
	RMSE(θ)	0.2347^{1}	0.2421^{2}	0.3332^{5}	0.2810^{4}	0.3383^{6}	0.2693^{3}
	Bias(α)	0.0768^{5}	-0.0302^{1}	0.0427^{2}	0.0544^{4}	0.0960^{6}	0.0518^{3}
	RMSE(α)	0.2713^{2}	0.2349^{1}	0.3469^{5}	0.3101^{4}	0.3776^{6}	0.2928^{3}
	D_{abs}	0.0249^{1}	0.0249^{2}	0.0263^{5}	0.0255^{4}	0.0263^{6}	0.0253^{3}
	$D_{\max}$	0.0407^{2}	0.0405^{1}	0.0443^{5}	0.0423^{4}	0.0448^{6}	0.0419^{3}
	Total	16^{2}	13^{1}	25^{5}	21^{4}	34^{6}	17^{3}
200	Bias(θ)	0.0231^{5}	-0.0442^{6}	-0.0084^{2}	0.0084^{3}	0.0196^{4}	0.0060^{1}
	RMSE(θ)	0.1614^{1}	0.1676^{2}	0.2334^{5}	0.1920^{4}	0.2346^{6}	0.1879^{3}
	Bias(α)	0.0372^{5}	-0.0243^{2}	0.0199^{1}	0.0287^{4}	0.0457^{6}	0.0252^{3}
	RMSE(α)	0.1739^{2}	0.1620^{1}	0.2303^{5}	0.2002^{4}	0.2404^{6}	0.1936^{3}
	D_{abs}	0.0176^{1}	0.0176^{2}	0.0186^{5}	0.0180^{4}	0.0187^{6}	0.0179^{3}
	$D_{\max}$	0.0286^{1}	0.0287^{2}	0.0315^{5}	0.0298^{4}	0.0316^{6}	0.0296^{3}
	Total	15^{1}	15^{1}	23^{4}	23^{4}	34^{6}	16^{3}

Table 2. Simulation results for $\theta = 1.0$ and $\alpha = 0.8$.

n	Qtd	MLE	MPS	OLS	WLS	CM	AD
20	Bias(θ)	0.1888^{5}	– 0.0916^{4}	– 0.0144^{2}	0.0137^{1}	0.1906^{6}	0.0655^{3}
	RMSE(θ)	0.5133^{2}	0.4526^{1}	0.6024^{5}	0.5770^{4}	0.6785^{6}	0.5207^{3}
	Bias(α)	0.7257^{5}	0.1283^{1}	0.4359^{2}	0.4754^{3}	0.9372^{6}	0.4954^{4}
	RMSE(α)	1.9591^{4}	1.1850^{1}	1.8712^{3}	2.0616^{5}	2.5606^{6}	1.7316^{2}
	D_{abs}	0.0568^{2}	0.0559^{1}	0.0580^{5}	0.0573^{4}	0.0590^{6}	0.0568^{3}
	$D_{\max}$	0.0950^{3}	0.0900^{1}	0.0968^{5}	0.0954^{4}	0.1018^{6}	0.0941^{2}
	Total	21^{3}	9^{1}	22^{5}	21^{3}	36^{6}	17^{2}
50	Bias(θ)	0.0710^{5}	– 0.0821^{6}	– 0.0215^{3}	0.0069^{1}	0.0651^{4}	0.0196^{2}
	RMSE(θ)	0.2874^{2}	0.2871^{1}	0.3747^{5}	0.3332^{4}	0.3878^{6}	0.3139^{3}
	Bias(α)	0.2288^{5}	– 0.0279^{1}	0.1271^{2}	0.1502^{3}	0.2819^{6}	0.1566^{4}
	RMSE(α)	0.6746^{2}	0.5188^{1}	0.7697^{5}	0.7230^{4}	0.9021^{6}	0.6766^{3}
	D_{abs}	0.0357^{2}	0.0356^{1}	0.0373^{5}	0.0364^{4}	0.0375^{6}	0.0362^{3}
	$D_{\max}$	0.0592^{2}	0.0583^{1}	0.0630^{5}	0.0609^{4}	0.0642^{6}	0.0602^{3}
	Total	18^{2}	11^{1}	25^{5}	20^{4}	34^{6}	18^{2}
100	Bias(θ)	0.0350^{5}	– 0.0564^{6}	– 0.0122^{3}	0.0074^{1}	0.0310^{4}	0.0097^{2}
	RMSE(θ)	0.1953^{1}	0.2016^{2}	0.2608^{5}	0.2248^{4}	0.2642^{6}	0.2176^{3}
	Bias(α)	0.1071^{5}	– 0.0391^{1}	0.0572^{2}	0.0748^{4}	0.1284^{6}	0.0741^{3}
	RMSE(α)	0.3964^{2}	0.3442^{1}	0.4743^{5}	0.4335^{4}	0.5150^{6}	0.4169^{3}
	D_{abs}	0.0252^{1}	0.0253^{2}	0.0265^{5}	0.0257^{4}	0.0265^{6}	0.0256^{3}
	$D_{\max}$	0.0416^{2}	0.0416^{1}	0.0449^{5}	0.0430^{4}	0.0452^{6}	0.0426^{3}
	Total	16^{2}	13^{1}	25^{5}	21^{4}	34^{6}	17^{3}
200	Bias(θ)	0.0173^{5}	– 0.0345^{6}	– 0.0062^{3}	0.0056^{2}	0.0151^{4}	0.0047^{1}
	RMSE(θ)	0.1352^{1}	0.1392^{2}	0.1814^{5}	0.1545^{4}	0.1825^{6}	0.1520^{3}
	Bias(α)	0.0520^{5}	– 0.0301^{2}	0.0273^{1}	0.0389^{4}	0.0613^{6}	0.0360^{3}
	RMSE(α)	0.2551^{2}	0.2372^{1}	0.3143^{5}	0.2817^{4}	0.3279^{6}	0.2754^{3}
	D_{abs}	0.0177^{1}	0.0178^{2}	0.0187^{5}	0.0181^{4}	0.0187^{6}	0.0181^{3}
	$D_{\max}$	0.0293^{1}	0.0294^{2}	0.0318^{5}	0.0302^{4}	0.0319^{6}	0.0301^{3}
	Total	15^{1}	15^{1}	24^{5}	22^{4}	34^{6}	16^{3}

Table 3. Simulation results for $\theta = 1.0$ and $\alpha = 1.5$.

n	Qtd	MLE	MPS	OLS	WLS	CM	AD
20	Bias(θ)	0.1329^{6}	– 0.0966^{4}	– 0.0362^{2}	– 0.0067^{1}	0.1239^{5}	0.0407^{3}
	RMSE(θ)	0.4009^{2}	0.3674^{1}	0.4609^{5}	0.4398^{4}	0.5007^{6}	0.4059^{3}
	Bias(α)	1.2371^{5}	0.1714^{1}	0.6473^{2}	0.7603^{3}	1.4537^{6}	0.8431^{4}
	RMSE(α)	3.4898^{5}	1.9987^{1}	2.9325^{2}	3.4810^{4}	3.9899^{6}	3.0004^{3}
	D_{abs}	0.0573^{2}	0.0568^{1}	0.0588^{5}	0.0580^{4}	0.0593^{6}	0.0575^{3}
	$D_{\max}$	0.0965^{3}	0.0926^{1}	0.0986^{5}	0.0971^{4}	0.1024^{6}	0.0961^{2}
	Total	23^{5}	9^{1}	21^{4}	20^{3}	35^{6}	18^{2}
50	Bias(θ)	0.0504^{5}	– 0.0686^{6}	– 0.0186^{3}	0.0035^{1}	0.0463^{4}	0.0144^{2}
	RMSE(θ)	0.2303^{1}	0.2335^{2}	0.2831^{5}	0.2555^{4}	0.2911^{6}	0.2451^{3}
	Bias(α)	0.3863^{5}	– 0.0421^{1}	0.2077^{2}	0.2520^{3}	0.4577^{6}	0.2723^{4}
	RMSE(α)	1.1813^{3}	0.9059^{1}	1.2732^{5}	1.2118^{4}	1.4940^{6}	1.1641^{2}
	D_{abs}	0.0361^{1}	0.0363^{2}	0.0377^{5}	0.0367^{4}	0.0377^{6}	0.0365^{3}
	$D_{\max}$	0.0605^{2}	0.0601^{1}	0.0638^{5}	0.0619^{4}	0.0647^{6}	0.0613^{3}
	Total	17^{2}	13^{1}	25^{5}	20^{4}	34^{6}	17^{2}
100	Bias(θ)	0.0247^{5}	– 0.0434^{6}	– 0.0095^{3}	0.0046^{1}	0.0224^{4}	0.0069^{2}
	RMSE(θ)	0.1576^{1}	0.1616^{2}	0.1956^{5}	0.1741^{4}	0.1982^{6}	0.1701^{3}
	Bias(α)	0.1795^{5}	– 0.0537^{1}	0.0948^{2}	0.1253^{3}	0.2074^{6}	0.1274^{4}
	RMSE(α)	0.6927^{2}	0.5995^{1}	0.7768^{5}	0.7282^{4}	0.8434^{6}	0.7100^{3}
	D_{abs}	0.0255^{1}	0.0256^{2}	0.0266^{5}	0.0259^{4}	0.0267^{6}	0.0258^{3}
	$D_{\max}$	0.0426^{1}	0.0426^{2}	0.0452^{5}	0.0436^{4}	0.0456^{6}	0.0434^{3}
	Total	15^{2}	14^{1}	25^{5}	20^{4}	34^{6}	18^{3}
200	Bias(θ)	0.0121^{5}	– 0.0263^{6}	– 0.0048^{3}	0.0034^{2}	0.0109^{4}	0.0034^{1}
	RMSE(θ)	0.1096^{1}	0.1122^{2}	0.1366^{5}	0.1206^{4}	0.1375^{6}	0.1192^{3}
	Bias(α)	0.0865^{5}	– 0.0430^{1}	0.0453^{2}	0.0641^{4}	0.0989^{6}	0.0617^{3}
	RMSE(α)	0.4455^{2}	0.4126^{1}	0.5136^{5}	0.4744^{4}	0.5357^{6}	0.4672^{3}
	D_{abs}	0.0180^{1}	0.0181^{2}	0.0188^{5}	0.0183^{4}	0.0188^{6}	0.0182^{3}
	$D_{\max}$	0.0300^{1}	0.0301^{2}	0.0320^{5}	0.0307^{4}	0.0321^{6}	0.0306^{3}
	Total	15^{2}	14^{1}	25^{5}	22^{4}	34^{6}	16^{3}

Table 4. Simulation results for $\theta = 1.0$ and $\alpha = 3.0$.

n	Qtd	MLE	MPS	OLS	WLS	CM	AD
20	Bias(θ)	0.0978^{6}	-0.0928^{5}	-0.0385^{3}	-0.0111^{1}	0.0848^{4}	0.0286^{2}
	RMSE(θ)	0.3242^{3}	0.3067^{1}	0.3564^{5}	0.3440^{4}	0.3763^{6}	0.3236^{2}
	Bias(α)	2.4791^{5}	0.3098^{1}	1.1323^{2}	1.4817^{3}	2.5623^{6}	1.6698^{4}
	RMSE(α)	7.3190^{6}	4.0607^{1}	5.1862^{2}	6.8683^{4}	6.9413^{5}	5.8643^{3}
	D_{abs}	0.0578^{1}	0.0580^{2}	0.0595^{5}	0.0587^{4}	0.0595^{6}	0.0581^{3}
	$D_{\max}$	0.0980^{3}	0.0951^{1}	0.0995^{5}	0.0984^{4}	0.1023^{6}	0.0974^{2}
	Total	24^{5}	11^{1}	22^{4}	20^{3}	33^{6}	16^{2}
50	Bias(θ)	0.0370^{5}	-0.0559^{6}	-0.0148^{3}	0.0018^{1}	0.0351^{4}	0.0108^{2}
	RMSE(θ)	0.1887^{1}	0.1909^{2}	0.2196^{5}	0.2022^{4}	0.2259^{6}	0.1963^{3}
	Bias(α)	0.7454^{5}	-0.0553^{1}	0.4061^{2}	0.4891^{3}	0.8704^{6}	0.5379^{4}
	RMSE(α)	2.3550^{4}	1.7769^{1}	2.4298^{5}	2.3343^{3}	2.8668^{6}	2.2813^{2}
	D_{abs}	0.0363^{1}	0.0367^{2}	0.0377^{5}	0.0369^{4}	0.0378^{6}	0.0367^{3}
	$D_{\max}$	0.0614^{2}	0.0611^{1}	0.0640^{5}	0.0623^{4}	0.0648^{6}	0.0619^{3}
	Total	18^{3}	13^{1}	25^{5}	19^{4}	34^{6}	17^{2}
100	Bias(θ)	0.0183^{5}	-0.0341^{6}	-0.0074^{3}	0.0032^{1}	0.0172^{4}	0.0054^{2}
	RMSE(θ)	0.1298^{1}	0.1323^{2}	0.1522^{5}	0.1389^{4}	0.1544^{6}	0.1365^{3}
	Bias(α)	0.3449^{5}	-0.0799^{1}	0.1861^{2}	0.2429^{3}	0.3925^{6}	0.2517^{4}
	RMSE(α)	1.3613^{2}	1.1675^{1}	1.4593^{5}	1.3910^{4}	1.5897^{6}	1.3692^{3}
	D_{abs}	0.0257^{1}	0.0259^{2}	0.0267^{5}	0.0260^{4}	0.0267^{6}	0.0259^{3}
	$D_{\max}$	0.0432^{1}	0.0433^{2}	0.0454^{5}	0.0440^{4}	0.0456^{6}	0.0438^{3}
	Total	15^{2}	14^{1}	25^{5}	20^{4}	34^{6}	18^{3}
200	Bias(θ)	0.0093^{5}	-0.0200^{6}	-0.0035^{3}	0.0027^{1}	0.0087^{4}	0.0029^{2}
	RMSE(θ)	0.0905^{1}	0.0921^{2}	0.1068^{5}	0.0968^{4}	0.1075^{6}	0.0959^{3}
	Bias(α)	0.1675^{5}	-0.0659^{1}	0.0908^{2}	0.1252^{4}	0.1885^{6}	0.1235^{3}
	RMSE(α)	0.8724^{2}	0.8033^{1}	0.9604^{5}	0.9060^{4}	1.0034^{6}	0.8966^{3}
	D_{abs}	0.0182^{1}	0.0182^{2}	0.0189^{5}	0.0184^{4}	0.0189^{6}	0.0184^{3}
	$D_{\max}$	0.0305^{1}	0.0306^{2}	0.0321^{5}	0.0311^{4}	0.0322^{6}	0.0310^{3}
	Total	15^{2}	14^{1}	25^{5}	21^{4}	34^{6}	17^{3}

Table 5. Simulation results for $\theta = 1.0$ and $\alpha = 5.0$.

n	Qtd	MLE	MPS	OLS	WLS	CM	AD
20	Bias(θ)	0.0840^{6}	-0.0826^{5}	-0.0382^{3}	-0.0092^{1}	0.0646^{4}	0.0244^{2}
	RMSE(θ)	0.2862^{3}	0.2700^{1}	0.3013^{5}	0.2974^{4}	0.3126^{6}	0.2811^{2}
	Bias(α)	4.4562^{6}	0.6396^{1}	1.6651^{2}	2.6590^{3}	3.8004^{5}	2.8700^{4}
	RMSE(α)	13.4363^{6}	7.4136^{1}	7.6407^{2}	11.9698^{5}	10.0108^{4}	9.7100^{3}
	D_{abs}	0.0582^{1}	0.0586^{3}	0.0595^{6}	0.0590^{4}	0.0591^{5}	0.0583^{2}
	$D_{\max}$	0.0987^{3}	0.0960^{1}	0.0993^{5}	0.0988^{4}	0.1012^{6}	0.0978^{2}
	Total	25^{5}	12^{1}	23^{4}	21^{3}	30^{6}	15^{2}
50	Bias(θ)	0.0316^{5}	-0.0488^{6}	-0.0120^{3}	0.0017^{1}	0.0310^{4}	0.0097^{2}
	RMSE(θ)	0.1676^{1}	0.1685^{2}	0.1894^{5}	0.1763^{4}	0.1953^{6}	0.1720^{3}
	Bias(α)	1.2940^{5}	-0.0454^{1}	0.7333^{2}	0.8626^{3}	1.5143^{6}	0.9525^{4}
	RMSE(α)	4.0846^{4}	3.0191^{1}	4.1388^{5}	3.9926^{3}	4.9254^{6}	3.9157^{2}
	D_{abs}	0.0367^{1}	0.0370^{3}	0.0379^{5}	0.0371^{4}	0.0379^{6}	0.0369^{2}
	$D_{\max}$	0.0620^{2}	0.0618^{1}	0.0642^{5}	0.0627^{4}	0.0650^{6}	0.0624^{3}
	Total	18^{3}	14^{1}	25^{5}	19^{4}	34^{6}	16^{2}
100	Bias(θ)	0.0156^{5}	-0.0295^{6}	-0.0061^{3}	0.0027^{1}	0.0151^{4}	0.0048^{2}
	RMSE(θ)	0.1155^{1}	0.1170^{2}	0.1317^{5}	0.1216^{4}	0.1338^{6}	0.1199^{3}
	Bias(α)	0.5920^{5}	-0.1052^{1}	0.3319^{2}	0.4215^{3}	0.6741^{6}	0.4402^{4}
	RMSE(α)	2.3165^{3}	1.9649^{1}	2.4381^{5}	2.3316^{4}	2.6687^{6}	2.3055^{2}
	D_{abs}	0.0259^{1}	0.0261^{2}	0.0268^{6}	0.0262^{4}	0.0268^{5}	0.0261^{3}
	$D_{\max}$	0.0437^{1}	0.0437^{2}	0.0455^{5}	0.0443^{4}	0.0458^{6}	0.0441^{3}
	Total	16^{2}	14^{1}	26^{5}	20^{4}	33^{6}	17^{3}
200	Bias(θ)	0.0078^{5}	-0.0173^{6}	-0.0029^{3}	0.0022^{1}	0.0076^{4}	0.0025^{2}
	RMSE(θ)	0.0806^{1}	0.0817^{2}	0.0924^{5}	0.0848^{4}	0.0931^{6}	0.0841^{3}
	Bias(α)	0.2854^{5}	-0.0936^{1}	0.1604^{2}	0.2133^{3}	0.3213^{6}	0.2141^{4}
	RMSE(α)	1.4725^{2}	1.3465^{1}	1.5872^{5}	1.5070^{4}	1.6621^{6}	1.4955^{3}
	D_{abs}	0.0183^{1}	0.0184^{2}	0.0189^{6}	0.0185^{4}	0.0189^{5}	0.0184^{3}
	$D_{\max}$	0.0308^{1}	0.0309^{2}	0.0322^{5}	0.0313^{4}	0.0323^{6}	0.0312^{3}
	Total	15^{2}	14^{1}	26^{5}	20^{4}	33^{6}	18^{3}

$$D_{\text{abs}} = \frac{1}{B \times n} \sum_{i=1}^{B} \sum_{j=1}^{n} |F(y_{ij}|\theta, \alpha) - F(y_{ij}|\hat{\theta}, \hat{\alpha})|, \tag{38}$$

$$D_{\max} = \frac{1}{B} \sum_{i=1}^{B} \max_{j} |F(y_{ij}|\theta, \alpha) - F(y_{ij}|\hat{\theta}, \hat{\alpha})|. \tag{39}$$

In Tables 1–5, we show the calculated values of (36)–(39). The superscript values indicate the rank obtained by each method and the line named as *total* shows the global rank for each method based on measures (35)–(38). For example, in Table 1 and $n = 20$, the MPS method has the superscript value equal to 1 and this value means that the ranks sum of the measures (35)–(38) was the lowest among all methods.

4. Conclusions

In this paper, we compared, by intensive simulation experiments, the parameters estimation of the MOEL distribution using six methods, namely, the maximum likelihood, ordinal and weighted least-squares, maximum product of spacings, Cramér–von Mises and Anderson–Darling.

From the simulations we have observed that, in all scenarios, the maximum product of spacings method (MPS) had the lowest overall rank. However, as the sample size increases the maximum likelihood method (MLE) has the second lowest rank and sometimes equalling the MPS. Therefore, the MPS method can be regarded as the best method to estimate the parameters of MOEL distribution. For large samples the MLE method is also good. An important observation is that for $\alpha > 1$ and $n = 20$, the MLE had the second highest rank, better than CM only, since the method CM showed the highest rank in all cases.

In general, the MPS method showed the lowest root mean-squared error. For the α parameter, the estimated RMSE(α) by the MPS method was the lowest for all α and n. In the case of θ parameter, the estimated RMSE(θ) by the MPS method was lower only for $n = 20, 50$ and $\alpha = 0.5, 0.8$; otherwise, the MLE method obtained a better result.

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