J Math Chem (2016) 54:1783–1791
DOI 10.1007/s10910-016-0635-6

ORIGINAL PAPER

# From the generalized Morse potential to a unified treatment of the $D$-dimensional singular harmonic oscillator and singular Coulomb potentials

**Pedro H. F. Nogueira**[1] · **Antonio S. de Castro**[1]

Received: 28 March 2016 / Accepted: 21 April 2016 / Published online: 6 June 2016
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**Abstract** Bound-state solutions of the singular harmonic oscillator and singular Coulomb potentials in arbitrary dimensions are generated in a simple way from the solutions of the one-dimensional generalized Morse potential. The nonsingular harmonic oscillator and nonsingular Coulomb potentials in arbitrary dimensions with their additional accidental degeneracies are obtained as particular cases. Added bonuses from these mappings are the straightforward determination of the critical attractive singular potential, the proper boundary condition on the radial eigenfunction at the origin and the inexistence of bound states in a pure inversely quadratic potential.

**Keywords** Morse potential · Singular harmonic oscillator · Singular Coulomb potential · $D$-dimensional Schrödinger equation

## 1 Introduction

Some exactly soluble systems with importance in atomic and molecular physics have been approached in the literature on quantum mechanics with a myriad of methods. Among such systems is the Morse potential $a(e^{-\alpha x} - 2e^{-2\alpha x})$ [1–19], the $D$-dimensional pseudoharmonic potential $a\,(x/b - b/x)^2$ [4,5,20–31], and the $D$-dimensional Kratzer–Fues potential $a\left(b^2/x^2 - 2b/x\right)$ and its modified version $a\left(b^2/x^2 - b/x\right)$ [4,5,12,30–34]. More general exactly soluble systems have also been appreciated: the generalized Morse potential $Ae^{-\alpha x} + Be^{-2\alpha x}$ [31,35–39], the sin-

✉ Antonio S. de Castro
castro@pq.cnpq.br

1 Departamento de Física e Química, Universidade Estadual Paulista - Campus de Guaratinguetá, Guaratinguetá, SP 12516-410, Brazil

gular harmonic oscillator $Ax^2 + Bx^{-2}$ [3–5,19,36,40–51] and the singular Coulomb potential $Ax^{-1} + Bx^{-2}$ [2–5,36,40,48,50,52–57].

In a recent paper [58], it was shown that the Schrödinger equation for all those exactly solvable problems mentioned above can be reduced to the confluent hypergeometric equation in such a way that it can be solved via Laplace transform method with closed-form eigenfunctions expressed in terms of generalized Laguerre polynomials. Connections between the Morse and those other potentials have also been reported. The three-dimensional Coulomb potential has been mapped into the one-dimensional Morse potential and into the three-dimensional singular Coulomb potential via change of function and variable [59]. The Morse potential with particular parameters has been mapped into the two-dimensional harmonic oscillator [60] and into the three-dimensional Coulomb potential [61]. Later, the generalized Morse potential was mapped into the three-dimensional harmonic oscillator and Coulomb potentials [62,63]. Furthermore, a certain mapping between the Morse potential with particular parameters and the three-dimensional Kratzer–Fues potential has been found with fulcrum on the algebra $so(2,1)$ and its representations [64]. Although the Morse and the Kratzer–Fues potentials are well-known systems, the map connecting them was used recently to obtain the Wigner distribution functions for the Kratzer–Fues potential from the Wigner distribution functions of the Morse potential [65].

In this paper, an alternative and more general approach for the mapping is developed. We show that bound-state solutions of the singular harmonic oscillator and singular Coulomb potentials in arbitrary dimensions can be generated in a simple way from the bound states of the one-dimensional generalized Morse potential via Langer transformation [59]. Links with the nonsingular harmonic oscillator and nonsingular Coulomb potentials in arbitrary dimensions with their additional accidental degeneracies are obtained as particular cases. Added bonuses from these interrelationships are the straightforward determination of the critical attractive singular potential (that one which avoids the famous "fall of a particle to the centre" [3]) and the proper boundary condition on the radial eigenfunction at the origin [that one which excludes spurious solutions coming from the Laplacian operator (see, e.g. [66,67])]. As a mere epiphenomenon of our approach, it is shown that a pure inversely quadratic potential can not hold bound states.

In Sect. 2 we present a detailed analysis of the bound-state solutions in a one-dimensional generalized Morse potential. In Sect. 3 we present a few relevant properties of the Schrödinger equation in $D$ dimensions for spherically symmetric potentials. We then proceed to show that the bound states in the singular harmonic oscillator and singular Coulomb potentials are linked to the bound states in the generalized Morse potential. Final remarks comprise Sect. 4.

## 2 Bound states in a generalized Morse potential

The time-independent Schrödinger equation is an eigenvalue equation for the characteristic pair $(E, \psi)$ with $E \in \mathbb{R}$. For a particle of mass $m$ embedded in a one-dimensional potential $V(x)$ it is given by

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}\left[E - V(x)\right]\psi(x) = 0, \tag{1}$$

where $\hbar$ is Planck's constant, and $\int_{-\infty}^{+\infty} dx\,|\psi|^2 = 1$ for bound states. For the generalized Morse potential

$$V(x) = V_1 e^{-\alpha x} + V_2 e^{-2\alpha x}, \quad \alpha > 0, \tag{2}$$

the substitution

$$\xi = \frac{2\sqrt{2mV_2}\,e^{-\alpha x}}{\hbar\alpha} \tag{3}$$

and the definitions

$$s = \frac{\sqrt{-2mE}}{\hbar\alpha}, \quad a = \frac{mV_1}{\hbar\alpha\sqrt{2mV_2}} + s + \frac{1}{2} \tag{4}$$

convert Eq. (1) into

$$\frac{d^2\psi(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{d\psi(\xi)}{d\xi} + \left(-\frac{1}{4} + \frac{s - a + 1/2}{\xi} - \frac{s^2}{\xi^2}\right)\psi(\xi) = 0, \tag{5}$$

whose solutions have asymptotic limits expressed as $\psi(\xi) \underset{|\xi|\to 0}{\to} \xi^{\pm s}$ and $\psi(\xi) \underset{|\xi|\to\infty}{\to} e^{\pm\xi/2}$. On account of the normalization condition, $\int_0^\infty d|\xi|\,|\psi(\xi)|^2/|\xi| = \alpha$, one has that $\psi$ behaves like $\xi^s$ as $|\xi| \to 0$ and like $e^{-\xi/2}$ as $|\xi| \to \infty$ with $\xi \in \mathbb{R}$ ($V_2 > 0$) and $s > 0$ ($E < 0$). We write $\psi = e^{-\xi/2}\xi^s w$, where $w$ satisfies Kummer's equation

$$\xi\frac{d^2 w(\xi)}{d\xi^2} + (2s + 1 - \xi)\frac{dw(\xi)}{d\xi} - aw(\xi) = 0 \tag{6}$$

with general solution expressed as

$$w(\xi) = AM(a, 2s + 1, \xi) + BU(a, 2s + 1, \xi). \tag{7}$$

Here, $A$ and $B$ are arbitrary constants, $M(a, b, z) = {}_1F_1(a, b, z)$ is the confluent hypergeometric function [68], and

$$U(a, b, z) = \frac{\pi}{\sin\pi b}\left[\frac{M(a, b, z)}{\Gamma(1 + a - b)} - z^{1-b}\frac{M(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)}\right], \tag{8}$$

where $\Gamma(z)$ is the gamma function. One has to search particular solutions of Eq. (6) such that $w(\xi) \underset{\xi\to 0}{\to} C$ and $w(\xi) \underset{\xi\to\infty}{\to} \xi^{\alpha_1} e^{\alpha_2\xi^{\alpha_3}}$, where $C$ is a nonvanishing constant, $\alpha_1$ and $\alpha_2$ are arbitrary constants, and $\alpha_3 < 1$. This occurs because $\xi^{\alpha_1} e^{\alpha_2\xi^{\alpha_3} - \xi/2} \to e^{-\xi/2}$ as $\xi \to \infty$. For the reason that [68]

$$M(a,b,z) \underset{|z|\to 0}{\to} 1, \quad U(a,b,z) \underset{|z|\to 0}{\to} \frac{\Gamma(b-1)\, z^{1-b}}{\Gamma(a)} \quad \text{for} \quad b \neq 1, \tag{9}$$

one has $B = 0$. On the other hand [68],

$$\frac{M(a,b,z)}{\Gamma(b)} \underset{|z|\to\infty}{\to} \frac{e^{i\pi a} z^{-a}}{\Gamma(b-a)} + \frac{e^{z} z^{a-b}}{\Gamma(a)}, \quad -\pi/2 < \arg z < 3\pi/2, \tag{10}$$

so that $w$ diverges as $e^{\xi}$ for large $\xi$. Due to the poles of the gamma function in (10), this bad behaviour can be remedied making $-a = n \in \mathbb{N}$. It follows from (4) that $V_1 < 0$ and therefore the generalized Morse potential is able to hold bound states only if it has a well structure ($V_1 < 0$ and $V_2 > 0$). Furthermore, $M(-n,b,z)$ is proportional to the generalized Laguerre polynomial $L_n^{(b-1)}(z)$, a polynomial of degree $n$ [68]. Therefore,

$$\psi_n(\xi) = N_n\, \xi^s e^{-\xi/2} L_n^{(2s)}(\xi), \tag{11}$$

where $N_n$ is a normalization constant. Substituting $a = -n$ in Eq. (4), one finds the quantization condition

$$n + s + \frac{1}{2} = \frac{m|V_1|}{\hbar\alpha\sqrt{2mV_2}}, \tag{12}$$

and because $s > 0$ one gets

$$n < \frac{m|V_1|}{\hbar\alpha\sqrt{2mV_2}} - \frac{1}{2}. \tag{13}$$

This restriction on $n$ limits the number of allowed states and requires $m|V_1|/\left(\hbar\alpha\sqrt{2mV_2}\right) > 1/2$ to make the existence of a bound state possible. Finally, the solution of the quantization condition is expressed as

$$E_n = -\frac{V_1^2}{4V_2}\left[1 - \frac{\hbar\alpha\sqrt{2mV_2}}{m|V_1|}\left(n + \frac{1}{2}\right)\right]^2. \tag{14}$$

These results for the generalized Morse potential is in agreement with those ones obtained in Ref. [58] via Laplace transform method.

## 3 Bound states in D dimensions

The $D$-dimensional time-independent Schrödinger equation is expressed as (see, e.g. [19,69])

$$-\frac{\hbar^2}{2m}\nabla_D^2 \psi(\vec{r}) + V(\vec{r})\,\psi(\vec{r}) = \varepsilon\psi(\vec{r}), \tag{15}$$

where $\nabla_D^2$ is the $D$-dimensional Laplacian operator. In spherical coordinates $\vec{r} = (r, \Omega)$. Here, $r = |\vec{r}| \in [0, \infty)$, and $\Omega$ denotes a set of $D-1$ angular variables. Equation (15) is an eigenvalue equation for the characteristic pair $(\varepsilon, \psi)$ with $\varepsilon \in \mathbb{R}$ and $\int d\tau\, |\psi|^2 = 1$ for bound states. In this last formula $d\tau = r^{D-1} dr d\Omega$ is the volume element and the integral is taken over the whole hyperspace.

For spherically symmetric potentials one can write (see, e.g. [19,69])

$$\psi\left(\vec{r}\right)=r^{(1-D)/2}\,u(r)Y\left(\Omega\right),\tag{16}$$

where $u$ obeys the radial equation

$$\frac{d^2u\left(r\right)}{dr^2}+\frac{2m}{\hbar^2}\left[\varepsilon-V\left(r\right)-\frac{L\left(L+1\right)\hbar^2}{2mr^2}\right]u\left(r\right)=0\tag{17}$$

with $\int_0^\infty dr\,|u|^2=1$ for bound-state solutions, and

$$L=l+\left(D-3\right)/2\quad\text{or}\quad L=-l-\left(D-1\right)/2,\tag{18}$$

in which $l=0,1,2,\ldots$ In (16), $Y$ denotes the normalized hyperspherical harmonics ($\int d\Omega\,|Y|^2=1$) labeled by $D-1$ quantum numbers:

$$\begin{aligned}
&l_{D-1}=l,\quad l_{D-2}=0,1,2,\ldots,l_{D-1},\quad l_{D-3}=0,1,2,\ldots,l_{D-2},\\
&\qquad\vdots\\
&l_4=0,1,2,\ldots,l_5,\quad l_3=0,1,2,\ldots,l_4,\quad l_2=0,1,2,\ldots,l_3,\\
&l_1=-l_2,-l_2+1,\ldots,+l_2-1,+l_2.
\end{aligned}\tag{19}$$

Hence, the essential degeneracy of the spectrum for a given $l$ is expressed by Avery [69]

$$d_l\left(D\right)=\frac{\left(D+2l-2\right)\left(D+l-3\right)!}{l!\left(D-2\right)!}.\tag{20}$$

With potentials expressed as

$$V(r)=Zr^{\delta}+\frac{\hbar^2\beta}{2mr^2},\tag{21}$$

the Langer transformation [59]

$$u=\sqrt{r/r_0}\,\phi,\quad r/r_0=e^{-\Lambda\alpha x},\tag{22}$$

with $r_0>0$ and $\Lambda>0$, transmutes the radial equation (17) into

$$\frac{d^2\phi\left(x\right)}{dx^2}+\frac{2m}{\hbar^2}\left\{-\frac{\left(\hbar\Lambda\alpha S\right)^2}{2m}-\left(\Lambda\alpha r_0\right)^2\left[Zr_0^{\delta}e^{-\Lambda\alpha(\delta+2)x}-\varepsilon e^{-2\Lambda\alpha x}\right]\right\}\phi\left(x\right)=0,\tag{23}$$

with

$$S=\sqrt{\beta+\left(L+1/2\right)^2}.\tag{24}$$

At this point, it is instructive to note that $S$ is insensible to the different choices of $L$ as prescribed by (18). Besides that, Eq. (23) is precisely the Schrödinger equation

for the 'Morse potential' with $V_1 = 0$ or $V_2 = 0$ when (21) is the pure inversely quadratic potential ($\delta = 0$ or $\delta = -2$). In this case there is no bound-state solution. Nevertheless, a connection with the bound states of the generalized Morse potential, with $\int_{-\infty}^{+\infty} dx\, e^{-2\Lambda\alpha x}|\phi|^2 = (\Lambda\alpha r_0)^{-1}$, might be reached if the pair $(\delta, \Lambda)$ is equal to $(2, 1/2)$ or $(-1, 1)$. As an immediate consequence of the mapping for bound states $S^2 > 0$. Thus,

$$\beta > -(D-2)^2/4. \tag{25}$$

Furthermore, the existence of bound states also demands $\phi(x) \underset{x\to+\infty}{\to} e^{-\Lambda\alpha S x}$ with $\Lambda S > 0$, in such a way that

$$u(r) \underset{r\to 0}{\to} r^{1/2+S}. \tag{26}$$

The above restriction on the coupling constant $\beta$ and the boundary condition $u(0) = 0$ represent important pieces for the determination of bound states. The first one excludes strongly attractive singular potentials and can be obtained by recurring to a regularization of the potential at the origin (see, e.g. [3]). The second one, well-grounded even for nonsingular potentials, can be legitimated by ruling out the Dirac delta function $\delta(\vec{r})$ coming from the Laplacian operator in (15) (see, e.g. [66,67]).

### 3.1 The singular harmonic oscillator

With $\delta = 2$ plus the definition $Z = m\omega^2/2$, the potential (21) is written as

$$V(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2\beta}{2mr^2}. \tag{27}$$

In order to complete the identification of the bound-state solutions with those ones from the generalized Morse potential one must choose $\Lambda = 1/2$, $V_1 = -\alpha^2 r_0^2\varepsilon/4$ and $V_2 = \alpha^2 r_0^4 m\omega^2/8$. For the reason that $V_1 < 0$ and $V_2 > 0$ one can see that bound-state solutions require $\varepsilon > 0$ and $\omega^2 > 0$, respectively, and choosing $\omega > 0$ one can write

$$\xi = m\omega r^2/\hbar. \tag{28}$$

Furthermore, (13) implies $\varepsilon > 2\hbar\omega(n + 1/2)$. Using (14) and (24) one can write the complete solution of the problem as

$$\begin{aligned} \varepsilon_{nL} &= \hbar\omega(2n + 1 + S), \\ u_{nL}(r) &= A_{nL} r^{1/2+S} e^{-m\omega r^2/(2\hbar)} L_n^{(S)}\left(m\omega r^2/\hbar\right). \end{aligned} \tag{29}$$

When $\beta = 0$, the case of a pure harmonic oscillator, one can write

$$\varepsilon_N = \hbar\omega(N + D/2), \quad N = 0, 1, 2, \ldots, \tag{30}$$

where $N = 2n + l$. The radial eigenfunction $u$, though, is labelled with the quantum numbers $N$ and $l$, with $l$ even (odd) for $N$ even (odd) and $l \le N$.

### 3.2 The singular Coulomb potential

Now, $\delta = -1$ and

$$V(r) = \frac{Z}{r} + \frac{\hbar^2 \beta}{2mr^2}. \tag{31}$$

Comparison of the bound states with those ones from the generalized Morse potential is done by choosing $\Lambda = 1$, $V_1 = \alpha^2 r_0 Z$ and $V_2 = -\alpha^2 r_0^2 \varepsilon$. The conditions $V_1 < 0$ and $V_2 > 0$ imply $Z < 0$ and $\varepsilon < 0$, respectively. Now,

$$\xi = 2\sqrt{2m|\varepsilon|}\, r/\hbar \tag{32}$$

and (13) implies $\varepsilon > -\hbar^2/[2ma^2 (n + 1/2)^2]$. Here, $a = \hbar^2/(m|Z|)$. Using (14) and (24) one can write

$$\begin{aligned} \varepsilon_{nL} &= -\frac{\hbar^2}{2ma^2 (n + 1/2 + S)^2}, \\ u_{nL}(r) &= B_{nL} r^{1/2+S} e^{-r/[a(n+1/2+S)]} L_n^{(2S)} \left(2r/[a(n + 1/2 + S)]\right). \end{aligned} \tag{33}$$

In the case of a pure Coulomb potential ($\beta = 0$), one can write

$$\varepsilon_N = -\frac{\hbar^2}{2ma^2 [N + (D-3)/2]^2}, \quad N = 1, 2, 3, \ldots \tag{34}$$

Here $N = n + l + 1$ and the radial eigenfunction $u$ is labelled with the quantum numbers $N$ and $l$, with $l \leq N - 1$.

## 4 Final remarks

We have shown that the complete infinite sets of bound-state solutions of the singular harmonic oscillator and singular Coulomb potentials (and their higher degenerate nonsingular counterparts) in arbitrary dimensions can be extracted from the finite set of bound-state solutions of the one-dimensional generalized Morse potential in a simple way. Surprisingly, the determination of the critical coupling constant $\beta_c = -(D-2)^2/4$ as well as the proper boundary condition $u(0) = 0$ emerged in a natural manner. As a by-product, we have shown that there is no bound state in a pure inversely quadratic potential.

**Acknowledgments** We thank Prof. Dr. M. G. Garcia for useful discussions. This work was supported in part by means of funds provided by FAPESP and CNPq (Grant 304743/2015-1).

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