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Asymptotic stability and bifurcations of 3D piecewise smooth vector fields

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Abstract. The paper deals with the analysis of the behavior of a nonsmooth three-dimensional vector field around a typal singularity. We focus on a class of generic one-parameter families Z_λ of Filippov systems and address the persistence problem for the asymptotic stability when the parameter varies near the bifurcation value $\lambda = 0$.

Mathematics Subject Classification. Primary 34A36 · 34C23 · 37G35.

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1. Introduction

This paper is part of a general program involving the asymptotic stability at typical singularities of systems represented by the following equation

$$\dot{u} = F(u) + \mathrm{sgn}(h)G(u) \tag{1}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n, h : \mathbb{R}^n \to \mathbb{R}$ and $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are smooth mappings.

We start with some historical facts. Anosov (see [2]) studied the asymptotic stability of systems of the form

$$\dot{u} = Au + \mathrm{sgn}(u_1)k,$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$, A is an $n \times n$ real-valued matrix and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{R}^n$ is a constant vector. In [23], conditions were established for the asymptotic stability of 3D systems represented by vector fields having the form (1) where

$$F(x, y, z) = 1/2(a_1 + b_1, a_2 + b_2, x + y),$$
$$G(x, y, z) = 1/2(a_1 - b_1, a_2 - b_2, x - y),$$

for selected real numbers a_1, a_2, b_1 and b_2.

In this paper, we analyze the bifurcation diagram and the asymptotic stability of the following family of piecewise smooth vector fields (PSVFs for short):

$$Z_\lambda(x, y, z) = (\dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\big((a + c, \lambda + d, b(y + x^2) + x) + \mathrm{sgn}(z)\,(a - c, \lambda - d, b(y + x^2) - x)\big)$$

or, equivalently,

$$Z_\lambda(x, y, z) = \begin{cases} X^\lambda(x, y, z) = (a, \lambda, b(y + x^2)) & \text{if } z \geq 0, \\ Y(x, y, z) = (c, d, x) & \text{if } z \leq 0, \end{cases} \tag{2}$$

with $a, b, c, d, \lambda \in \mathbb{R}, b \cdot c \neq 0$ and λ arbitrarily small. Moreover, system (2), with the parameter nearby $\lambda = 0$, represents an important class of PSVFs exhibiting some interesting properties.

The main result of the paper is:

Theorem 1. *Let Z_λ given by (2). If $a < 0$, $b < 0$, $c > 0$, $d < 0$, $a + bd > 0$ and λ is arbitrarily small, then:*

- *Z_λ is asymptotically stable at the origin when $\lambda \geq 0$ and*
- *Z_λ is not Lyapunov stable at the origin when $\lambda < 0$.*

In [9], the asymptotic stability of system (2) is analyzed in the case $bc > 0$.

The paper is organized as follows: In Sect. 2, we give an overall description of the problem and formalize some basic concepts on PSVFs. In Sect. 3, some auxiliary results are stated and we pave the way in order to prove the main results in Sect. 4. In Sect. 5, we establish the equivalence between a whole class of PSVFs exhibiting certain intrinsic properties and system (2), with $\lambda = 0$.

2. Preliminaries

This work fits in a general program for understanding the local qualitative behavior around typical singularities of PSVFs in dimension $n > 2$. In [24], a list of future directions concerning this program is presented. This program (also called the *Thom-Smale program*) has as a first step the classification of codimension zero singularities. In [14], a list of such codimension zero 3D singularities and their respective normal forms is presented.

However, we recommend the perusal of the paper [10] where we improve and complement such a list. Bifurcation problems of PSVFs also are considered in [14]. But in this pioneer work, the main goal of our paper, the asymptotic stability, is not carried on. In fact, as long as we know, the paper [9] is the first one that addresses this topic for the 3D codimension one PSVFs singularities.

The orbit solutions of the system through points on the switching region $\Sigma = \{(x, y, z); z = 0\}$, when they exist, are defined by Filippov's convention, see [14]. Such systems are widely used to model phenomena in Electrical and Electronic Engineering, Physics, Economics, Biology among other areas (examples of applications can be found in [3,4,13,20] and references therein).

The main tool used in this paper is the theory of the contact between a vector field and the boundary of a manifold, since the traditional methods involving Lyapunov functions do not apply here, see [6–8,15,19,24].

2.1. Filippov's convention

Consider $K = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 < \delta\}$ where $\delta > 0$ is arbitrarily small and $\Sigma = \{(x, y, z) \in K \,|\, h(x, y, z) = 0\}$, where in this paper we consider $h(x, y, z) = z$. Clearly the switching manifold Σ is the separating boundary of the regions $\Sigma_+ = \{(x, y, z) \in K \,|\, z \geq 0\}$ and $\Sigma_- = \{(x, y, z) \in K \,|\, z \leq 0\}$.

Let $\mathfrak{X}^r$ be the space of C^r-vector fields on K endowed with the C^r-topology with $r = \infty$ or $r > 1$ large enough for our purposes. Call Ω^r the space of vector fields $Z_\lambda : K \to \mathbb{R}^3$ such that

$$Z_\lambda(x, y, z) = \begin{cases} X^\lambda(x, y, z), & \text{for } (x, y, z) \in \Sigma_+, \\ Y(x, y, z), & \text{for } (x, y, z) \in \Sigma_-, \end{cases}$$

where $X^\lambda = (X_1^\lambda, X_2^\lambda, X_3^\lambda)$ and $Y = (Y_1, Y_2, Y_3)$ are in $\mathfrak{X}^r$. We consider $\Omega^r = \mathfrak{X}^r \times \mathfrak{X}^r$ endowed with the product topology and denote any element in Ω^r by $Z_\lambda = (X^\lambda, Y)$, which we will accept to be multivalued in points of Σ. The basic results of differential equations, in this context, were stated by Filippov in [14]. Related theories can be found in [3,22,24] and references therein. On Σ, we distinguish the following regions:

- Crossing region: $\Sigma^c = \{p \in \Sigma \,|\, X_3^\lambda(p) \cdot Y_3(p) > 0\}$. Moreover, we denote $\Sigma^{c+} = \{p \in \Sigma \,|\, X_3^\lambda(p) > 0, Y_3(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma \,|\, X_3^\lambda(p) < 0, Y_3(p) < 0\}$.
- Sliding region: $\Sigma^s = \{p \in \Sigma \,|\, X_3^\lambda(p) < 0, Y_3(p) > 0\}$.

- Escaping region: $\Sigma^e = \{p \in \Sigma \,|\, X_3^\lambda(p) > 0, Y_3(p) < 0\}$.

When $q \in \Sigma^s$, following the Filippov's convention, the **sliding vector field** associated with $Z_\lambda \in \Omega^r$ is the vector field $\widehat{Z}_\lambda^s$ tangent to Σ^s expressed in coordinates as

$$\widehat{Z}_\lambda^s(q) = \frac{1}{(Y_3 - X_3^\lambda)(q)} \left((X_1^\lambda Y_3 - Y_1 X_3^\lambda)(q), (X_2^\lambda Y_3 - Y_2 X_3^\lambda)(q), 0\right). \tag{3}$$

Associated with (3), there exists the planar **normalized sliding vector field**

$$Z_\lambda^s(q) = \left((X_1^\lambda Y_3 - Y_1 X_3^\lambda)(q), (X_2^\lambda Y_3 - Y_2 X_3^\lambda)(q)\right). \tag{4}$$

Note that if $q \in \Sigma^s$, then $X_3^\lambda(q) < 0$ and $Y_3(q) > 0$. So, $(Y_3 - X_3^\lambda)(q) > 0$ and therefore, $\widehat{Z}_\lambda^s$ and Z_λ^s are topologically equivalent in Σ^s, Z_λ^s has the same orientation as $\widehat{Z}_\lambda^s$, and it can be C^r-extended to the closure $\overline{\Sigma^s}$ of Σ^s.

The points $q \in \Sigma^s$ such that $Z_\lambda^s(q) = 0$ are called **pseudo-equilibria of Z**, and the points $p \in \Sigma$ such that $X_3^\lambda(p) \cdot Y_3(p) = 0$ are called **tangential singularities of $\mathbf{Z}_\lambda$** (i.e., the trajectory through p is tangent to Σ).

2.2. Distinguished singularities

In our approach, we deal with two important distinguished tangential singularities: the points where the contact is either quadratic or cubic, which are called **fold and cusp singularities**, respectively. When p is a fold singularity of both smooth vector fields, we say that p is a **twofold singularity**, and when p is a cusp singularity for one smooth vector field and a fold singularity for the other one, we say that p is a **cusp-fold singularity**, see Fig. 1. In [11,12,17,18], twofold singularities are studied, and in [4,5], applications of such theory in electrical and control systems, respectively, are exhibited.

In $\mathbb{R}^3$, through a generic cusp singularity emanate two branches of fold singularities, see Fig. 1. In one side of this branch, it appears visible fold singularities and in the other one invisible fold singularities.

The contact between the smooth vector field X^λ and the switching manifold $\Sigma = h^{-1}(0)$ is characterized by the expression $X^\lambda h(p) = \left\langle X^\lambda(p), \nabla h(p)\right\rangle$ (note that in this paper we have $X^\lambda h(p) = X_3^\lambda(p)$) and $(X^\lambda)^i h(p) = \left\langle X^\lambda(p), \nabla (X^\lambda)^{i-1} h(p)\right\rangle, i \geq 2$, where $\langle .,. \rangle$ is the canonical inner product in $\mathbb{R}^3$. We emphasize that this notation is useful to characterize the kind of contact between the trajectories of X^λ and Σ. Moreover, it has nothing to do with the composition of $(X^\lambda)^i$ and h (see the pioneering text [14], Chapter 2, page 52). So, in a formal language, p is a fold point of X^λ if $X^\lambda h(p) = 0$ and $(X^\lambda)^2 h(p) \neq 0$. Moreover, p is a cusp point of X^λ if $X^\lambda h(p) = (X^\lambda)^2 h(p) = 0$, $(X^\lambda)^3 h(p) \neq 0$ and $\{dh(p), d(X^\lambda h)(p), d((X^\lambda)^2 h)(p)\}$ is a linearly independent set. Considering the expression (2), the kind of contact of X^λ, Y with Σ can be characterized in terms of the parameters a, b, c, d and λ. More precisely, we have $X^\lambda h(p) = b(y+x^2)$, $(X^\lambda)^2 h(p) = 2abx + \lambda b$, $(X^\lambda)^3 h(p) = 2a^2 b$, $Yh(p) = x$ and $Y^2 h(p) = c$. Therefore, we get that the origin is a twofold singularity for Z_λ if $bc\lambda \neq 0$ and it is a cusp-fold singularity if $\lambda = 0$ and $abc \neq 0$.

We define the sets of tangential singularities $S_{X^\lambda} = \{p \in \Sigma \,|\, X_3^\lambda(p) = 0\}$ and $S_Y = \{p \in \Sigma \,|\, Y_3(p) = 0\}$.

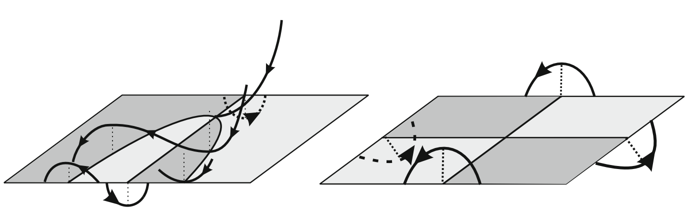

FIG. 1. On the *left* it appears a cusp-fold singularity and on the *right* a twofold singularity

Remark 1. *Consider* (4)*. It is easy to see that a twofold or a cusp-fold singularity is an equilibrium point of the normalized sliding vector field.*

Notations

- We denote the flow of a vector field $W \in \mathfrak{X}^r$ by $\phi_W(t,p)$ where $t \in I$ with $I = I(p,W) \subset \mathbb{R}$ being an interval depending on $p \in K$ and W.
- Given a vector field W defined in $A \subset K$, we denote the **backward trajectory** $\phi_W^-(A)$ (respectively, **forward trajectory** $\phi_W^+(A)$) the set of all negative (respectively, positive) orbits of W through points of A.
- We denote the boundary of an arbitrary set $A \subset K$ by ∂A.

2.3. The first return map

In this section, we consider the dynamics given by the interaction of the smooth vector fields X^λ and Y out of the sliding region $\Sigma^{s,e}$. Our purpose is to define a first return map on this region, involving the flows of X^λ and Y.

Consider $p \in \Sigma^{c+}$ and suppose that there exists $t_1(p)$, the positive return time of the trajectory of X passing through p.

We put $\phi_{X^\lambda}(t_1(p),p) = p_1 \in \Sigma$. Let $t_2(p_1)$ be the positive return time of trajectory of Y passing through $p_1 \in \Sigma$. Note that we want $p_1 \in \Sigma^{c-}$. Observe that the domain of the first return map is an open subset $U \subset \Sigma^{c+}$ such that $\phi_{X^\lambda}(t_1(p),U) \subset \Sigma^{c-}, t_1(p) > 0$ and $t_2(p) > 0$.

The *first return map* associated with $Z_\lambda = (X^\lambda, Y)$ is defined by the composition $\varphi_{Z_\lambda}(p) = \phi_Y(t_2(p_1), \phi_{X^\lambda}(t_1(p),p))$.

The next result provides the explicit expression of such map:

Lemma 1. *Considering the PSVF given in* (2)*, the expression of the first return map* $\varphi_{Z_\lambda} : U \subset \Sigma^{c+} \to \Sigma^{c+}$ *is given by*

$$\varphi_{Z_\lambda}(x,y) = \left(\frac{2ax+\Delta_1}{4a}, y + \frac{d(2ax+\Delta_1)}{2ac} + \frac{\lambda(-6ax-\Delta_1)}{4a^2} \right), \tag{5}$$

where $\Delta_1 = 3\lambda - \sqrt{9\lambda^2 + 36a\lambda x - 12a^2(x^2+4y)}$.

Proof. Let be $p = (x,y,0) \in \Sigma^{c+} = \{(x,y,0); X^\lambda h(x,y,0) > 0, Yh(x,y,0) > 0\} = \{(x,y,0); b(y+x^2) > 0, x > 0\}$. The flows of X^λ and Y are given by $\left(at + x, \lambda t + y, b\left(\frac{a^2t^3}{3} + at^2x + \frac{\lambda t^2}{2} + tx^2 + ty\right)\right)$ and $(ct + x, dt + y, c/2t^2 + xt)$, respectively. Solving $\phi_{X^\lambda}(t_1(p),p) \in \Sigma^{c-}$ and $\phi_Y(t_2(p_1),p_1) \in \Sigma^{c+}$, we obtain the expressions of the positive return times $t_1(p) = \frac{-6ax-\Delta_1}{4a^2}$ and $t_2(p_1) = \frac{2ax+\Delta_1}{2ac}$. Composing the flows, i.e., computating $\phi_Y(t_2(p_1), \phi_{X^\lambda}(t_1(p),p))$, we get the explicit expression of first return map. □

Note that we can extend smoothly φ_{Z_λ} to the boundary of Σ^c. In this way, the unique fixed point of φ_{Z_λ}, in a neighborhood of the origin, is the origin. Let $\Delta_2 = (ad)^2 - adc\lambda$. When $\lambda \neq 0$, the eigenvalues of $D\varphi_{Z_\lambda}$ at the origin are

$$\xi_\pm^\lambda = \frac{2ad - c\lambda \pm 2\sqrt{\Delta_2}}{c\lambda}, \tag{6}$$

the eigenvectors associated with ξ_+^λ and ξ_-^λ, respectively, are $v_\pm^\lambda = (\omega_\pm^\lambda, 1)$, where $\omega_\pm^\lambda = \frac{ac}{ad \pm \sqrt{\Delta_2}}$ and the eigenspaces associated with $\xi_\pm^\lambda$, respectively, are tangent to the straight lines

$$S_\pm^\lambda = \left\{ (x,y,0) \in \Sigma | x = \frac{ac}{ad \pm \sqrt{\Delta_2}} y \right\}. \tag{7}$$

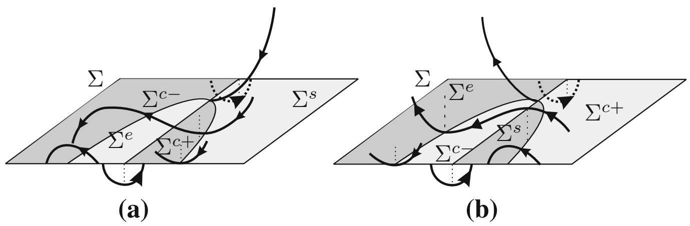

FIG. 2. The two possible local dynamics of Z_0 with hypothesis H_1 and H_2

3. Auxiliary results

3.1. The case $\boldsymbol{\lambda = 0}$

Consider (2), with $\lambda = 0$. In this case, Z_0 presents a cusp-fold singularity at the origin, since $abc \neq 0$, see Sect. 2.2. Note that $S_{X^0} = \{(x, y, 0) \in \Sigma \,|\, y = -x^2\}$ and $S_Y = \{(x, y, 0) \in \Sigma \,|\, x = 0\}$ are the sets of tangential singularities of X^0 and Y, respectively. At this moment, we stress that the notation X^0 represents the smooth vector field X^λ with $\lambda = 0$.

3.1.1. Local dynamics of the normalized sliding vector field. From (4), the normalized sliding vector field is given by

$$Z_0^s = (ax - bc(y + x^2), -db(y + x^2)).$$

So, the eigenvalues of Z_0^s are $\lambda_1^0 = a$ and $\lambda_2^0 = -db$ and the eigenspaces associated with λ_1^0 and λ_2^0, respectively, are

$$E_1^0 = \{(x, y, 0) \in \Sigma \,|\, y = 0\} \quad \text{and} \quad E_2^0 = \left\{(x, y, 0) \in \Sigma \,|\, y = \frac{(a + bd)x}{bc}\right\}. \tag{8}$$

In order to get Z_0 asymptotically stable at the cusp-fold singularity, some extra hypotheses must be imposed on the parameters:

Hypothesis 1. *(H_1): $c > 0$, i.e., the fold point generated by the vector field Y must be invisible.*

Hypothesis 2. *(H_2): $\lambda_1^0 = a < 0$ and $-\lambda_2^0 = bd > 0$, i.e., the origin must be asymptotically stable for Z_0^s.*

Note that, since $a < 0$ and $c > 0$, the flow of X^0 goes from the right to the left and the flow of Y goes from the left to the right with respect to the $x-$axis. Following H_1 and H_2, the phase portraits of Z_0, in Σ^s, are given by one of the following illustrations, in Fig. 2.

However, just at Case (a) of Fig. 2, the asymptotic stability is expected. In fact, in Case (b), it is easy to check that the smooth vector fields X^0 are not Lyapunov stable at the origin. So we consider the following hypothesis:

Hypothesis 3. *(H_3): $b < 0$, i.e., the cusp singularity generated by the vector field X^0 must be of the same topological type as described in Fig. 2,* Case (a).

By consequence of H_2 and H_3, we conclude that $d < 0$.

Lemma 2. *The eigenspace E_1^0 associated with λ_1^0 is tangent to the curve S_{X^0}, in Σ, at the origin.*

Proof. Straightforward according to (8). □

Hypothesis 4. *(H_4): $-a < bd \Rightarrow 0 < a + bd.$, i.e., E_2^0 is stronger than E_1^0 because $|\lambda_1| < |\lambda_2|$.*

Remark 2. *As an immediate consequence of H_4, we get $(bc)/(a + bd) < 0$ and $E_2^0 \cap \Sigma^s = \emptyset$, see Fig. 3.* In particular, this implies that the sliding region Σ^s is Z_0^s-invariant.

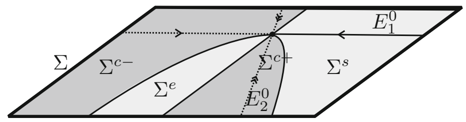

Fig. 3. Local dynamic of Z_0^s

3.1.2. Local dynamics of the first return map. Now, in order to determine the dynamics of the forward trajectories of Z_0, we consider the expression of the first return map, given in (5), with $\lambda = 0$. We get

$$\varphi_{Z_0}(x,y) = \left(\frac{ax - \sqrt{-3a^2(x^2+4y)}}{2a}, y + \frac{d(ax - \sqrt{-3a^2(x^2+4y)})}{ac} \right).$$

Given a point $p \in \mathbb{R}^3$, it is easy to see that the forward trajectory $\phi^+_{Z_0}(p)$ of Z passing through p intersects $\overline{\Sigma^s} \cup \overline{\Sigma^{c+}}$. In what follows, we prove that $\phi^+_{Z_0}(p) \cap \overline{\Sigma^s} \neq \emptyset$.

Lemma 3. *The image of the curve $y = -x^2$, with $x > 0$, by φ_{Z_0} is the curve $y = -\frac{x^2}{4} + 2\frac{d}{c}x$ with $x > 0$, i.e.,*

$$\varphi_{Z_0}(\{y = -x^2,\ \textit{with } x > 0\}) = \left\{ y = -\frac{x^2}{4} + 2\frac{d}{c}x,\ \textit{with } x > 0 \right\}.$$

Proof. Consider $p_0 = (u, -u^2, 0)$, with $u > 0$. The trajectory of X^0 through p_0 intersects Σ at $p_1 = (-2u, -u^2, 0)$ after a time $t_1 = -3u/a$. The trajectory of Y through p_1 intersects Σ at $p_2 = (2u, 4du/c - u^2, 0)$ after a time $t_2 = 4u/c$. Considering the change of variables $x = 2u$, after a time $\bar{t} = t_1 + t_2 = \frac{(4a-3c)u}{ac}$, the curve $y = -x^2$ returns to Σ at the curve $y = -\frac{x^2}{4} + 2\frac{d}{c}x$. □

Lemma 4. *The image of the curve $x = 0$, with $y < 0$, by φ_{Z_0} is the curve $y = -\frac{x^2}{3} + 2\frac{d}{c}x$ with $x > 0$, i.e.,*

$$\varphi_{Z_0}(\{x = 0,\ \textit{with } y < 0\}) = \left\{ y = -\frac{x^2}{3} + 2\frac{d}{c}x,\ \textit{with } x > 0 \right\}.$$

Proof. The proof is analogous to that one presented in the previous lemma, considering the change of variables $x = \sqrt{-3y_0}$. □

Lemma 5. *The image of the set Σ^{c+} by φ_{Z_0} remains between the curves $y = -\frac{x^2}{3} + 2\frac{d}{c}x$ and $y = -\frac{x^2}{4} + 2\frac{d}{c}x$, with $x > 0$, i.e.,*

$$\varphi_{Z_0}(\Sigma^{c+}) \subset \left\{ (x,y,0) \in \Sigma \,|\, -\frac{x^2}{3} + 2\frac{d}{c}x < y < -\frac{x^2}{4} + 2\frac{d}{c}x,\ \textit{with } x > 0 \right\}.$$

Proof. Given a point $p_0 = (x_0, y_0, 0) \in \Sigma^{c+}$ (where $x_0 > 0$ and $y_0 < 0$), the trajectory of X^0 by p_0 intersects Σ at $p_1 \in \Sigma^{c-}$ and the trajectory of Y by p_1 intersects Σ at p_2, where p_2 is situated between the curves $y = -\frac{x^2}{3} + 2\frac{d}{c}x$ and $y = -\frac{x^2}{4} + 2\frac{d}{c}x$ which correspond to the images of the curves $x = 0$, with $y < 0$ and $y = -x^2$, with $x > 0$, respectively. □

Lemma 6. *Given $p_0 = (x_0, y_0, 0) \in \overline{\Sigma^{c+}}$, call $p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0)$ and $p_n = (x_n, y_n, 0) = \varphi^n_{Z_0}(p_0)$, when it is well defined. Then $x_1 > x_0$ and $x_n \to \infty$ when $n \to \infty$.*

Proof. Given $p_0 = (x_0, y_0, 0) \in \overline{\Sigma^{c+}}$, a straightforward calculus shows that $x_1 = \frac{x_0}{2} + \frac{\sqrt{3}\sqrt{-(x_0^2+4y_0)}}{2}$ where $p_1 = (x_1, y_1, 0) = \varphi_{Z_0}(p_0)$. Since $p_0 \in \overline{\Sigma^{c+}}$ we conclude that $y_0 \leq -x_0^2 < -x_0^2/3$. So,

$$y_0 < -x_0^2/3 \Rightarrow -4x_0^2 - 12y_0 > 0 \Rightarrow (-3(x_0^2 + 4y_0)) > x_0^2$$
$$\Rightarrow \frac{\sqrt{-3(x_0^2+4y_0)}}{2} > \frac{x_0}{2} \Rightarrow x_1 > x_0.$$

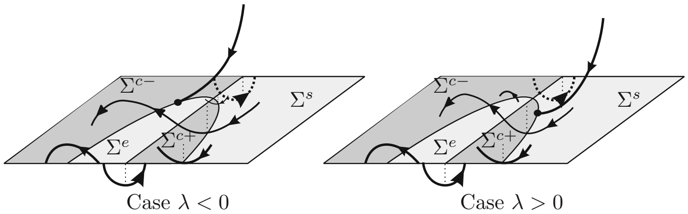

Fig. 4. The local dynamic of Z_λ, with hypotheses H_1 and H_3

A recursive analysis shows that $x_{n+1} > x_n$. In fact, repeating the previous argument

$$x_{n+1} = \frac{x_n + \sqrt{-3(x_n^2 + 4y_n)}}{2} > 2x_n \Rightarrow \frac{x_{n+1}}{x_n} > 2.$$

Since $\frac{x_{n+1}}{x_n} > 1$, from a test of convergence of sequences, we get $x_n \to \infty$. □

Proposition 1. *For all $p \in K$, it happens $\phi^+_{Z_0}(p) \cap \overline{\Sigma^s} \neq \emptyset$.*

Proof. As we observed above, given a point $p \in K$, it is easy to see that $\phi^+_{Z_0}(p) \cap [\overline{\Sigma^s} \cup \overline{\Sigma^{c+}}] \neq \emptyset$. So, it is enough to prove that $\varphi^{n_0}_{Z_0}(\overline{\Sigma^{c+}}) \subset \overline{\Sigma^s}$ for some $n_0 > 0$. By Lemmas 3, 4 and 5 we obtain that

$$\varphi_{Z_0}(\overline{\Sigma^{c+}}) \subset \left\{(x, y, 0) \in \Sigma \,|\, \frac{x^2}{3} + 2\frac{d}{c}x \leq y \leq -\frac{x^2}{4} + 2\frac{d}{c}x, \text{ with } x > 0\right\}.$$

By Lemma 6, there exists $n_0 > 0$ such that $p_{n_0} = (x_{n_0}, y_{n_0}, 0) = \varphi^{n_0}_{Z_0}(p)$ satisfies $y_{n_0} + x^2_{n_0} \geq 0$, since $y_{n_0-1} > \frac{-64d^2}{9c^2}$ by Lemma 5. Therefore $p_{n_0} \in \overline{\Sigma^s}$. □

3.2. The case $\boldsymbol{\lambda \neq 0}$

When $\lambda \neq 0$, we consider the normal form (2), presenting a twofold singularity at the origin, since $bc \neq 0$, see Sect. 2.2. The local dynamics for Z_λ is given in Fig. 4. The tangential sets S_{X^λ} and S_Y remain the same as the ones established in Sect. 3.1.

In fact, in previous works, there were considered the asymptotic stability of PSVFs that present a twofold singularity at origin. More precisely, in [23] it was proved the asymptotic stability of PSVFs presenting a twofold singularity in the case where the first return map is of elliptical type, i.e., has non-real eigenvalues. In [17], it was proved that PSVFs presenting a twofold singularity with real eigenvalues of first return map are not Lyapunov stable (similarly when $\lambda < 0$), but the basin of attraction is exhibited. Nevertheless, in the present work, by means of a variation on the parameter λ it is possible to observe changes on the eigenvectors of the sliding vector fields and on the stabilities of the first return maps and sliding vector fields. Roughly speaking, the variation of λ produces a variation on the stability of the twofold singularity.

3.2.1. Local dynamics of the normalized sliding vector fields. According to (4), the normalized sliding vector field is given by

$$Z^s_\lambda = (ax - bc(y + x^2), \lambda x - db(y + x^2)). \tag{9}$$

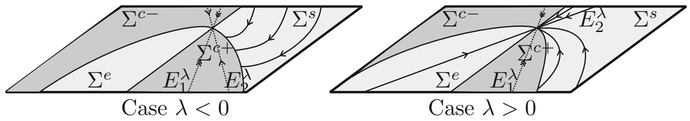

FIG. 5. The local dynamics of Z^s_λ with hypothesis H_1–H_4

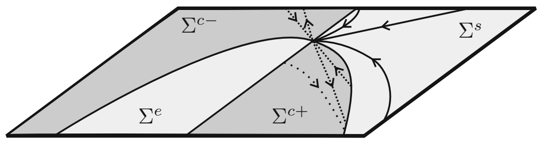

FIG. 6. Dynamics of φ_{Z_λ} and Z^s_λ, under the hypothesis H_1–H_4 with $\lambda > 0$

Let $\Delta_3 = (a+bd)^2 - 4bc\lambda$. The eigenvalues of $DZ^s_\lambda(0,0)$ are $\lambda^\lambda_1 = \frac{a-bd-\sqrt{\Delta_3}}{2}$ and $\lambda^\lambda_2 = \frac{a-bd+\sqrt{\Delta_3}}{2}$, and the eigenspaces associated with λ^λ_1 and λ^λ_2, respectively, are

$$\begin{aligned} E^\lambda_1 &= \left\{(x,y,0) \in \Sigma \,|\, y = \tfrac{2\lambda}{a+bd-\sqrt{\Delta_3}}x\right\} \\ E^\lambda_2 &= \left\{(x,y,0) \in \Sigma \,|\, y = \tfrac{2\lambda}{a+bd+\sqrt{\Delta_3}}x\right\}. \end{aligned} \tag{10}$$

Under the hypotheses H_1–H_4, we get that $\lambda^\lambda_{1,2}$ are negative and E^λ_1 is stronger than E^λ_2. Besides, we obtain the following results:

Lemma 7. *The eigenspace $E^\lambda_1 \subset \Sigma^c$ and*
- (a) $E^\lambda_2 \subset [\Sigma^s \cup \Sigma^e]$ *when* $\lambda > 0$*;*
- (b) $E^\lambda_2 \subset \Sigma^c$ *when* $\lambda < 0$*, see Fig. 5.*

Proof. It is straightforward according to (10). □

Note that in case $\lambda < 0$, the sliding vector fields have a transient behavior in Σ^s, and as a consequence, all the orbits in Σ^s will be iterated by the first return map, whereas in case $\lambda > 0$, Z^s_λ is asymptotically stable at the origin.

3.2.2. Local dynamics of the first return map. Now, in order to determine the dynamics of the positive trajectories of Z_λ, we consider the first return map φ_{Z_λ} of Z_λ, whose expression is given in (5).

Lemma 8. *Under the hypothesis H_1–H_4, the origin is a hyperbolic saddle fixed point for φ_{Z_λ} and*
- (a) $S^\lambda_\pm \subset \Sigma^c$ *when* $\lambda > 0$ *and*
- (b) $S^\lambda_+ \subset \Sigma^c, S^\lambda_- \subset [\Sigma^e \cup \Sigma^s]$ *when* $\lambda < 0$.

Besides, S^λ_+ (resp. S^λ_-) is an expansive (resp. contractive) direction.

Proof. It follows by the expressions (6) and (7), of the eigenvalues and the eigenspaces of $D\varphi_{Z_\lambda}(0)$, respectively. □

By Lemma 8, when $\lambda > 0$, we get that given $p \in \Sigma^{c+}$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}_{Z_\lambda}(p) \in \Sigma^s$. And Lemma 7, under the hypothesis H_1–H_4, provides that Z^s_λ is asymptotically stable at the origin. See Fig. 6, when the dotted lines in Σ^{c+} represent the iterated of φ_{Z_λ} and the line in Σ^s the dynamic of Z^s_λ.

In this case, we get that Z_λ is asymptotically stable at the origin, under the hypothesis H_1–H_4.

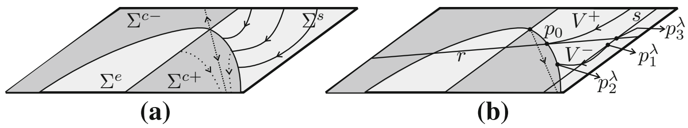

FIG. 7. In **a**, we have the local dynamic of φ_{Z_λ} (*dotted line*) and Z_λ^s (in Σ^s). In **b** are presented the straight lines r and s, the points $p_0, p_1^\lambda, p_2^\lambda$ and p_3^λ and the regions V^+ and V^-

When $\lambda < 0$, Lemma 7 provides that the trajectories of the sliding vector field Z_λ^s have a transient behavior in Σ^s. In fact, in this case, we shall prove that Z_λ is not Lyapunov stable at the origin (which corresponds to a twofold singularity).

Lemma 9. *Given $p_0 = (x_0, -x_0^2, 0)$ (under the curve $y = -x^2$), with $x_0 > 0$, we get*

$$\varphi_{Z_\lambda}(x_0, -x_0^2, 0) = \left(2x_0 + \frac{3\lambda}{2a}, -x_0^2 - \frac{3\lambda(\lambda + 2ax_0)}{2a^2} + \frac{d(3\lambda + 4ax_0)}{ac}, 0\right).$$

Proof. Straightforward. □

We denote $\varphi_{Z_\lambda}(p_0) = p_1^\lambda = (x_1^\lambda, y_1^\lambda, 0)$, which can be situated at $\overline{\Sigma^{c+}}$ and in this case, by Lemma 8, and its distance to the origin increases when compared to p_0. Otherwise, p_1^λ can be situated at Σ^s, and in this case, the trajectory by this point slides to the parabola $y = -x^2$. The intersection point will be called $p_2^\lambda = (x_2^\lambda, y_2^\lambda, 0) = (x_2^\lambda, -(x_2^\lambda)^2, 0)$. As the origin is an attractor for Z_λ^s, we have to discuss the behavior of the mapping φ_{Z_λ} at the origin.

Denote by $d(p, 0)$ the euclidian distance between the point p to the origin.

Lemma 10. *Under the hypotheses H_1–H_4 with $\lambda < 0$ and with the previous notation,*

$$d(p_2^\lambda, 0) > d(p_0, 0).$$

Proof. From (9), the straight line

$$r : (x(\alpha), y(\alpha), 0) = (x_0, -x_0^2, 0) + \alpha(ax_0, \lambda x_0, 0) \text{ with } \alpha \in \mathbb{R},$$

is tangent to the trajectory of Z_λ^s by $p_0 = (x_0, -x_0^2, 0)$.

Note that r splits Σ^s into two regions, denoted by V^+ and V^-. Consider the vertical straight line $s : p = p_1^\lambda + \beta(0, 1, 0)$, with $\beta \in \mathbb{R}$, see Fig. 7. We get that $r \cap s = p_3^\lambda$, where $p_3^\lambda = (x_3^\lambda, y_3^\lambda, 0) = \left(x_1^\lambda, -x_0^2 + \frac{\lambda}{a}\left(\frac{3\lambda}{2a} + x_0\right), 0\right)$. Observe that $y_1^\lambda < y_3^\lambda$. Therefore p_1^λ and, consequently p_2^λ, are situated at the region V^- described in Fig. 7. So, $d(p_2^\lambda, 0) > d(p_0, 0)$. □

Lemma 11. *Z_λ is not Lyapunov stable at the origin for $\lambda < 0$.*

Proof. From Lemma 7, we get that Z_λ^s has a transient behavior, i.e., Σ^{c+} is an attractor set for Z_λ^s, and by Lemma 8, we conclude that all points in Σ^{c+} converge to $\overline{\Sigma^s}$. So, in order to analyze the stability of Z_λ at the origin, it is sufficient to study the intersection of the trajectories of Z_λ with $\partial\Sigma^s$. By Lemma 10, we obtain that the distance between the origin and a point in $\overline{\Sigma^s}$ increases along the time. Therefore, we conclude that Z_λ is not Lyapunov stable at the origin for this case. □

Remark 3. *As consequence of Lemmas 2, 7 and 8, we get that Z_0 has codimension of at least two, because the eigenspaces of the normalized sliding vector field and the first return map are tangent to S_{X^0}.*

4. Proof of Theorem 1

4.1. Case $\lambda = 0$

When $\lambda = 0$, by Proposition 1, the trajectories of all points in $\mathbb{R}^3$ intersect $\overline{\Sigma^s}$. By hypotheses H_2 and H_4, the ω-limit set of all trajectories in $\overline{\Sigma^s}$ is the origin. So, Z_0 is asymptotically stable at the origin.

4.2. Case $\lambda > 0$

When $\lambda > 0$, by Item (a) of Lemma 8 the trajectories of all points in K intersect $\overline{\Sigma^s}$. Moreover, the origin is a hyperbolic attractor for Z^s_λ and by Lemma 7 we get $E_1^\lambda \subset \Sigma^c$ and $E_2^\lambda \subset \Sigma^s$, for $x > 0$. Therefore, the positive orbits of Z_λ follows the orbits of Z^s_λ. So, Z_λ is asymptotically stable at the origin.

4.3. Case $\lambda < 0$

When $\lambda < 0$, the result is an immediate consequence of Lemma 11.

5. The mild equivalence

In this section, we prove that all PSVFs presenting a cusp-fold singularity p, with some intrinsic properties, are topologically equivalent to (2), with $\lambda = 0$. First of all, let us announce the relation of equivalence that we are considering. Our intention with this kind of equivalence is to provide a simple characterization of generic (typical) singularities. For more details, see [9,16,21].

As stated in [9], the **topological type** of $Z \in \Omega^r$ at $p \in \Sigma$ is characterized by all oriented orbits passing through or tending to p (in positive or negative time).

Definition 1. *We say that $Z = (X, Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \Omega^r$ presenting switching manifolds Σ and $\widetilde{\Sigma}$, respectively, are* **mild equivalent** *if the following conditions are satisfied:*

(i) *$X|_{\Sigma_+}$ is topologically equivalent to $\widetilde{X}|_{\widetilde{\Sigma}_+}$,*

(ii) *$Y|_{\Sigma_-}$ is topologically equivalent to $\widetilde{Y}|_{\widetilde{\Sigma}_-}$ and*

(iii) *There is a homeomorphism $\xi : \Sigma \to \widetilde{\Sigma}$ such that the topological types of Z at $p \in \Sigma$ and of $\widetilde{Z}$ at $\widetilde{p} = \xi(p) \in \widetilde{\Sigma}$ are equivalent (coincide).*

From this definition, the concept of **mild structural stability** *in Ω^r is naturally obtained.*

Now, we write the homeomorphism that provides this equivalence.

Proposition 2. *Let $\overline{Z} = (\overline{X}, \overline{Y}) \in \Omega^r$ such that $\overline{X}$ has a cusp singularity at p, $\overline{Y}$ has an invisible fold singularity at p, the sliding vector field $\overline{Z}^s$ has an attractor node at p, a branch of the weak manifold and the strong manifold associated with the node of $\overline{Z}^s$ are all placed in Σ^c, and a branch of the weak manifold is placed in Σ^s. Then, $\overline{Z}$ is mild equivalent to Z given by (2), with $\lambda = 0$.*

Proof. Let $\xi(p) = 0$. See Fig. 8. Since $\overline{X}$ has a cusp at p, there exists on Σ a branch $S_{\overline{X}_i}$ (resp. $S_{\overline{X}_v}$) of invisible (resp. visible) fold points of $\overline{X}$ starting at p. The same holds for X. By arc length parametrization, consider the identification $\xi(S_{\overline{X}_i} \cap \overline{V}) = S_{X_i} \cap V$ (resp. $\xi(S_{\overline{X}_v} \cap \overline{V}) = S_{X_v} \cap V$).

Also, since $\overline{Y}$ has a fold at p, there exist on Σ branches $S_{\overline{Y}_-}$ and $S_{\overline{Y}_+}$ of invisible fold points of $\overline{Y}$ starting at p. W.l.g. consider that $S_{\overline{Y}_-}$ (resp. $S_{\overline{Y}_+}$) is the branch that belongs to the boundary of $\overline{\Sigma}^e$ (resp.

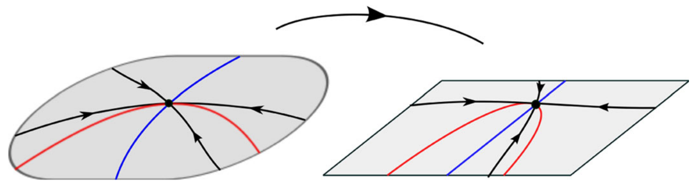

FIG. 8. The homeomorphism that provides the mild equivalence

Σ^s). The same holds for Y. By arc length parametrization, consider the identification $\xi(S_{\overline{Y}_-} \cap \overline{V}) = S_{Y_-} \cap V$ (resp. $\xi(S_{\overline{Y}_+} \cap \overline{V}) = S_{Y_+} \cap V$).

Consider $\overline{U}^s$ (resp. $\overline{U}^w$) the strong (resp. weak) manifold associated with the node of the sliding vector field $\overline{Z}^s$. The same for Z^s. By arc length parametrization, consider the identification $\xi(\overline{U}^s \cap \overline{\Sigma}^{c^+} \cap \overline{V}) = U^s \cap \Sigma^{c^+} \cap V$ (resp. $\xi(\overline{U}^s \cap \overline{\Sigma}^{c^-} \cap \overline{V}) = U^s \cap \Sigma^{c^-} \cap V$, $\xi(\overline{U}^w \cap \overline{\Sigma}^{c^-} \cap \overline{V}) = U^w \cap \Sigma^{c^-} \cap V$ and $\xi(\overline{U}^w \cap \overline{\Sigma}^s \cap \overline{V}) = U^w \cap \Sigma^s \cap V$).

Let $\overline{V}_i$, $i = 1, \dots, 8$, be the part of the curve $\overline{V} \cap \overline{\Sigma}$ between $\overline{q}_i$ and $\overline{q}_{i+1}$ where $\overline{q}_1 = S_{\overline{Y}_-} \cap \overline{V}, \overline{q}_2 = \overline{U}^s \cap \overline{\Sigma}^{c^+} \cap \overline{V}, \overline{q}_3 = S_{\overline{X}_v} \cap \overline{V}, \overline{q}_4 = \overline{U}^w \cap \overline{\Sigma}^s \cap \overline{V}, \overline{q}_5 = S_{\overline{Y}_+} \cap V, \overline{q}_6 = \overline{U}^s \cap \overline{\Sigma}^{c^-} \cap \overline{V}, \overline{q}_7 = \overline{U}^w \cap \overline{\Sigma}^e \cap \overline{V}, \overline{q}_8 = S_{\overline{X}_i} \cap \overline{V}$ and $\overline{q}_9 = \overline{q}_1$.

Consider the same for V and $Z = (X, Y)$. Since p (resp. the origin) is a node for $\overline{Z}^s$ (resp. Z^s) and by the position of the invariant stable manifolds, the negative trajectory of $\overline{Z}^s$ (resp. Z^s) by a point $\overline{q}$ of $S_{\overline{Y}_-}$ (resp. q of S_{Y_-}) meets $\overline{V}_1$ at a point $\overline{r}$ (resp. V_1 at a point r). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{q}\,\overline{r}}$ and $\widehat{q\,r}$. So, $\xi(\overline{A}_1) = A_1$ where $\overline{A}_1$ is the region of Σ bounded by $V_1 \cup (\overline{U}^s \cap \overline{\Sigma}^{c^+}) \cup S_{\overline{Y}_-}$ and A_1 is the analogous for $Z = (X, Y)$. The positive trajectory of $\overline{Z}^s$ (resp. Z^s) by a point $\overline{q}$ of $S_{\overline{Y}_-}$ (resp. q of S_{Y_-}) meets $S_{\overline{X}_i}$ at a point $\overline{s}$ (resp. S_{X_i} at a point s). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{q}\,\overline{s}}$ and $\widehat{q\,s}$. So, $\xi(\overline{A}_8) = A_8$ where $\overline{A}_8$ is the region of Σ bounded by $V_8 \cup S_{\overline{X}_i} \cup S_{\overline{Y}_-}$ and A_8 is the analogous for $Z = (X, Y)$. The positive trajectory of $\overline{Z}^s$ (resp. Z^s) by a point $\overline{s}$ of $S_{\overline{X}_i}$ (resp. s of S_{X_i}) converges to $\overline{p}$ (resp. the origin 0). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{s}\,\overline{p}}$ and $\widehat{s\,0}$. So, $\xi(\overline{A}_7) = A_7$ where $\overline{A}_7$ is the region of Σ bounded by $V_7 \cup (\overline{U}^w \cap \overline{\Sigma}^{c^-}) \cup S_{\overline{X}_i}$ and A_7 is the analogous for $Z = (X, Y)$. Repeat this argumentation and $\xi(\overline{A}_j) = A_j$ for $j = 2, \dots, 6$. In this way, $\xi(\overline{\Sigma}) = \Sigma$ and the topological types of Z at 0 and of $\overline{Z}$ at $\overline{p}$ are equivalent (coincide).

Reduce, if necessary, the neighborhood $\overline{V}$ (resp. V) in such a way that $\phi_{\overline{Y}}(\overline{V}_5 \cup \overline{V}_6 \cup \overline{V}_7 \cup \overline{V}_8) = \overline{V}_1 \cup \overline{V}_2 \cup \overline{V}_3 \cup \overline{V}_4$ (resp. $\phi_Y(V_5 \cup V_6 \cup V_7 \cup V_8) = V_1 \cup V_2 \cup V_3 \cup V_4$). So, given a point $\overline{q} \in (\overline{A}_5 \cup \overline{A}_6 \cup \overline{A}_7 \cup \overline{A}_8)$ (resp. $q \in (A_5 \cup A_6 \cup A_7 \cup A_8)$), the positive trajectory of $\overline{Y}$ by $\overline{q}$ (resp. Y by q) meets $\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3 \cup \overline{A}_4$ (resp. $A_1 \cup A_2 \cup A_3 \cup A_4$) at a point $\overline{r}$ (resp. r). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{q}\,\overline{r}}$ and $\widehat{q\,r}$. So, $\xi(\overline{\Sigma}_-) = \Sigma_-$.

Now, let us construct the homeomorphism in Σ_+. Since $\xi(S_{\overline{X}_v}) = S_{X_v}$, given $\overline{q} \in S_{\overline{X}_v}$ (resp. $q \in S_{X_v}$) the trajectory of $\overline{X}$ by $\overline{q}$ (resp. X by q) meets either $\overline{\Sigma}^{c-}$ or $\overline{V}$ (resp. Σ^{c-} or V) for positive time at $\overline{r}$ (resp. r) and meets $\overline{V}$ (resp. V) for negative time at $\overline{s}$ (resp. s). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{r}\,\overline{q}}$ and $\widehat{r\,q}$ (resp. $\widehat{\overline{q}\,\overline{s}}$ and $\widehat{q\,s}$). Let $H_{\overline{X}}$ (resp. H_X) be given by $H_{\overline{X}} = \phi_{\overline{X}}(S_{\overline{X}_v}) \cap \overline{\Sigma}^{c-}$ (resp. $H_X = \phi_X(S_{X_v}) \cap \Sigma^{c-}$), $\overline{q}_{10} = H_{\overline{X}} \cap \overline{V}$ (resp. $q_{10} = H_X \cap V$) and $\overline{A}_9$ (resp. A_9) the region of $\overline{\Sigma}^{c-}$ (resp. Σ^{c-}) bounded by $H_{\overline{X}} \cap S_{\overline{X}_i} \cap \overline{L}$ where $\overline{L}$ (resp. $H_X \cap S_{X_i} \cap L$ where L) is the part of the arc $\overline{V}_8$ between $\overline{q}_{10}$ and $\overline{q}_9 = \overline{q}_1$ (resp. V_8 between q_{10} and $q_9 = q_1$). Given a point $\overline{q} \in \overline{A}_9$ (resp. $q \in A_9$), the negative trajectory of $\overline{X}$ by $\overline{q}$ (resp. X by q) meets $\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_8$ (resp. $A_1 \cup A_2 \cup A_8$) at a point $\overline{r}$ (resp. r). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{q}\,\overline{r}}$ and $\widehat{q\,r}$. Given a point

$\overline{q} \in (\overline{A}_3 \cup \overline{A}_4 \cup \overline{A}_5 \cup \overline{A}_6 \cup (\overline{A}_7 \backslash \overline{A}_9))$ (resp. $q \in (A_3 \cup A_4 \cup A_5 \cup A_6 \cup (A_7 \backslash A_9)))$, the negative trajectory of $\overline{X}$ by $\overline{q}$ (resp. X by q) meets $\overline{V}$ (resp. $A_1 \cup$) at a point $\overline{s}$ (resp. s). By arc length parametrization, identify the arcs of trajectory $\widehat{\overline{q}\,\overline{s}}$ and $\widehat{q\,s}$. So, $\xi(\overline{\Sigma}_+) = \Sigma_+$.

This finishes the proof, and we conclude that $\overline{Z}$ and Z are mild equivalent. □

6. Asymptotic stability in a perturbed relay system

In this section, we illustrate Theorem 1 through a model found in the theory of nonlinear oscillations. We point out that such model was discussed in detail in [1,2], under another point of view. Consider the relay system expressed as $z''' = \alpha\,\mathrm{sgn}(z)$ where $\alpha \in \mathbb{R}$. This system can be rewritten as

$$Z(x,y,z) = \begin{cases} X(x,y,z) = (y, -\alpha, x) & \text{if } z \geq 0, \\ Y(x,y,z) = (y, \alpha, x) & \text{if } z \leq 0. \end{cases} \tag{11}$$

Consider the perturbation of (11) given by

$$\overline{Z}(x,y,z) = \begin{cases} \overline{X}(x,y,z) = \left(y - \frac{\alpha\lambda}{a^2}, -\alpha, x\right) & \text{if } z \geq 0, \\ Y(x,y,z) = \left(y + c, \alpha, x + \frac{1}{\alpha} y^2\right) & \text{if } z \leq 0. \end{cases} \tag{12}$$

where $1/\alpha$ and $\alpha\lambda/a^2$ are small enough such that (11) and (12) are sufficiently C^r-close. First, let us apply the following change of variables on $\overline{X}$:

$$(u,v,w) = \left(\frac{-a^2}{\alpha}\left(x + \frac{1}{2\alpha}y^2\right), -\sqrt{2}\frac{a}{\alpha}y, z\right).$$

So, we get

$$X(u,v,w) = (\dot{u},\dot{v},\dot{w}) = \left(\tfrac{-a^2}{\alpha}\left(\dot{x} + \tfrac{2}{2\alpha}y\dot{y}\right), -\sqrt{2}\tfrac{a}{\alpha}\dot{y}, \dot{z}\right)$$
$$= \left(\lambda, \sqrt{2}a, \tfrac{-\alpha}{a^2}\left(u + \tfrac{1}{2}v^2\right)\right).$$

Now let us apply the following change of variables on $X(u,v,w)$:

$$(U,V,W) = \left(\frac{1}{\sqrt{2}}v, u, \frac{-ba^2}{\alpha}w\right).$$

So, we get

$$X(U,V,W) = (\dot{U},\dot{V},\dot{W}) = \left(\frac{1}{\sqrt{2}}\dot{v}, \dot{u}, \frac{-ba^2}{\alpha}\dot{w}\right) = (a, \lambda, b(V + U^2)). \tag{13}$$

In a similar way, let us apply the following change of variables on $\overline{Y}$:

$$(U,V,W) = \left(x + \frac{1}{2\alpha}y^2, \frac{d}{\alpha}y, z\right).$$

So, we get

$$Y(U,V,W) = (\dot{U},\dot{V},\dot{W}) = \left(\dot{x} + \frac{2}{2\alpha}y\dot{y}, \frac{d}{\alpha}\dot{y}, \dot{z}\right) = (c, d, U). \tag{14}$$

By (13) and (14), we get that $Z(U,V,W) = (X(U,V,W), Y(U,V,W))$ is a small perturbation of the relay system (11) which is in the normal form (2). So, its stability can be obtained from Theorem 1.

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