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On path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{F}P^n)$

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Abstract. We estimate the number of homotopy types of path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{F}P^n)$ from the m-sphere $\mathbb{S}^m$ to the projective space $\mathbb{F}P^n$ for $\mathbb{F}$ being the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the skew algebra $\mathbb{H}$ of quaternions. Then, the homotopy types of path-components of the mapping spaces $M(E\Sigma^m, \mathbb{F}P^n)$ for the suspension $E\Sigma^m$ of a homology m-sphere Σ^m are studied as well.

Introduction

Let $M(X, Y)$ be the space of all continuous (not necessarily based) maps between connected spaces X and Y with the compact-open topology. The space $M(X, Y)$ is in general disconnected with path-components in one-to-one correspondence with the set $\langle X, Y\rangle$ of (free) homotopy classes of maps. Furthermore, different components may—and frequently do—have distinct homotopy types. A basic problem in homotopy theory is to determine whether two path-components are homotopy equivalent or, more generally, to classify the path-components of $M(X, Y)$ up to homotopy type. For a basepoint $x_0 \in X$, we have the evaluation map $\omega\colon M(X, Y) \to Y$ defined by $\omega(g) = g(x_0)$ for $g \in M(X, Y)$, which is a fibration [13]. Let $M_f(X, Y)$ be the path-component of $M(X, Y)$ that contains a given map $f\colon X \to Y$, and write $\omega_f\colon M_f(X, Y) \to Y$ for the restriction of ω to $M_f(X, Y)$.

Works on these classification problems date back to the 1940s. Whitehead [24, Theorem 2.8] considered the case $X = \mathbb{S}^m$, the m-sphere, and proved that $M_f(\mathbb{S}^m, Y)$ is homotopy equivalent to $M_0(\mathbb{S}^m, Y)$ if and only if the evaluation

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fibration $\omega_f \colon M_f(\mathbb{S}^m, Y) \to Y$ admits a section, where 0 denotes the constant map. Then, results by Hansen [10,11], and later by McClendon [19] treated this classification problem as well.

Hansen [12] obtained a classification of path-components of $M(M^n, \mathbb{S}^n)$, where M^n is a suitably restricted n-manifold. The aim of [21] was to describe an approach which fits in well with Hansen's and leads to some further progress. In [6] the authors made use of Gottlieb groups of spheres to deal with path-components of the spaces $M(\mathbb{S}^{n+k}, \mathbb{S}^n)$ for $8 \le k \le 13$.

Let now G be a Lie group and $P \to X$ a principal G-bundle over a space X with gauge group $\mathcal{G}$. Atiyah and Bott [1] proved the following. *Let BG be the classifying space for G. Then in homotopy theory $B\mathcal{G} = M_P(X, BG)$. Here the subscript P denotes the path-component of a map of X into BG which induces P* (i.e., the principal G-bundle $P \to X$).

The case in which X is a manifold and $Y = BG$ has been the subject of extensive recent research by Crabb, Kono, Sutherland, Tsukuda, and others (see e.g., [4,14,22]).

Then, Lupton and Smith [17] gave a general method that may be effectively applied to the question of whether two path-components of a function space $M(X, Y)$ have the same homotopy type provided X is a co-H-space.

Now, let $\mathbb{F}P^n$ denote the projective n-space with $n \ge 1$ for $\mathbb{F}$ being the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the skew algebra $\mathbb{H}$ of quaternions. The purpose of this paper is to classify homotopy types of path-components of the space $M(\mathbb{S}^m, \mathbb{F}P^n)$ for certain m, n.

Section 1 fixes some notations and definitions, and necessary results as well. In particular, we recall a very important result on $\omega_f \colon M_f(\mathbb{S}^m, X) \to X$ for our further investigations which was obtained by G.W. Whitehead [24] with a correction by J.H.C. Whitehead [25] and then generalized in [16] for $\omega_f \colon M_f(EA, X) \to X$, where EA is the reduced suspension of the space A. Next, we make use of some results from [7, Chapter 1] to identify path-components of $M(\mathbb{S}^m, \mathbb{F}P^n)$ for some m, n. Section 2 takes up the systematic study of the path-components of $M(\mathbb{S}^m, \mathbb{R}P^n)$ and its main results are stated in Propositions 2.3 and 2.4. Section 3 is concentrated with the homotopy type of path-components of mapping spaces $M(\mathbb{S}^m, \mathbb{C}P^n)$ and the main result is presented in Proposition 3.2. Section 4 is devoted to the homotopy type of path-components of mapping spaces $M(\mathbb{S}^m, \mathbb{H}P^n)$ and some computations are presented in Proposition 4.2. Finally, Sect. 5 presents some background on homology m-spheres Σ^m for $m \ge 1$ to study path-components of the mapping spaces $M(\Sigma^m, \mathbb{F}P^n)$. The main result of that section is Theorem 5.5 which uses results from [7, Chapter 2] and Sect. 2 to estimate homotopy types of path-components of the mapping spaces $M(E\Sigma^m, \mathbb{F}P^n)$.

1. Prerequisites

For topological spaces X and Y, let $M(X, Y)$ be the space of all continuous maps equipped with the compact-open topology. In the pointed case, for this space we write $M_*(X, Y)$. Let $M_f(X, Y)$ (resp. $M_{*f}(X, Y)$) be the path-component of

$M(X, Y)$ (resp. $M_*(X, Y)$) containing a (resp. pointed) map $f\colon X \to Y$ and denote by $M_0(X, Y)$ (resp. $M_{*0}(X, Y)$) the path-component containing the constant map.

Given pointed spaces X and Y, write $\langle X, Y\rangle$ and $[X, Y]$ for the sets of homotopy classes of free and pointed maps, respectively. It is well known that there is an action of the fundamental group $\pi_1(Y)$ on $[X, Y]$ and there is a bijection $\langle X, Y\rangle \approx [X, Y]/\pi_1(Y)$.

Let $[f]$ denote the homotopy class of a map $f\colon X \to Y$. If X is a Hausdorff space then for any homotopy $H\colon I \times X \to Y$ there is an associated continuous map $\overline{H}\colon I \to M(X, Y)$ and consequently $[f] \subseteq M_f(X, Y)$.

If the evaluation map $\mathrm{ev}\colon M(X, Y) \times X \to Y$ is continuous and $\sigma\colon I \to M(X, Y)$ is a path then its adjoint $\hat{\sigma}\colon I \times X \xrightarrow{\sigma\times \mathrm{id}_X} M(X, Y) \times X \xrightarrow{\mathrm{ev}} Y$ is also continuous and so $[f] \supseteq M_f(X, Y)$.

In particular, if X is a compactly generated space then for any path $\sigma\colon I \to M(X, Y)$ its adjoint $\hat{\sigma}\colon I \times X \to Y$ is continuous and $[f] \supseteq M_f(X, Y)$. Because X is also Hausdorff we derive that $[f] = M_f(X, Y)$ and the set $\langle X, Y\rangle$ coincides with the set of all path-components of the space $M(X, Y)$.

As pointed out by Whitehead [24] all path-components of $M_*(\mathbb{S}^n, X)$ for the n-sphere $\mathbb{S}^n$ have the same homotopy type. Moreover, Lang [16, Lemma 2.1] generalized this result for the space $M_*(EA, X)$, where EA is the reduced suspension of the pointed space A. In general, distinct path-components of the space $M_*(X, Y)$ need not be homotopy equivalent.

Therefore, the following problem naturally arises.

Problem 1.1. Given spaces X and Y, classify all path-components of the spaces $M(X, Y)$ and $M_*(X, Y)$ up to homotopy type.

We consider a variety of cases beginning with the most classical, mentioning progress on Problem 1.1, when appropriate.

Throughout the rest of this paper, all spaces are assumed to be pointed compactly generated and all maps are pointed maps. Further, we do not distinguish between a map and its homotopy class and we freely use notation from Toda's book [23].

Given a space X, denote by $\pi_n(X)$ its nth homotopy group. Further, for $f \in \pi_n(X)$ there is the evaluation fibration ω_f (at the base point of $\mathbb{S}^n$)

$$M_{*f}(\mathbb{S}^n, X) \hookrightarrow M_f(\mathbb{S}^n, X) \xrightarrow{\omega_f} X.$$

A very important result for our further investigations was obtained by G.W. Whitehead [24] with a correction by J.H.C. Whitehead [25] and then generalized by Lang [16]. It describes the boundary operator in the exact homotopy sequence for the evaluation fibration by means of the Whitehead product.

Theorem 1.2. (G.W. Whitehead) *If $f \in \pi_n(X)$ then for any $i \geq 1$ there is a commutative diagram*

$$\begin{array}{ccccccc}
\cdots \longrightarrow \pi_{i+1}(X) & \xrightarrow{\partial_f} & \pi_i(M_{*f}(\mathbb{S}^n, X)) & \longrightarrow & \pi_i(M_f(\mathbb{S}^n, X)) & \longrightarrow \cdots \\
 & {\scriptstyle p_f}\searrow & \approx \downarrow {\scriptstyle H_f} & & & \\
 & & \pi_{i+n}(X), & & &
\end{array}$$

where ∂_f is the boundary operator in the exact homotopy sequence for the evaluation fibration ω_f, the map H_f is the adjoint isomorphism (also called the Hurewicz isomorphism), and p_f is the Whitehead product with f, i.e., $p_f(g) = [f, g]$ for any $g \in \pi_{i+1}(X)$.

Recall that the *Gottlieb groups* $G_n(X)$ for $n \geq 1$ of a space X have been defined in [9] as evaluation subgroups

$$G_n(X) = \operatorname{Im}(\mathrm{ev}_*\colon \pi_n(M_*(X, X), \mathrm{id}_X) \to \pi_n(X)).$$

For a wide class of spaces X, the group $G_n(X)$ is the subgroup of the nth homotopy group $\pi_n(X)$ containing all elements $f\colon \mathbb{S}^n \to X$ such that $f \vee \mathrm{id}_X\colon \mathbb{S}^n \vee X \to X$ extends (up to homotopy) to a map $F\colon \mathbb{S}^n \times X \to X$, i.e., the diagram

$$\begin{array}{ccc} \mathbb{S}^n \vee X & \xrightarrow{f \vee \mathrm{id}_X} & X \\ \big\downarrow & \nearrow_{F} & \\ \mathbb{S}^n \times X & & \end{array}$$

commutes (up to homotopy). It is easy to observe that $G_n(X) = \pi_n(X)$ provided X is an H-space.

Further, the *Whitehead center groups* $P_n(X)$ for $n \geq 1$ defined in [9] consist of all elements $f \in \pi_n(X)$ such that the Whitehead product $[f, g] = 0$ for any $g \in \pi_m(X)$ and $m \geq 1$. Certainly, $G_n(X) \subseteq P_n(X)$ for all $n \geq 1$. Notice that for any $f \in P_n(X)$ and $i \geq 1$, Theorem 1.2 implies that there is a short exact sequence

$$0 \to \pi_{i+n}(X) \to \pi_i(M_f(\mathbb{S}^n, X)) \to \pi_i(X) \to 0. \tag{1.1}$$

More generally, a map $f\colon A \to X$ such that $f \vee \mathrm{id}_X\colon A \vee X \to X$ extends (up to homotopy) to a map $F\colon A \times X \to X$, i.e., the diagram

$$\begin{array}{ccc} A \vee X & \xrightarrow{f \vee \mathrm{id}_X} & X \\ \big\downarrow & \nearrow_{F} & \\ A \times X & & \end{array}$$

commutes (up to homotopy) is called *cyclic*. Write $G(A, X)$ for the set of homotopy classes of all cyclic maps $A \to X$ and notice that $G(A, X)$ is a subgroup of $[A, X]$ provided A is a co-H-space.

Yoon [26] observed a connection between $G(EA, X)$ and path-components of $M(EA, X)$. In particular, [26, Theorems 4.5 and 4.9] imply:

Proposition 1.3. *For any $f \in \pi_n(X)$ the following are equivalent:*

(i) *the evaluation fibrations $(M_f(\mathbb{S}^n, X), \omega_f, X)$ and $(M_0(\mathbb{S}^n, X), \omega_0, X)$ are fibre homotopy equivalent;*

(ii) *the evaluation fibration* $(M_f(\mathbb{S}^n, X), \omega_f, X)$ *has a section;*
(iii) $f \in G_n(X)$.

Proposition 1.4. *If* $f, f' \in \pi_n(X)$ *and* $f + f' \in G_n(X)$ *or* $f - f' \in G_n(X)$ *then the corresponding evaluation fibrations* $(M_f(\mathbb{S}^n, X), \omega_f, X)$ *and* $(M_{f'}(\mathbb{S}^n, X), \omega_g, X)$ *are fibre homotopy equivalent. In particular, the path-components* $M_f(\mathbb{S}^n, X)$ *and* $M_{\pm f'}(\mathbb{S}^n, X)$ *are homotopy equivalent.*

Since $G_n(X) \subseteq P_n(X)$, by Proposition 1.3 the sequence (1.1) splits and we get an isomorphism

$$\pi_i(M_f(\mathbb{S}^n, X)) \approx \pi_{i+n}(X) \oplus \pi_i(X) \tag{1.2}$$

for any $f \in G_n(X)$ and $i \geq 1$.

Further, since path-components of the space $M(A, X)$ are in one-to-one correspondence with the set $\langle A, X \rangle$, we may say that the quotient map $[A, X] \to \langle A, X \rangle$ is given by $[f] \mapsto M_f(A, X)$ for $[f] \in [A, X]$. Then, Lupton–Smith's results [17, Corollary 2.5 and (8)] imply that $[A, X] \to \langle A, X \rangle$ yields a surjection

$$[A, X]/G(A, X) \to \{M_f(A, X);\ f \in \langle A, X \rangle\}/\simeq,$$

provided A is a co-H-space, where $\simeq$ is the homotopy equivalence relation. In particular, if X is an H-space then all path-components of $M(A, X)$ are homotopy equivalent. Further, for $A = \mathbb{S}^n$ there is also a surjection

$$\pi_n(X)/G_n(X) \to \{M_f(\mathbb{S}^n, X);\ f \in \langle \mathbb{S}^n, X \rangle\}/\simeq. \tag{1.3}$$

Now, write $\pi_n(X)/\pm G_n(X)$ for the quotient set of $\pi_n(X)$ by the relation $\sim$ with $f \sim g$ provided $f - g \in G_n(X)$ or $f + g \in G_n(X)$ for $f, g \in \pi_n(X)$. Since, by Proposition 1.4, the path-components $M_f(\mathbb{S}^n, X)$ and $M_{-f}(\mathbb{S}^n, X)$ are homotopy equivalent for any $f \in \pi_n(X)$, formula (1.3) leads to a surjection

$$\pi_n(X)/\pm G_n(X) \to \{M_f(\mathbb{S}^n, X);\ f \in \langle \mathbb{S}^n, X \rangle\}/\simeq. \tag{1.4}$$

To make use of formula (1.4) in the sequel, given real numbers x, y, we write

$$\chi(x, y) = \left\lceil \frac{(x-1)(y-1)}{2} \right\rceil + \left\lceil \frac{(x-1)}{2} \right\rceil + \left\lceil \frac{(y-1)}{2} \right\rceil + 1,$$

where $\lceil r \rceil = \min\{k \in \mathbb{Z};\ k \geq r\}$ for any real r, where $\mathbb{Z}$ is the integers. Further, we recall:

Lemma 1.5. ([6, Lemma 1]) *For positive integers* m, m', n, n' *with* $m \mid n$, $m' \mid n'$ *and* $n, n' \geq 1$, *let* $\mathbb{Z}_m \times \mathbb{Z}_{m'} < \mathbb{Z}_n \times \mathbb{Z}_{n'}$, $m\mathbb{Z} \times \mathbb{Z}_{m'} < \mathbb{Z} \times \mathbb{Z}_{n'}$ *and* $m\mathbb{Z} \times m'\mathbb{Z} < \mathbb{Z} \times \mathbb{Z}$ *be the obvious inclusions. Then:*

$$|(\mathbb{Z}_n \times \mathbb{Z}_{n'})/\pm(\mathbb{Z}_m \times \mathbb{Z}_{m'})| = \chi(\tfrac{n}{m}, \tfrac{n'}{m'});$$
$$|(\mathbb{Z} \times \mathbb{Z}_{n'})/\pm(m\mathbb{Z} \times \mathbb{Z}_{m'})| = \chi(m, \tfrac{n'}{m'});$$
$$|(\mathbb{Z} \times \mathbb{Z})/\pm(m\mathbb{Z} \times m'\mathbb{Z})| = \chi(m, m').$$

In particular, $|(\mathbb{Z}_n \times \mathbb{Z}_{n'})/\pm(\mathbb{Z}_m \times \mathbb{Z}_{n'})| = |\mathbb{Z}_n/\pm\mathbb{Z}_m| = \chi(\frac{n}{m}, 1)$.

Let now $\mathbb{F}$ be the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, or the quaternion algebra $\mathbb{H}$. We write $\mathbb{F}P^n$ for the projective n-space over $\mathbb{F}$, $\gamma_n \colon \mathbb{S}^{d(n+1)-1} \to \mathbb{F}P^n$ for the quotient map where $d = \dim_{\mathbb{R}} \mathbb{F}$, and $i_{\mathbb{F}} \colon \mathbb{S}^d = \mathbb{F}P^1 \hookrightarrow \mathbb{F}P^n$ for the canonical inclusion.

2. Path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{R}P^n)$

We study the homotopy type of path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{R}P^n)$.

Since $\mathbb{R}P^1$ is a topological group, any two path-components of the space $M(\mathbb{S}^1, \mathbb{R}P^1)$ are homeomorphic and, by (1.1) the evaluation map $M_{ki_{\mathbb{R}}}(\mathbb{S}^1, \mathbb{R}P^1) \to \mathbb{R}P^1$ is a (weak) homotopy equivalence, for any $ki_{\mathbb{R}} \in \pi_1(\mathbb{R}P^1) = \mathbb{Z}\{i_{\mathbb{R}}\}$, the infinite cyclic group. Further, because $\pi_m(\mathbb{R}P^1) = 0$ for $m \geq 2$, the space $M(\mathbb{S}^m, \mathbb{R}P^1)$ is path-connected and, again by (1.1) the evaluation map $M(\mathbb{S}^m, \mathbb{R}P^1) \to \mathbb{R}P^1$ is a (weak) homotopy equivalence.

In view of [7, Corollary 2.34] it holds $G_m(\mathbb{R}P^2) = \pi_m(\mathbb{R}P^2)$ for $m \geq 3$. Notice that $G_m(\mathbb{R}P^n) = \pi_m(\mathbb{R}P^n)$ for $m \geq 1$ and $n = 3, 7$ because $\mathbb{R}P^3$ and $\mathbb{R}P^7$ are H-spaces. Hence, by (1.3) all path-components of the space $M(\mathbb{S}^m, \mathbb{R}P^n)$ are homotopy equivalent for $m \geq 3$ and $n = 2$, or $m \geq 1$ and $n = 3, 7$. Further, the space $M(\mathbb{S}^m, \mathbb{R}P^n)$ is path-connected for $2 \leq m < n$. Consequently (1.2) leads to:

Remark 2.1. There are isomorphisms:

(1) $\pi_i(M_f(\mathbb{S}^m, \mathbb{R}P^2)) \approx \pi_{i+m}(\mathbb{R}P^2) \oplus \pi_i(\mathbb{R}P^2)$ for any $f \in \pi_m(\mathbb{R}P^2)$, $i \geq 1$ and $m \geq 3$;

(2) $\pi_i(M_f(\mathbb{S}^m, \mathbb{R}P^n)) \approx \pi_{i+m}(\mathbb{R}P^n) \oplus \pi_i(\mathbb{R}P^n)$ for any $f \in \pi_m(\mathbb{R}P^n)$, $i, m \geq 1$ and $n = 1, 3, 7$;

(3) $\pi_i(M(\mathbb{S}^m, \mathbb{R}P^n)) \approx \pi_{i+m}(\mathbb{R}P^n) \oplus \pi_i(\mathbb{R}P^n)$ for $i \geq 1$ and $2 \leq m < n$.

Next, by [8] we have

$$G_1(\mathbb{R}P^n) = \begin{cases} 0, & \text{for } n \text{ even}, \\ \pi_1(\mathbb{R}P^n), & \text{for } n \text{ odd}, \end{cases}$$

and (1.3) implies that any two path-components of $M(\mathbb{S}^1, \mathbb{R}P^n)$ are homotopy equivalent for $n \geq 3$ odd. Further, by [20] it holds

$$G_n(\mathbb{R}P^n) = \begin{cases} 0, & \text{for } n \text{ even}, \\ 2\pi_n(\mathbb{R}P^n), & \text{for } n \neq 1, 3, 7 \text{ odd}. \end{cases}$$

Hence, by Proposition 1.3 the spaces $M_f(\mathbb{S}^n, \mathbb{R}P^n)$ and $M_{f'}(\mathbb{S}^n, \mathbb{R}P^n)$ are homotopy equivalent for $n \neq 1, 3, 7$ odd and any $f, f' \in 2\pi_n(\mathbb{R}P^n)$.

Now, we extend the discussion above for the missing cases.

Lemma 2.2. (1) *The path-components $M_f(\mathbb{S}^{2m}, \mathbb{R}P^{2m})$ and $M_{f'}(\mathbb{S}^{2m}, \mathbb{R}P^{2m})$ are homotopy equivalent if and only if $f = \pm f'$ for $m \geq 1$.*

(2) *The number of homotopy types of path-components of $M(\mathbb{S}^{4m-1}, \mathbb{R}P^{2m})$ is finite, for $m \neq 1, 2, 4$.*

(3) *The path-components $M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ and $M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ are homotopy equivalent if and only if $m = 1, 2, 4$.*

(4) *$M(\mathbb{S}^1, \mathbb{R}P^{2m})$ has two path-components, $M_0(\mathbb{S}^1, \mathbb{R}P^{2m})$ and $M_{i_{\mathbb{R}}}(\mathbb{S}^1, \mathbb{R}P^{2m})$, and they are not homotopy equivalent for $m \geq 1$.*

Proof. We mainly use the result stated in Theorem 1.2.

(1): If $f = \pm f'$ then, by Proposition 1.4, the path-components $M_f(\mathbb{S}^{2m}, \mathbb{R}P^{2m})$ and $M_{f'}(\mathbb{S}^{2m}, \mathbb{R}P^{2m})$ are homotopy equivalent for $m \geq 1$.
Conversely, given $f \in \pi_{2m}(\mathbb{R}P^{2m})$ the evaluation fibration

$$M_{*f}(\mathbb{S}^{2m}, \mathbb{R}P^{2m}) \hookrightarrow M_f(\mathbb{S}^{2m}, \mathbb{R}P^{2m}) \xrightarrow{\omega_f} \mathbb{R}P^{2m}$$

yields the exact sequence

$$\pi_{2m}(\mathbb{R}P^{2m}) \xrightarrow{[f,-]} \pi_{4m-1}(\mathbb{R}P^{2m}) \to \pi_{2m-1}(M_f(\mathbb{S}^{2m}, \mathbb{R}P^{2m})) \to 0$$

for $m > 1$. Then, mimicking the proof of [11, Theorem 3.1], we get that both groups $\pi_{2m-1}(M_f(\mathbb{S}^{2m}, \mathbb{R}P^{2m}))$ and $\pi_{2m-1}(M_{f'}(\mathbb{S}^{2m}, \mathbb{R}P^{2m}))$ are not isomorphic provided $f \neq \pm f'$.
Finally, for $m = 1$ we have the exact sequence

$$\pi_2(\mathbb{R}P^2) \xrightarrow{[f,-]} \pi_3(\mathbb{R}P^2) \to \pi_1(M_f(\mathbb{S}^2, \mathbb{R}P^2)) \to \pi_1(\mathbb{R}P^2) \to 0.$$

Because $[\iota_2, \iota_2] = 2\eta_2$, we deduce that $\mathrm{Im}[f, -]$ and $\mathrm{Im}[f', -]$ are different subgroups of $\pi_3(\mathbb{R}P^2)$ provided $f \neq \pm f'$. Consequently, the groups $\pi_1(M_f(\mathbb{S}^2, \mathbb{R}P^2))$ and $\pi_1(M_{f'}(\mathbb{S}^2, \mathbb{R}P^2))$ are not isomorphic as they are finite groups with different orders.

(2) [1]: We show that for $m \neq 1, 2, 4$ we have $a_m(2m-1)!/8\pi_{4m-1}(\mathbb{R}P^{2m}) \subseteq G_{4m-1}(\mathbb{R}P^{2m})$, where $a_m = 1$ for m even and $a_m = 2$ for m odd.
Let $SO(m)$ be the m-special orthogonal group and write $\Delta\colon \pi_i(\mathbb{S}^{2m}) \to \pi_{i-1}(SO(2m))$ for the boundary map coming from the fibration $SO(2m) \hookrightarrow SO(2m+1) \to \mathbb{S}^{2m}$. Then, by [18] the order of $\Delta[\iota_{2m}, \iota_{2m}]$ is $a_m(2m-1)!/8$, where $a_m = 1$ is m is even and $a_m = 2$ if m is odd, provided $m \neq 1, 2, 4$. In view of [7, Corollary 2.34] we deduce $a_m(2m-1)!/8\pi_{4m-1}(\mathbb{R}P^{2m}) \subseteq G_{4m-1}(\mathbb{R}P^{2m})$. Then, (2) follows by formula (1.3).

(3): Obviously $m = 1, 2, 4$ implies that both path-components $M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ and $M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ are homotopy equivalent.
Conversely, notice that the evaluation fibration

$$M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}) \hookrightarrow M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}) \xrightarrow{\omega_0} \mathbb{R}P^{2m-1}$$

leads to the exact sequence

$$\pi_{2m-1}(\mathbb{R}P^{2m-1}) \xrightarrow{[0,-]} \pi_{4m-3}(\mathbb{R}P^{2m-1}) \to \pi_{2m-2}(M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})) \to 0$$

and consequently, to an isomorphism $\pi_{4m-3}(\mathbb{R}P^{2m-1}) \xrightarrow{\approx} \pi_{2m-2}(M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}))$.
Further, the evaluation fibration

$$M_{*\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}) \hookrightarrow M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}) \xrightarrow{\omega_{\gamma_{2m-1}}} \mathbb{R}P^{2m-1}$$

[1] We are deeply grateful to Juno Mukai for indicating reference [18] needed in this proof.

leads to the exact sequence

$$\pi_{2m-1}(\mathbb{R}P^{2m-1}) \xrightarrow{[\gamma_{2m-1},-]} \pi_{4m-3}(\mathbb{R}P^{2m-1}) \\ \to \pi_{2m-2}(M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})) \to 0.$$

But, the supposed homotopy equivalence of the path-components $M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ and $M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})$ determines an isomorphism

$$\pi_{2m-2}(M_0(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})) \xrightarrow{\approx} \pi_{2m-2}(M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1})).$$

Since the group $\pi_{4m-3}(\mathbb{R}P^{2m-1})$ is finite, we deduce that the epimorphism

$$\pi_{4m-3}(\mathbb{R}P^{2m-1}) \to \pi_{2m-2}(M_{\gamma_{2m-1}}(\mathbb{S}^{2m-1}, \mathbb{R}P^{2m-1}))$$

above is an isomorphism. Hence, $[\gamma_{2m-1}, \gamma_{2m-1}] = 0$ and consequently $m = 1, 2, 4$.

(4): Recall that $\pi_1(\mathbb{R}P^{2m}) = \mathbb{Z}_2\{i_{\mathbb{R}}\}$, the cyclic group of order two for $m \geq 1$. Given $f \in \pi_1(\mathbb{R}P^{2m})$ with $m > 1$, the evaluation fibration

$$M_{*f}(\mathbb{S}^1, \mathbb{R}P^{2m}) \hookrightarrow M_f(\mathbb{S}^1, \mathbb{R}P^{2m}) \xrightarrow{\omega_f} \mathbb{R}P^{2m}$$

leads to the exact sequence

$$\pi_{2m}(\mathbb{R}P^{2m}) \xrightarrow{[f,-]} \pi_{2m}(\mathbb{R}P^{2m}) \to \pi_{2m-1}(M_f(\mathbb{S}^1, \mathbb{R}P^{2m})) \to 0.$$

If $f = 0$ then the sequence above implies an isomorphism

$$\pi_{2m}(\mathbb{R}P^{2m}) \xrightarrow{\approx} \pi_{2m-1}(M_0(\mathbb{S}^1, \mathbb{R}P^{2m})).$$

If $f = i_{\mathbb{R}}$ then by [2, (4.1–3)] it holds $[i_{\mathbb{R}}, \gamma_{2m}] = -2\gamma_{2m} \neq 0$. Then, the isomorphism

$$\pi_{2m}(\mathbb{R}P^{2m})/\operatorname{Im}([i_{\mathbb{R}}, -]) \xrightarrow{\approx} \pi_{2m-1}(M_{i_{\mathbb{R}}}(\mathbb{S}^1, \mathbb{R}P^{2m}))$$

shows that the path-components $M_0(\mathbb{S}^1, \mathbb{R}P^{2m})$ and $M_{i_{\mathbb{R}}}(\mathbb{S}^1, \mathbb{R}P^{2m})$ are not homotopy equivalent.
Finally, for $m = 1$ and $f \in \pi_1(\mathbb{R}P^2)$ the evaluation fibration

$$M_{*f}(\mathbb{S}^1, \mathbb{R}P^2) \hookrightarrow M_f(\mathbb{S}^1, \mathbb{R}P^2) \xrightarrow{\omega_f} \mathbb{R}P^2$$

leads to the exact sequence

$$0 \to \pi_2(\mathbb{R}P^2)/\operatorname{Im}([f, -]) \to \pi_1(M_f(\mathbb{S}^1, \mathbb{R}P^2)) \to \pi_1(\mathbb{R}P^2) \to 0.$$

Since $[i_{\mathbb{R}}, \gamma_2] = -2\gamma_2 \neq 0$, we deduce again that the path-components $M_0(\mathbb{S}^1, \mathbb{R}P^2)$ and $M_{i_{\mathbb{R}}}(\mathbb{S}^1, \mathbb{R}P^2)$ are not homotopy equivalent and the proof is complete. □

Proposition 2.3. *Let $f \in \pi_{m+i}(\mathbb{R}P^m)$ and suppose that $M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m)$ and $M_0(\mathbb{S}^{m+i}, \mathbb{R}P^m)$ are homotopy equivalent for some $i \geq 0$ and $m \geq 1$. If $\gamma_{m*}G_{m+i}(\mathbb{S}^m) = P_{m+i}(\mathbb{R}P^m)$ then $f \in P_{m+i}(\mathbb{R}P^m)$. In particular, it holds if $m \geq 1$ is odd and $i \geq 0$, or $m = 2$ and $i \geq 0$, or $m \geq 4$ is even and $i \leq m-2$, or $(i, m) = (6, 6), (7, 4), (7, 6)$.*

Proof. Certainly, we may assume that $m \geq 2$. Let $f \in \pi_{m+i}(\mathbb{R}P^m)$. If $i = 0$ and m is even, by Lemma 2.2(1) we deduce that $f = 0$. For other cases, consider the evaluation fibration

$$M_{*f}(\mathbb{S}^{m+i}, \mathbb{R}P^m) \hookrightarrow M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m) \to \mathbb{R}P^m$$

which by Theorem 1.2, yields the following short exact homotopy sequence

$$0 \to \pi_{j+m+i}(\mathbb{R}P^m) \to \pi_j(M_0(\mathbb{S}^{m+i}, \mathbb{R}P^m)) \to \pi_j(\mathbb{R}P^m) \to 0$$

for $f = 0$ and $j \geq 1$. In particular, for $j = m-1$ with $m > 2$ there is an isomorphism

$$\pi_{2m+i-1}(\mathbb{R}P^m) \xrightarrow{\approx} \pi_{m-1}(M_0(\mathbb{S}^{m+i}, \mathbb{R}P^m)). \tag{2.1}$$

But the evaluation fibration above leads also to the exact homotopy sequence

$$\pi_m(\mathbb{R}P^m) \xrightarrow{[f,-]} \pi_{2m+i-1}(\mathbb{R}P^m) \to \pi_{m-1}(M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m)) \to 0$$

and consequently we get an isomorphism

$$\pi_{2m+i-1}(\mathbb{R}P^m)/\operatorname{Im}([f,-]) \approx \pi_{m-1}(M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m)).$$

Because of $\pi_{m-1}(M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m)) \xrightarrow{\approx} \pi_{m-1}(M_0(\mathbb{S}^{m+i}, \mathbb{R}P^m))$, the isomorphism in (2.1) yields $[f, \gamma_m] = 0$ since the group $\pi_{2m+i-1}(\mathbb{R}P^m)$ is finite provided $i \neq 0$ or m is odd. Since $f = \gamma_m \alpha$ for some $\alpha \in \pi_{m+i}(\mathbb{S}^m)$ we conclude that $\alpha \in G_{m+i}(\mathbb{S}^m)$ and then $f \in \gamma_{m*}G_{m+i}(\mathbb{S}^m) = P_{m+i}(\mathbb{R}P^m)$.

Now, for $m = 2$ the evaluation fibration above leads to the exact homotopy sequence

$$\pi_2(\mathbb{R}P^2) \xrightarrow{[f,-]} \pi_{i+3}(\mathbb{R}P^2) \to \pi_1(M_f(\mathbb{S}^{i+2}, \mathbb{R}P^2)) \to \pi_1(\mathbb{R}P^2) \to 0$$

which implies the sequence

$$0 \to \pi_{i+3}(\mathbb{R}P^2) \to \pi_1(M_0(\mathbb{S}^{i+2}, \mathbb{R}P^2)) \to \pi_1(\mathbb{R}P^2) \to 0.$$

Because of $\pi_1(M_f(\mathbb{S}^{i+2}, \mathbb{R}P^2)) \xrightarrow{\approx} \pi_1(M_0(\mathbb{S}^{i+2}, \mathbb{R}P^2))$, we obtain $[f, \gamma_2] = 0$. Since $f = \gamma_2 \alpha$ for some $\alpha \in \pi_{i+2}(\mathbb{S}^2)$, we conclude that $f \in \gamma_{2*}G_{i+2}(\mathbb{S}^2) = P_{i+2}(\mathbb{R}P^2)$.

In view of [7, Proposition 2.16, Theorem 2.19], we have that $\gamma_{m*}G_{m+i}(\mathbb{S}^m) = P_{m+i}(\mathbb{R}P^m)$ under conditions on i, m as in the statement of the current proposition and the proof follows. □

Proposition 2.4. *Consider the space $M(\mathbb{S}^{m+i}, \mathbb{R}P^m)$ for $1 \le i \le 7$ and $m \ge 1$. If $(i, m) \ne (3, 4)$, $(3, 2^k - 3)$ for $k \ge 4$, $(4, 4)$, $(5, 4)$, $(6, 4)$, $(5, 6)$, $(6, 2^k - 5)$ for $k \ge 5$, $(7, 8)$, $(7, 11)$ then the cardinality of the set $\{M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+i}, \mathbb{R}P^m \rangle\}/\simeq$ is:*

(i) *one for $m = 2, 6$ or $m \equiv 3 \pmod 4$, and two otherwise, for $i = 1$;*
(ii) *one for $m \equiv 2, 3 \pmod 4$, and two otherwise, for $i = 2$;*
(iii) *one for $m = 2, 3, 5$ or $m \equiv 7 \pmod 8$, two for $m \equiv 1, 3, 5 \pmod 8$ with $m \ge 9$, at most seven for $m \equiv 2 \pmod 4$ and $m \ge 6$ or $m = 12$, at most thirteen for $m \equiv 0 \pmod 4$ and $m \ge 8$ with $m \ne 12$, for $i = 3$;*
(iv) *one for $i = 4, 5$;*
(v) *one for $m = 2, 3$ or $m \equiv 4, 5, 7 \pmod 8$ with $m \ne 4$, two otherwise, for $i = 6$;*
(vi) *one for $m = 2, 3, 5, 7$ or $m \equiv 15 \pmod{16}$, at most eight or thirty one, for $m = 4, 6$ respectively, two for $m \ge 9$ odd and $m \not\equiv 15 \pmod{16}$, at most one hundred twenty one for $m \ge 10$ even, for $i = 7$.*

Proof. Recall that by [7, Proposition 2.16, Theorems 2.19, 2.41], $P_{m+i}(\mathbb{R}P^m) = G_{m+i}(\mathbb{R}P^m) = \gamma_{m*} G_{m+i}(\mathbb{S}^m)$ except for the following pairs: $(i, m) = (3, 4)$, $(3, 2^k - 3)$ for $k \ge 4$, $(4, 4)$, $(5, 4)$, $(6, 4)$, $(5, 6)$, $(6, 2^k - 5)$ for $k \ge 5$, $(7, 8)$, $(7, 11)$.

Hence, for the non-exceptional pairs (i, m), by formula (1.3) the cardinality of the set $\{M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+i}, \mathbb{R}P^m \rangle\}/\simeq$ is bounded above by the order of the quotient group

$$\pi_{m+i}(\mathbb{R}P^m)/\gamma_{m*} G_{m+i}(\mathbb{S}^m) \approx \pi_{m+i}(\mathbb{S}^m)/G_{m+i}(\mathbb{S}^m).$$

Furthermore, by Proposition 1.3 and [17, Theorem 3.10] the path-components $M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m)$ and $M_0(\mathbb{S}^{m+i}, \mathbb{R}P^m)$ are homotopy equivalent if and only if $f \in G_{m+i}(\mathbb{R}P^m)$.

If $i = 1$ then $\pi_{m+1}(\mathbb{S}^m) = \mathbb{Z}_2\{\eta_m\}$ for $m \ge 3$ and by [7, (1.15)] the cardinality of $\{M_f(\mathbb{S}^{m+1}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+1}, \mathbb{R}P^m \rangle\}/\simeq$ is one for $m = 2, 6$ or $m \equiv 3 \pmod 4$, and two otherwise.

If $i = 2$ then $\pi_{m+2}(\mathbb{S}^m) = \mathbb{Z}_2\{\eta_m^2\}$ for $m \ge 2$ and by [7, (1.16)] the cardinality of $\{M_f(\mathbb{S}^{m+2}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+2}, \mathbb{R}P^m \rangle\}/\simeq$ is one for $m \equiv 2, 3 \pmod 4$, and two otherwise.

If $i = 3$ and $m = 2, 3, 5$ or $m \equiv 7 \pmod 8$ then by [7, (1.32)] the cardinality of the set $\{M_f(\mathbb{S}^{m+3}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+3}, \mathbb{R}P^m \rangle\}/\simeq$ is one. If $m \equiv 1, 3, 5 \pmod 8$ with $m \ge 9$, then by [7, (1.32)] $\pi_{m+3}(\mathbb{S}^m) \approx \mathbb{Z}_{24}$ and $\nu_m \notin G_{m+3}(\mathbb{S}^m)$, so the cardinality is two. If $m \equiv 2 \pmod 4$ and $m \ge 6$ or $m = 12$ then by [7, (1.32)] the cardinality is at most seven. If $m \equiv 0 \pmod 4$ and $m \ge 8$ with $m \ne 12$ then again by [7, (1.32)] and (1.4) the cardinality is at most thirteen.

If $i = 4, 5$ then by [7, Proposition 1.30] the cardinality of the set $\{M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+i}, \mathbb{R}P^m \rangle\}/\simeq$ is one.

If $i = 6$ and $m = 2, 3$ or $m \equiv 4, 5, 7 \pmod 8$ with $m \ne 4$, then by [7, Proposition 1.16] the cardinality of $\{M_f(\mathbb{S}^{m+6}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+6}, \mathbb{R}P^m \rangle\}/\simeq$ is one, and it is two otherwise.

If $i = 7$ and $m = 2, 3, 5, 7$ or $m \equiv 15 \pmod{16}$, then by [7, Proposition 1.16 and (1.51)] the cardinality of $\{M_f(\mathbb{S}^{m+7}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+7}, \mathbb{R}P^m\rangle\}/\simeq$ is one. For $m = 4, 6$ then again by [7, Proposition 1.16] and (1.4) it is at most eight or thirty one, respectively. For $m \geq 9$ odd and $m \not\equiv 15 \pmod{16}$ by [7, (1.51)] and Proposition 2.3 the cardinality is two. For $m \geq 10$ even, it is at most one hundred twenty one, by [7, (1.51)], Proposition 2.3 and (1.4). □

The pairs (i, m) excluded on Proposition 2.4 are considered below.

Proposition 2.5. *The cardinality of the set* $\{M_f(\mathbb{S}^{m+i}, \mathbb{R}P^m);\ f \in \langle \mathbb{S}^{m+i}, \mathbb{R}P^m\rangle\}/\simeq$ *is:*

(i) *at most seventy four for* $(i, m) = (3, 4)$*;*
(ii) *at most two for* $(i, m) = (3, 2^k - 3)$*, for* $k \geq 4$*;*
(iii) *at least two for* $(i, m) = (4, 4)$*;*
(iv) *at least two for* $(i, m) = (5, 4)$*;*
(v) *at most five for* $(i, m) = (6, 4)$*;*
(vi) *at most sixteen for* $(i, m) = (5, 6)$*;*
(vii) *two for* $(i, m) = (6, 2^k - 5)$ *for* $k \geq 5$*;*
(viii) *at most one thousand two hundred sixty one for* $(i, m) = (7, 8)$*;*
(ix) *at most five for* $(i, m) = (7, 11)$.

Proof. (i): Because $\pi_7(\mathbb{R}P^4) = \mathbb{Z}\{\gamma_4\nu_4\} \oplus \mathbb{Z}_{12}\{\gamma_4 E\nu'\}$ hence $\pi_7(\mathbb{R}P^4)/12\pi_7(\mathbb{R}P^4) \approx \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}$. Since $G_7(\mathbb{R}P^4) \supseteq 12\pi_7(\mathbb{R}P^4)$ [7, Theorem 2.41(1)] and (1.4) the cardinality of the set $\{M_f(\mathbb{S}^7, \mathbb{R}P^4);\ f \in \langle \mathbb{S}^7, \mathbb{R}P^4\rangle\}/\simeq$ is at most seventy four.

(ii): Because $\pi_{2^k}(\mathbb{R}P^{2^k-3}) = \mathbb{Z}_{24}\{\gamma_{2^k-3}\nu_{2^k-3}\}$ hence $\pi_{2^k}(\mathbb{R}P^{2^k-3})/2\pi_{2^k}(\mathbb{R}P^{2^k-3}) \approx \mathbb{Z}_2$ for $k \geq 4$. Since $G_{2^k}(\mathbb{R}P^{2^k-3}) \supseteq 2\pi(\mathbb{R}P^{2^k-3})$ [7, Theorem 2.41(2)] and (1.4) the cardinality of $\{M_f(\mathbb{S}^{2^k}, \mathbb{R}P^{2^k-3});\ f \in \langle \mathbb{S}^{2^k}, \mathbb{R}P^{2^k-3}\rangle\}/\simeq$ is at most two.

(iii)–(iv): Recall that $\pi_8(\mathbb{R}P^4) = \mathbb{Z}_2\{\gamma_4\nu_4\eta_7\} \oplus \mathbb{Z}_2\{\gamma_4(E\nu')\eta_7\}$ and $\pi_9(\mathbb{R}P^4) = \mathbb{Z}_2\{\gamma_4\nu_4\eta_7^2\} \oplus \mathbb{Z}_2\{\gamma_4(E\nu')\eta_7^2\}$. By [7, Theorem 2.19(5)–(6)], $P_8(\mathbb{R}P^4) = \mathbb{Z}_2\{\gamma_4(E\nu')\eta_7\}$ and $P_9(\mathbb{R}P^4) = \mathbb{Z}_2\{\gamma_4(E\nu')\eta_7^2\}$. Thus, $\gamma_4\nu_4\eta_7 \notin P_8(\mathbb{R}P^4)$ and $\gamma_4\nu_4\eta_7^2 \notin P_9(\mathbb{R}P^4)$ and then the homotopy groups $\pi_1(M_0(\mathbb{S}^\epsilon, \mathbb{R}P^4))$ and $\pi_1(M_f(\mathbb{S}^\epsilon, \mathbb{R}P^4))$ are not isomorphic, where $f = \gamma_4\nu_4\eta_7$ or $f = \gamma_4\nu_4\eta_7^2$, for $\epsilon = 8, 9$, respectively. Consequently, the path-components $M_0(\mathbb{S}^\epsilon, \mathbb{R}P^4)$ and $M_f(\mathbb{S}^\epsilon, \mathbb{R}P^4)$ are not homotopy equivalent and the cardinality of $\{M_f(\mathbb{S}^\epsilon, \mathbb{R}P^4);\ f \in \langle \mathbb{S}^\epsilon, \mathbb{R}P^4\rangle\}/\simeq$ is at least two, for f and ϵ as above.

(v): Since $\pi_{10}(\mathbb{R}P^4) = \mathbb{Z}_8\{\gamma_4\nu_4^2\} \oplus \mathbb{Z}_3\{\gamma_4\alpha_1(4)\alpha_1(7)\} \oplus \mathbb{Z}_3\{\gamma_4[\iota_4, \iota_4]\alpha_1(7)\}$ we conclude that $\pi_{10}(\mathbb{R}P^4)/3\pi_{10}(\mathbb{R}P^4) \approx \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and since $G_{10}(\mathbb{R}P^4) \supseteq 3\pi_{10}(\mathbb{R}P^4)$ [7, Theorem 2.41(3)] by (1.4) the cardinality of the set $\{M_f(\mathbb{S}^{10}, \mathbb{R}P^4);\ f \in \langle \mathbb{S}^{10}, \mathbb{R}P^4\rangle\}/\simeq$ is at most five.

(vi): Since $\pi_{11}(\mathbb{R}P^6) = \mathbb{Z}\{\gamma_6[\iota_6, \iota_6]\}$ we obtain $\pi_{11}(\mathbb{R}P^6)/30\pi_{11}(\mathbb{R}P^6) \approx \mathbb{Z}_{30}$ and by [7, Theorem 2.41(4)] and (1.4) the cardinality of $\{M_f(\mathbb{S}^{11}, \mathbb{R}P^6);\ f \in \langle \mathbb{S}^{11}, \mathbb{R}P^6\rangle\}/\simeq$ is at most sixteen.

(vii): Because $\pi_{2^i+1}(\mathbb{R}P^{2^i-5}) = \mathbb{Z}_2\{\gamma_{2^i-5}\nu^2_{2^i-5}\}$ and $P_{2^i+1}(\mathbb{R}P^{2^i-5}) = 0$ by [7, Theorem 2.19(7)], the cardinality of $\{M_f(\mathbb{S}^{2^i+1}, \mathbb{R}P^{2^i-5});\ f \in \langle \mathbb{S}^{2^i+1}, \mathbb{R}P^{2^i-5}\rangle\}/\simeq$ is two.

(viii): Because $\pi_{15}(\mathbb{R}P^8) = \mathbb{Z}\{\gamma_8\sigma_8\} \oplus \mathbb{Z}_8\{\gamma_8 E\sigma'\} \oplus \mathbb{Z}_3\{\gamma_8\alpha_2(8)\} \oplus \mathbb{Z}_5\{\gamma_8\alpha_1(8)\}$ hence $\pi_{15}(\mathbb{R}P^8)/2520\pi_{15}(\mathbb{R}P^8) \approx \mathbb{Z}_{2520}$. Thus, by [7, Theorem 2.41(5)] and (1.4) the cardinality of the set $\{M_f(\mathbb{S}^{15}, \mathbb{R}P^8);\ f \in \langle \mathbb{S}^{15}, \mathbb{R}P^8\rangle\}/\simeq$ is at most one thousand two hundred sixty one.

(ix): Because $\pi_{18}(\mathbb{R}P^{11}) = \mathbb{Z}_{16}\{\gamma_{11}\sigma_{11}\} \oplus \mathbb{Z}_3\{\gamma_{11}\alpha_2(11)\} \oplus \mathbb{Z}_5\{\gamma_{11}\alpha_1(11)\}$ we conclude that $\pi_{18}(\mathbb{R}P^{11})/2\pi_{18}(\mathbb{R}P^{11}) \approx \mathbb{Z}_2$. Thus, by [7, Theorem 2.41(6)] and (1.4) the cardinality of the set $\{M_f(\mathbb{S}^{18}, \mathbb{R}P^{11});\ f \in \langle \mathbb{S}^{18}, \mathbb{R}P^{11}\rangle\}/\simeq$ is at most two. □

3. Path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{C}P^n)$

We study homotopy types of path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{C}P^n)$.

By [11, Theorem 5.1], the path-components $M_f(\mathbb{S}^2, \mathbb{C}P^1)$ and $M_{f'}(\mathbb{S}^2, \mathbb{C}P^1)$ are homotopy equivalent if and only if $f = \pm f'$. If $m \geq 3$, then [7, Corollary 1.3] gives $\pi_m(\mathbb{C}P^1) = G_m(\mathbb{C}P^1)$ and by (1.3) all path-components of the space $M(\mathbb{S}^m, \mathbb{C}P^1)$ are homotopy equivalent.

The space $M(\mathbb{S}^m, \mathbb{C}P^n)$ is path-connected for $m = 1$ and $n \geq 1$, or $3 \leq m < 2n + 1$, because $\pi_m(\mathbb{C}P^n) = 0$ under these conditions. Consequently (1.2) leads to:

Remark 3.1. There are isomorphisms $\pi_i(M(\mathbb{S}^m, \mathbb{C}P^n)) \approx \pi_{i+m}(\mathbb{C}P^n) \oplus \pi_i(\mathbb{C}P^n)$ for $i \geq 1$, where $m = 1$ and $n \geq 1$, or $3 \leq m < 2n + 1$. In particular, $\pi_1(M(\mathbb{S}^1, \mathbb{C}P^n)) \approx \mathbb{Z}$, $\pi_1(M(\mathbb{S}^{2n}, \mathbb{C}P^n)) \approx \mathbb{Z}$, and the space $M(\mathbb{S}^m, \mathbb{C}P^n)$ is 1-connected for $3 \leq m < 2n$.

Next, by [15, Theorem III.8] it holds $n!\pi_{2n+1}(\mathbb{C}P^n) \subseteq G_{2n+1}(\mathbb{C}P^n)$. Then, (1.3) and (1.4) imply that the number of homotopy types of path-components of $M(\mathbb{S}^{2n+1}, \mathbb{C}P^n)$ is bounded above by $\frac{n!}{2} + 1$.

Now, we examine the homotopy type of path-components of the space $M(\mathbb{S}^m, \mathbb{C}P^2)$ for $m = 1, \ldots, 12$. If $m = 1, 3, 4$, the space $M(\mathbb{S}^m, \mathbb{C}P^2)$ was considered in Remark 3.1. For the remaining cases we make use of (1.3), (1.4) and [7, Theorems 2.20 and 2.25] to state the following:

Proposition 3.2. (1) *If $n \geq 1$ and the path-components $M_{ki_\mathbb{C}}(\mathbb{S}^2, \mathbb{C}P^{2n})$ and $M_{li_\mathbb{C}}(\mathbb{S}^2, \mathbb{C}P^{2n})$ are homotopy equivalent then $k \equiv l \pmod 2$.*
(2) *The cardinality of the set $\{M_f(\mathbb{S}^m, \mathbb{C}P^2);\ f \in \langle \mathbb{S}^m, \mathbb{C}P^2\rangle\}/\simeq$ is:*
(i) *two for $m = 5, 6, 7$;*
(ii) *at most two for $m = 8, 9$;*
(iii) *one for $m = 10, 11, 12$.*

Proof. (1): Recall that $\pi_2(\mathbb{C}P^{2n}) = \mathbb{Z}\{i_\mathbb{C}\}$ and by [2, (4.1–3)] it holds $[i_\mathbb{C}, \gamma_{2n}] = \gamma_{2n}\eta_{4n+1}$. Given $f \in \pi_2(\mathbb{C}P^{2n})$, by Theorem 1.2 the evaluation fibration

$$M_{*f}(\mathbb{S}^2, \mathbb{C}P^{2n}) \hookrightarrow M_f(\mathbb{S}^2, \mathbb{C}P^{2n}) \xrightarrow{\omega_f} \mathbb{C}P^{2n}$$

leads to the exact sequence

$$\pi_{4n+1}(\mathbb{C}P^{2n}) \xrightarrow{[f,-]} \pi_{4n+2}(\mathbb{C}P^{2n}) \to \pi_{4n}(M_f(\mathbb{S}^2, \mathbb{C}P^{2n})) \to 0.$$

If $f = ki_{\mathbb{C}}$ and k is even then $[ki_{\mathbb{C}}, \gamma_{2n}] = k\gamma_{2n}\eta_{4n+1} = 0$. Hence, by the sequence above there is an isomorphism

$$\pi_{4n+2}(\mathbb{C}P^{2n}) \xrightarrow{\approx} \pi_{4n}(M_{ki_{\mathbb{C}}}(\mathbb{S}^2, \mathbb{C}P^{2n})).$$

If $f = li_{\mathbb{C}}$ and l is odd then $[li_{\mathbb{C}}, \gamma_{2n}] = \gamma_{2n}\eta_{4n+1} \neq 0$. Then, since $\pi_{4n+2}(\mathbb{C}P^{2n}) \approx \mathbb{Z}_2$ the isomorphism

$$\pi_{4n+2}(\mathbb{C}P^{2n})/\operatorname{Im}([li_{\mathbb{C}}, -]) \xrightarrow{\approx} \pi_{4n}(M_{li_{\mathbb{C}}}(\mathbb{S}^2, \mathbb{C}P^{2n}))$$

shows that the path-components $M_{ki_{\mathbb{C}}}(\mathbb{S}^2, \mathbb{C}P^{2n})$ and $M_{li_{\mathbb{C}}}(\mathbb{S}^2, \mathbb{C}P^{2n})$ are not homotopy equivalent.

Now we prove part (2).

(i): Let $m = 5$. We have $G_5(\mathbb{C}P^2) = P_5(\mathbb{C}P^2) = 2\pi_5(\mathbb{C}P^2)$ by [7, Theorems 2.20(1) and 2.44] and so (1.3) implies that there are at most two homotopy types on the set of path-components of $M(\mathbb{S}^5, \mathbb{C}P^2)$. Next, the evaluation fibration $M_{*\gamma_2}(\mathbb{S}^5, \mathbb{C}P^2) \hookrightarrow M_{\gamma_2}(\mathbb{S}^5, \mathbb{C}P^2) \to \mathbb{C}P^2$ leads, in view of Theorem 1.2, to the exact sequence

$$\pi_5(\mathbb{C}P^2) \xrightarrow{[\gamma_2,-]} \pi_9(\mathbb{C}P^2) \to \pi_4(M_{\gamma_2}(\mathbb{S}^5, \mathbb{C}P^2)) \to 0.$$

Since $[\gamma_2, \gamma_2] = \gamma_2[\iota_5, \iota_5] \neq 0$, we conclude that $M_{\gamma_2}(\mathbb{S}^5, \mathbb{C}P^2)$ and $M_0(\mathbb{S}^5, \mathbb{C}P^2)$ are not homotopy equivalent. Consequently, there are exactly two homotopy types of path-components of $M(\mathbb{S}^5, \mathbb{C}P^2)$.

Let $m = 6$. Since $\pi_6(\mathbb{C}P^2) = \mathbb{Z}_2\{\gamma_2\eta_5\}$ and $G_6(\mathbb{C}P^2) = 0$ by [7, Theorem 2.45], the equation (1.3) implies that there are at most two homotopy types of path-components of $M(\mathbb{S}^6, \mathbb{C}P^2)$. The same argument as for $m = 5$ and [7, (1.15)] show that $M_{\gamma_2\eta_5}(\mathbb{S}^6, \mathbb{C}P^2)$ and $M_0(\mathbb{S}^6, \mathbb{C}P^2)$ are not homotopy equivalent, and consequently there are exactly two homotopy types of path-components of $M(\mathbb{S}^6, \mathbb{C}P^2)$.

Let $m = 7$. Since $\pi_7(\mathbb{C}P^2) = \mathbb{Z}_2\{\gamma_2\eta_5^2\}$ and $G_7(\mathbb{C}P^2) = P_7(\mathbb{C}P^2) = 0$ by [7, Theorem 2.20(2)], the equation (1.3) guarantees that there are at most two homotopy types of path-components of $M(\mathbb{S}^7, \mathbb{C}P^2)$. The same argument as above and [7, (1.16)] show that $M_{\gamma_2\eta_5^2}(\mathbb{S}^7, \mathbb{C}P^2)$ and $M_0(\mathbb{S}^7, \mathbb{C}P^2)$ are not homotopy equivalent.

(ii): Let $m = 8, 9$. Recall that $\pi_8(\mathbb{C}P^2) = \mathbb{Z}_{24}\{\gamma_2\nu_5\}$ and $\pi_9(\mathbb{C}P^2) = \mathbb{Z}_2\{\gamma_2\nu_5^2\}$, and by [7, Theorem 2.45(2)] we know that $G_8(\mathbb{C}P^2) \supseteq 2\pi_8(\mathbb{C}P^2)$. Then, (1.3) implies the result.

(iii): Since $G_m(\mathbb{C}P^2) = \pi_m(\mathbb{C}P^2)$ by [7, Corollary 2.47(1)] for $m = 10, 11, 12$, by (1.3) the proof is completed. □

4. Path-components of the mapping spaces $M(\mathbb{S}^m, \mathbb{H}P^n)$

We study the homotopy type of path-components of mapping spaces $M(\mathbb{S}^m, \mathbb{H}P^n)$.

Since $\pi_m(\mathbb{H}P^n) = 0$ for $m = 1, 2, 3$ and $n \geq 1$, the space $M(\mathbb{S}^m, \mathbb{H}P^n)$ is path-connected under these conditions. In view of (1.2), this implies:

Remark 4.1. There are isomorphisms $\pi_i(M(\mathbb{S}^m, \mathbb{H}P^n)) \approx \pi_{i+m}(\mathbb{H}P^n) \oplus \pi_i(\mathbb{H}P^n)$ for $i \geq 1$, $m = 1, 2, 3$ and $n \geq 1$. In particular, $\pi_1(M(\mathbb{S}^3, \mathbb{H}P^n)) \approx \mathbb{Z}$ for $n \geq 1$, and the space $M(\mathbb{S}^m, \mathbb{H}P^n)$ is 1-connected for $m = 1, 2$ and $n \geq 1$.

By [7, Theorem 2.49(1)] it holds:

$$G_{4n+3}(\mathbb{H}P^n) \supseteq \begin{cases} (2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}), & \text{for } n \text{ even}, \\ 2(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}), & \text{for } n \text{ odd}. \end{cases}$$

Since

$$\pi_k(\mathbb{H}P^n) = \gamma_{n*}\pi_k(\mathbb{S}^{4n+3}) \oplus i_{\mathbb{H}*}E\pi_{k-1}(\mathbb{S}^3), \tag{4.1}$$

by (1.3) the number of homotopy types of path-components of $M(\mathbb{S}^{4n+3}, \mathbb{H}P^n)$ is bounded above by the orders of the finite groups

$$\pi_{4n+3}(\mathbb{H}P^n)/(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}) \approx \mathbb{Z}_{(2n+1)!} \oplus i_{\mathbb{H}*}E\pi_{4n+2}(\mathbb{S}^3)$$

and

$$\pi_{4n+3}(\mathbb{H}P^n)/2(2n+1)!\gamma_{n*}\pi_{4n+3}(\mathbb{S}^{4n+3}) \approx \mathbb{Z}_{2(2n+1)!} \oplus i_{\mathbb{H}*}E\pi_{4n+2}(\mathbb{S}^3),$$

for n even and odd, respectively.

By [11, Theorem 5.1], the path-components $M_f(\mathbb{S}^4, \mathbb{H}P^1)$ and $M_{f'}(\mathbb{S}^4, \mathbb{H}P^1)$ are homotopy equivalent if and only if $f = \pm f'$. Now, we study the homotopy type of path-components of the spaces $M(\mathbb{S}^4, \mathbb{H}P^n)$ for $n > 1$ and $M(\mathbb{S}^m, \mathbb{H}P^1)$ for $m = 5, \ldots, 17$. In order to do this we make use of (1.3) and [7, Chapter I].

Proposition 4.2. (1) *If $n > 1$ and the path-components $M_{ki_{\mathbb{H}}}(\mathbb{S}^4, \mathbb{H}P^n)$ and $M_{li_{\mathbb{H}}}(\mathbb{S}^4, \mathbb{H}P^n)$ are homotopy equivalent then* $(24, k(n+1)) = (24, l(n+1))$.
(2) *The cardinality of the set* $\{M_f(\mathbb{S}^m, \mathbb{H}P^1);\ f \in \pi_m(\mathbb{H}P^1)\}/\simeq$ *is:*
 (i) *one for* $m = 5, 6, 8, 9, 10, 16, 17$*;*
 (ii) *at most nineteen for* $m = 7$*;*
 (iii) *at most eight for* $m = 11$*;*
 (iv) *two for* $m = 12$*;*
 (v) *at most four for* $m = 13$*;*
 (vi) *at most four for* $m = 14$*;*
 (vii) *at most twenty two for* $m = 15$.

Proof. (1): By (4.1), $\pi_4(\mathbb{H}P^n) = \mathbb{Z}\{i_{\mathbb{H}}\}$ for $n > 1$. Given $f \in \pi_4(\mathbb{H}P^n)$, by Theorem 1.2 the evaluation fibration

$$M_{*f}(\mathbb{S}^4, \mathbb{H}P^n) \hookrightarrow M_f(\mathbb{S}^4, \mathbb{H}P^n) \to \mathbb{H}P^n$$

leads to the exact sequence

$$\pi_{4n+3}(\mathbb{H}P^n) \xrightarrow{[f,-]} \pi_{4n+6}(\mathbb{H}P^n) \to \pi_{4n+2}(M_f(\mathbb{S}^4, \mathbb{H}P^n)) \to \pi_{4n+2}(\mathbb{H}P^n) \xrightarrow{[f,-]} \pi_{4n+5}(\mathbb{H}P^n).$$

Since $\pi_{4n+2}(\mathbb{H}P^n) = i_{\mathbb{H}*}E\pi_{4n+1}(\mathbb{S}^3)$, the proof of [7, Lemma 2.8] shows that $[i_{\mathbb{H}}, i_{\mathbb{H}}E\beta] = 0$ for any $\beta \in \pi_{4n+1}(\mathbb{S}^3)$ and $n > 1$, so the map $\pi_{4n+2}(\mathbb{H}P^n) \xrightarrow{[f,-]} \pi_{4n+5}(\mathbb{H}P^n)$ is trivial. Consequently we derive the short exact sequence

$$0 \to \pi_{4n+6}(\mathbb{H}P^n)/\operatorname{Im}([f,-]) \to \pi_{4n+2}(M_f(\mathbb{S}^4, \mathbb{H}P^n)) \to i_{\mathbb{H}*}E\pi_{4n+1}(\mathbb{S}^3) \to 0.$$

Let now $f = ki_{\mathbb{H}}$. Because of [2, (4.1–3)] it holds $[i_{\mathbb{H}}, \gamma_n] = \pm(n+1)\gamma_n\nu_{4n+3}$. Hence since $[i_{\mathbb{H}}, i_{\mathbb{H}}E\beta] = 0$ for any $\beta \in \pi_{4n+2}(\mathbb{S}^3)$ and $n > 1$, we deduce that the group $\operatorname{Im}([ki_{\mathbb{H}}, -]) = \{k(n+1)\gamma_n\nu_{4n+3}\}$ has order $\frac{24}{(24,k(n+1))}$ since the order $\#\nu_{4n+3} = 24$.

From this we derive that the path-components $M_{ki_{\mathbb{H}}}(\mathbb{S}^4, \mathbb{H}P^n)$ and $M_{li_{\mathbb{H}}}(\mathbb{S}^4, \mathbb{H}P^n)$ are not homotopy equivalent provided $(24, k(n+1)) \neq (24, l(n+1))$.

(2): First, we notice that the generators of all Gottlieb groups below are from [7, Chapter I, Section 1.2].

(i): If $m = 5, 6, 8, 9, 10, 16, 17$ then $G_m(\mathbb{H}P^1) = \pi_m(\mathbb{H}P^1)$. Hence (1.3) leads to a single homotopy type of path-components of $M(\mathbb{S}^m, \mathbb{H}P^1)$.

(ii): If $m = 7$ then $\pi_7(\mathbb{H}P^1) = \mathbb{Z}\{\nu_4\} \oplus \mathbb{Z}_4\{E\nu'\} \oplus \mathbb{Z}_3\{\alpha_1(4)\}$ and it holds $G_7(\mathbb{H}P^1) = \{6\nu_4 + E\nu', 2E\nu'\}$. Thus, the canonical epimorphisms $\mathbb{Z}\{\nu_4\} \to \mathbb{Z}_{12}$ and $\mathbb{Z}_4\{E\nu'\} \to 6\mathbb{Z}_{12}$ lead to an isomorphism $\pi_7(\mathbb{H}P^1)/G_7(\mathbb{H}P^1) \approx \mathbb{Z}_{12} \oplus \mathbb{Z}_3$ and so (1.3) and (1.4) yields at most nineteen homotopy types of path-components of $M(\mathbb{S}^7, \mathbb{H}P^1)$.

(iii): If $m = 11$ then $\pi_{11}(\mathbb{H}P^1) = \mathbb{Z}_{15}\{\alpha_1(4) + \alpha_2(4)\}$ and $G_{11}(\mathbb{H}P^1) = 0$. Hence (1.3) and (1.4) yields at most eight homotopy types of path-components of $M(\mathbb{S}^{11}, \mathbb{H}P^1)$.

(iv): If $m = 12$ then $\pi_{12}(\mathbb{H}P^1) = \mathbb{Z}_2\{\varepsilon_4\}$ and $G_{12}(\mathbb{H}P^1) = 0$. Hence (1.3) and (1.4) yields two homotopy types of path-components of $M(\mathbb{S}^{12}, \mathbb{H}P^1)$.

(v): If $m = 13$ then $\pi_{13}(\mathbb{H}P^1) = \mathbb{Z}_2\{\varepsilon_4^3\} \oplus \mathbb{Z}_2\{\mu_4\} \oplus \mathbb{Z}_2\{\eta_4\varepsilon_5\}$ and $G_{13}(\mathbb{H}P^1) = \mathbb{Z}_2\{\varepsilon_4^3\}$. Hence (1.3) yields at most four homotopy types of path-components of $M(\mathbb{S}^{13}, \mathbb{H}P^1)$.

(vi): If $m = 14$ then $\pi_{14}(\mathbb{H}P^1) = \mathbb{Z}_8\{\nu_4\sigma'\} \oplus \mathbb{Z}_4\{E\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_4\mu_5\} \oplus \mathbb{Z}_3\{\alpha_1(4)\alpha_2(7)\} \oplus \mathbb{Z}_3\{\nu_4\alpha_2(7)\} \oplus \mathbb{Z}_5\{\nu_4\alpha_1(7)\}$ and $G_{14}(\mathbb{H}P^1) = \{\nu_4\sigma' \pm E\varepsilon', 2E\varepsilon', \alpha_1(4)\alpha_2(7), \nu_4\alpha_2(7), \nu_4\alpha_1(7)\}$. Thus, the canonical epimorphisms $\mathbb{Z}_8\{\nu_4\sigma'\} \to \mathbb{Z}_2$ and $\mathbb{Z}_4\{E\varepsilon'\} \to \mathbb{Z}_2$ lead to an isomorphism $\pi_{14}(\mathbb{H}P^1)/G_{14}(\mathbb{H}P^1) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and by (1.3) and (1.4) there are at most four homotopy types of path-components of $M(\mathbb{S}^{14}, \mathbb{H}P^1)$.

(vii): If $m = 15$ then $\pi_{15}(\mathbb{H}P^1) = \mathbb{Z}_2\{\nu_4\sigma'\eta_{14}\} \oplus \mathbb{Z}_2\{\nu_4\bar{\nu}_7\} \oplus \mathbb{Z}_2\{\nu_4\varepsilon_7\} \oplus \mathbb{Z}_4\{E\nu'\} \oplus \mathbb{Z}_2\{\varepsilon_4\nu_{12}\} \oplus \mathbb{Z}_2\{(E\nu')\varepsilon_7\} \oplus \mathbb{Z}_3\{\alpha_3(3)\} \oplus \mathbb{Z}_7\{\alpha_1(3)\}$ and $G_{15}(\mathbb{H}P^1)$

$= \{\nu_4\sigma'\eta_{14}, \nu_4\bar{\nu}_7, \nu_4\varepsilon_7, 2E\nu', \varepsilon_4\nu_{12}, (E\nu')\varepsilon_7\}$. Because $\pi_{15}(\mathbb{H}P^1)/G_{15}(\mathbb{H}P^1) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{21}$, by (1.3) and (1.4) there are at most twenty two homotopy types of path-components of $M(\mathbb{S}^{15}, \mathbb{H}P^1)$. □

5. Path-components of the mapping spaces $M(E\Sigma^m, \mathbb{F}P^n)$

We study homotopy types of path-components of the mapping spaces $M(E\Sigma^m, \mathbb{F}P^n)$ for the suspension $E\Sigma^m$ of a homology m-sphere Σ^m.

First, recall that a map $f\colon X \to Y$ is called a *homology isomorphism* if it induces isomorphisms on all homology groups. Then, according to [5] a space Σ^n is called a *homology n-sphere* if there is an isomorphism of homologies $H_k(\Sigma^n) \approx H_k(\mathbb{S}^n)$ for all $k \geq 0$. Given a homology n-sphere Σ^n with $n \geq 1$, the fundamental group $\pi_1(\Sigma^n)$ is perfect for $n \geq 2$ and the abelianization $\pi_1(\Sigma^1)^{\mathrm{ab}} \approx \mathbb{Z}$, the infinite cyclic group. Thus, the result of Bousfield–Kan [3, Chapter VII, Proposition 3.2, p. 206] implies that there is a homology isomorphism $\mathfrak{h}_n\colon \Sigma^n \to \mathbb{S}^n$ for $n \geq 2$. It is not difficult to show that there is also a homology isomorphism $\mathfrak{h}_1\colon \Sigma^1 \to \mathbb{S}^1$.

Given a homology isomorphism $f\colon X \to Y$, let C_f be the mapping cone of f. Since the cofibration $X \xrightarrow{f} Y \to C_f$ induces the long homology exact sequence

$$\cdots \to H_m(X) \xrightarrow{H_m(f)} H_m(Y) \to H_m(C_f) \to \cdots$$

we derive that the reduced homologies $\tilde{H}_m(C_f) = 0$ for all $m \geq 0$.

Remark 5.1. Let Σ^n be a homology n-sphere with $n \geq 1$ and $\mathfrak{h}_n\colon \Sigma^n \to \mathbb{S}^n$ a homology isomorphism.

(1) If Σ^n is 1-connected and $n \geq 2$ then $\mathfrak{h}_n\colon \Sigma^n \to \mathbb{S}^n$ is a weak homotopy equivalence. This implies that the suspension $E\mathfrak{h}_n\colon E\Sigma^n \to E\mathbb{S}^n = \mathbb{S}^{n+1}$ is a weak homotopy equivalence for $n \geq 1$.
(2) Since the fundamental group of a co-H-space is free, a co-H-structure on Σ^n implies that Σ^n is 1-connected for $n \geq 2$ and $\pi_1(\Sigma^1) \approx \mathbb{Z}$. Consequently $\mathfrak{h}_n\colon \Sigma^n \to \mathbb{S}^n$ is a weak homotopy equivalence for $n \geq 1$, provided Σ^n has a co-H-structure.

Proposition 5.2. *If $f\colon X \to Y$ is a homology isomorphism then the induced maps $f^*(Z)\colon [Y, Z] \to [X, Z]$ and $(Ef)^*(Z)\colon [EY, Z] \to [EX, Z]$ are a surjection of sets and an epimorphism of groups, respectively for any space Z. In particular, if Σ^n is a homology n-sphere for $n \geq 1$ and $\mathfrak{h}_n\colon \Sigma^n \to \mathbb{S}^n$ a homology isomorphism then $\mathfrak{h}_n^*(Z)\colon [\mathbb{S}^n, Z] \to [\Sigma^n, Z]$ and $(E\mathfrak{h}_n)^*(Z)\colon [\mathbb{S}^{n+1}, Z] \to [E\Sigma^n, Z]$ are a surjection of sets and an epimorphism of groups, respectively for any space Z.*

Proof. Let $f\colon X \to Y$ be a homology isomorphism. Then, given $\beta\colon X \to Z$, the obstruction to find $\gamma\colon Y \to Z$ such that $\gamma f = \beta$ lies in $H^{m+1}(C_f; \pi_m(Z))$ for $m \geq 0$. Since $\tilde{H}_m(C_f) = 0$ for $m \geq 0$, the Universal Coefficient Theorem implies that $f^*(Z)$ is a surjection. The arguments as above applied to the cofibration $EX \xrightarrow{Ef} EY \to C_{Ef}$ imply an epimorphism $(Ef)^*(Z)\colon [EY, Z] \to [EX, Z]$ of groups and the proof follows. □

Notice that following the procedure above one can easily show that $(E\mathfrak{h}_n)^*(Z)$ is an isomorphism.

Corollary 5.3. *For any space X the map $E\mathfrak{h}_n\colon E\Sigma^n \to \mathbb{S}^{n+1}$ determines an epimorphism of groups $\pi_{n+1}(X)/G_{n+1}(X) \to [E\Sigma^n, X]/G(E\Sigma^n, X)$.*

Proof. In view of Proposition 5.2, the map $E\mathfrak{h}_n\colon E\Sigma^n \to \mathbb{S}^{n+1}$ implies an epimorphism of groups $(E\mathfrak{h}_n)^*(X)\colon \pi_{n+1}(X) \to [E\Sigma^n, X]$ which restricts to a homomorphism $(E\mathfrak{h}_n)^*(X)\colon G_{n+1}(X) \to G(E\Sigma^n, X)$. Thus, we get the required epimorphism $\pi_{n+1}(X)/G_{n+1}(X) \to [E\Sigma^n, X]/G(E\Sigma^n, X)$. □

To state the main result of this section, we need:

Proposition 5.4. *Let $G \times X \to X$ be a free and proper action of a path-connected topological group G on a space X.*

(1) *If $n \geq 2$ and $\pi_{n-1}(G) = 0$ then $q_*\colon [\Sigma^n, X] \to [\Sigma^n, X/G]$ is surjective.*
(2) *If $n \geq 2$, the quotient map $q\colon X \to X/G$ is a principal G-bundle and $[\Sigma^n, G] = 0$ then $q_*\colon [\Sigma^n, X] \to [\Sigma^n, X/G]$ is injective. In particular, if $\pi_n(G) = 0$ then q_* is injective.*

Proof. (1): If $\pi_{n-1}(G) = 0$ then $H^{m+1}(\Sigma^n; \pi_m(G)) = 0$ for all $m \geq 0$. By obstruction theory, this implies that $q_*\colon [\Sigma^n, X] \to [\Sigma^n, X/G]$ is surjective.
(2): Take $f, g \in [\Sigma^n, X]$ such that $qf \simeq qg$. But $q\colon X \to X/G$ is a principal G-bundle, so by the homotopy lifting property $f \simeq g'$ for some map $g'\colon \Sigma^n \to X$ with $qg' = qg$. Hence, there is a map $\alpha\colon \Sigma^n \to G$ such that $\alpha g \simeq g'$, where $(\alpha g)(s) = \alpha(s)g(s)$ for $s \in \Sigma^n$. Since $[\Sigma^n, G] = 0$, the map $\alpha\colon \Sigma^n \to G$ is homotopically trivial and consequently $f \simeq g' \simeq g$. Next, by Proposition 5.2, the map $\mathfrak{h}_n^*(G)\colon \pi_n(G) = [\mathbb{S}^n, G] \to [\Sigma^n, G]$ is a surjection. Hence, $\pi_n(G) = 0$ implies $[\Sigma^n, G] = 0$ and the proof is complete. □

Now, by [7, Corollary 3.34(2)] we have $G_3(\mathbb{R}P^2) = \pi_3(\mathbb{R}P^2)$, and Lemma 2.2(2) and Proposition 2.5 imply that the quotient group $\pi_{4n-1}(\mathbb{R}P^{2n})/G_{4n-1}(\mathbb{R}P^{2n})$ is finite for $n \geq 1$. Notice that by Proposition 5.4, the quotient maps $\mathbb{S}^n \to \mathbb{R}P^n$ and $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$ induce bijections $[\Sigma^m, \mathbb{S}^n] \xrightarrow{\approx} [\Sigma^m, \mathbb{R}P^n]$ and $[\Sigma^m, \mathbb{S}^{2n+1}] \xrightarrow{\approx} [\Sigma^m, \mathbb{C}P^n]$.

Then, using (1.3), results from Sect. 2, Proposition 5.2, and Corollary 5.3 we may state:

Theorem 5.5. (1) *The quotient map $\gamma_n\colon \mathbb{S}^n \to \mathbb{R}P^n$ induces a bijection*

$$[\Sigma^m, \mathbb{S}^n] \xrightarrow{\approx} [\Sigma^m, \mathbb{R}P^n]$$

for $m, n \geq 1$. The number of homotopy types of path-components of $M(E\Sigma^m, \mathbb{R}P^n)$ is bounded above by the order of the finite group $\pi_{m+1}(\mathbb{R}P^n)/\pm G_{m+1}(\mathbb{R}P^n)$ for $n \geq 1$, $m \geq 2$, with $m \neq 2n-1$ if n is even.
(2) *The quotient map $\gamma_n\colon \mathbb{S}^{2n+1} \to \mathbb{C}P^n$ induces a bijection*

$$[\Sigma^m, \mathbb{S}^{2n+1}] \xrightarrow{\approx} [\Sigma^m, \mathbb{C}P^n]$$

for $m \geq 3$, $n \geq 1$. *The number of homotopy types of path-components of* $M(E\Sigma^m, \mathbb{C}P^n)$ *is bounded above by the order of the finite group* $\pi_{m+1}(\mathbb{C}P^n)/\pm G_{m+1}(\mathbb{C}P^n)$.
(3) *All path-components of the spaces* $M(E\Sigma^m, \mathbb{H}P^n)$ *for* $m = 1, 2$ *are homotopy equivalent. The number of homotopy types of path-components of* $M(E\Sigma^m, \mathbb{H}P^n)$ *is bounded above by the order of the finite group* $\pi_{m+1}(\mathbb{H}P^n)/\pm G_{m+1}(\mathbb{H}P^n)$, *for* $n \geq 1$, $m \geq 4$.

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