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Electrical stimulation tracking control for paraplegic patients using T–S fuzzy models

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Abstract

This paper presents the design and simulation results considering the use of functional electrical stimulation to control the leg position of paraplegic patients. The plant is described by a nonlinear system using Takagi–Sugeno fuzzy models and a closed-loop control is presented. A transfer function represents the mathematical model related to the muscle torque and the electrical stimulation pulse width. Considering that, during the operation, the leg position is between 0° and 60°, then two fuzzy regulators were designed, assuming that the state vector is available or using an observer when only the output is available. This design was based on the Lyapunov stability theory and Linear Matrix Inequalities (LMIs). The simulation results show that the proposed procedures are efficient and offer good results for this control problem. Finally, new conditions regarding the design of the output tracking control, using a suitable nonlinear transformation for the description of the plant in an adequate form, is presented.

Keywords: Tracking control; T–S fuzzy models; LMI; State regulator; Functional electric stimulation; Paraplegic patients; Control of leg position.

1. Introduction

The presented study shows, through theoretical studies and simulations, the possibility of designing a nonlinear controller for tracking control of the leg position for paraplegic patients, using the description of the plant by a Takagi–Sugeno (T–S) fuzzy model. The control system was designed considering that, during the operation, the angle position of the joint knee is between 0° and 60°. Several researchers have used Functional Electrical Stimulation (FES) to restore some motion activities of persons with injured spinal cord [1–5]. However, FES is not yet a regular

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clinical method because the amount of effort involved in using actual stimulation systems still outweighs the functional benefits they provide. One serious problem using FES is that artificially activated muscles fatigue at a faster rate than those activated by the natural physiological processes. Due to these problems, a considerable effort has been directed toward developing FES systems based on closed-loop control.

The movement of the lower limb is measured in real time with several types of sensors and the stimulation pattern is modulated accordingly [5–8]. The dynamics of the lower limb were represented by a nonlinear second order model, which took into account the gravitational and inertial characteristics of the anatomical segment as into well as the damping and stiffness properties of the knee joint. In this paper, the authors present a Takagi–Sugeno nonlinear system [9–18] with the aim of controlling the position of the leg of a paraplegic patient [19,20]. The controller was designed in order to track a desired trajectory of the joint knee angle, that is between 0° and 60°. The input of the plant is an electrical stimulation applied to the quadriceps muscle. The leg mathematical model proposed by [5] was considered. The authors showed that, for the conditions considered in their experiments, a simple one-pole transfer function was able to model the relationship between the stimulus pulse width and the active muscle torque. The state space model of paraplegic patients to vary knee joint angle with stimulus pulse width and active muscle torque was developed by the authors [19–23].

The systematic procedure for fuzzy control systems design proposed in this manuscript involves fuzzy model construction for nonlinear systems [24], considering arbitrary operation points [18,25,26].

Muscle is a highly complex nonlinear system capable of producing the same output for a variety of inputs [27–32]. When exploiting physiolocally activated muscle, an important effort of the researches is to minimize fatigue [29,32,33]. When the quadriceps is electrically stimulated, its response is nonlinear. Thus, T–S fuzzy models are used to design a controller to vary the knee joint angle.

In [19,20,22,23], feedback strategies were proposed for controlling the leg position of paraplegic patients, considering a given operation point. The nonlinear plant was described by a Takagi–Sugeno (T–S) fuzzy model and an electrogoniometer was used fastened on the thigh and shank of the patient. The design method considered all plant nonlinearities and was efficient and reliable to control the leg position of a paraplegic patient, applying electric stimulation at the quadriceps muscle.

The design of tracking control systems for nonlinear plants described by T–S fuzzy systems has been studied in [34–38], and allows the design of the feedback gains, without the need of a new design of the gains for each point operation systems. The tracking procedure proposed in this manuscript allows the tracking of a broader class of signals than the control strategies presented in [39] and [40], where the operation point is not completely known but is in a given region.

For the design method proposed in this paper, the concept of virtual desired variables was used. In turn generalized kinematic constraint are introduced, to make the design procedure clear. The design procedure is split into two independent steps: i) determine the virtual desired variables from the desired output equation and the generalized kinematic constraint; and ii) determine the control feedback gains and observer gains by solving a set of Linear Matrix Inequalities (LMIs) [15,41,42], the same type of LMIs for stabilization problem [43]. The LMIs problems have been solved with Matlab Toolbox [44]. Simulation results illustrate the performance of the proposed procedure.

2. Takagi–Sugeno fuzzy representation

Some classes of nonlinear systems can be exactly represented by T–S fuzzy models [45]. The local description of the dynamic plant to be controlled, in terms of local linear models, is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t),$$
$$\mathbf{y}(t) = \mathbf{C}_i \mathbf{x}(t),$$

where $i = 1, 2, \ldots, r$ (r is the number of linear models), $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{u}(t) \in \Re^m$ is the input vector, $\mathbf{y}(t) \in \Re^q$ is the output vector, $\mathbf{A}_i \in \Re^{n\times n}$, $\mathbf{B}_i \in \Re^{n\times m}$ and $\mathbf{C}_i \in \Re^{q\times n}$. This information is represented by IF-THEN rules, where the i-th rule is given by:

$$\text{Rule } i: \text{IF } z_1(t) \text{ is } \mathcal{M}_1^i \text{ AND } \ldots \text{ AND } z_p(t) \text{ is } \mathcal{M}_p^i,$$
$$\text{THEN} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}_i \mathbf{x}(t). \end{cases} \tag{1}$$

Considering the fuzzy model (1), one has, by definition, that $\mathcal{M}^i_j$ ($j = 1, 2, \ldots, p$) is the fuzzy set j of the rule i, $z_1(t), \ldots, z_p(t)$ are the premise variables and $\mu^i_j(z_j(t))$ are the membership functions of the fuzzy rule $\mathcal{M}^i_j$. Define:

$$w^i(\mathbf{z}(t)) = \prod_{j=1}^{p} \mu^i_j(z_j(t)), \quad \mathbf{z}(t) = [z_1(t)\ z_2(t)\ \ldots\ z_p(t)]. \tag{2}$$

Since $\mu^i_j(z_j(t)) \geq 0$ one has, for $i = 1, 2, \ldots, r$,

$$w^i(\mathbf{z}(t)) \geq 0 \ \text{ and } \ \sum_{i=1}^{r} w^i(\mathbf{z}(t)) > 0. \tag{3}$$

To get a T–S fuzzy model for nonlinear systems, one usually adopts $\mathbf{z}(t) = \mathbf{x}(t)$, where $\mathbf{x}(t)$ is the nonlinear system state vector. Thus, for a given pair $(\mathbf{x}(t), \mathbf{u}(t))$, the resulting fuzzy system is the weighted mean of the local models described in (1):

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \frac{\sum_{i=1}^{r} w^i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t))}{\sum_{i=1}^{r} w^i(\mathbf{x}(t))} = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)) \\ &= \left(\sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{A}_i\right)\mathbf{x}(t) + \left(\sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i\right)\mathbf{u}(t) = \mathbf{A}(\boldsymbol{\alpha})\mathbf{x}(t) + \mathbf{B}(\boldsymbol{\alpha})\mathbf{u}(t),\end{aligned}$$

$$\alpha_i(\mathbf{x}(t)) = \frac{w^i(\mathbf{x}(t))}{\sum_{i=1}^{r} w^i(\mathbf{x}(t))}, \qquad \boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_r]^T, \tag{4}$$

for $i = 1, 2, \ldots, r$.

The nonforced system ($\mathbf{u}(t) = 0$) is defined as:

$$\dot{\mathbf{x}}(t) = \frac{\sum_{i=1}^{r} w^i(\mathbf{x}(t))\mathbf{A}_i\mathbf{x}(t)}{\sum_{i=1}^{r} w^i(\mathbf{x}(t))} = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{A}_i\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\alpha})\mathbf{x}(t). \tag{5}$$

For both cases, forced and nonforced, the output is given by:

$$\mathbf{y}(t) = \frac{\sum_{i=1}^{r} w^i(\mathbf{x}(t))\mathbf{C}_i\mathbf{x}(t)}{\sum_{i=1}^{r} w^i(\mathbf{x}(t))} = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{C}_i\mathbf{x}(t) = \mathbf{C}(\boldsymbol{\alpha})\mathbf{x}(t). \tag{6}$$

Observe that, for $i = 1, 2, \ldots, r$,

$$\alpha_i(\mathbf{x}(t)) \geq 0 \ \text{ and } \ \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t)) = 1. \tag{7}$$

3. Takagi–Sugeno fuzzy tracking control

Takagi–Sugeno fuzzy models can be used for tracking control systems design for nonlinear systems. The tracking control system proposed in this manuscript uses the concept of virtual variable [43], introduced to simplify the design procedure. This method can be divided into two steps:

- determination of the desired virtual variables of the system dynamics;
- determination of the control gains, using LMI-based design, for system stabilization.

At first, for fuzzy regulator design, the premise variables of fuzzy rules are assumed to be measurable. After that, this concept of virtual variable is used for immeasurable state variables in dynamic nonlinear systems, to design fuzzy observers [43]. Both of cases are applied to design controllers such that the joint knee angle follows a given trajectory.

3.1. Tracking control and state regulator

The basic architecture for output tracking control, presented in this section, considers that all state variables are available. The method was proposed in [43]. In a general form, consider a nonlinear dynamic equation given by:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t))\,\mathbf{u}(t),\\ \mathbf{y}(t) &= \mathbf{q}(\mathbf{x}(t)),\end{aligned} \tag{8}$$

where $\mathbf{x}(t) \in \Re^n$ is the state vector, $\mathbf{y}(t) \in \Re^m$ is the output variable, $\mathbf{u}(t) \in \Re^m$ is the input vector, $\bar{\mathbf{r}}(t) \in \Re^m$ is the desired output, $\mathbf{f}(x)$, $\mathbf{g}(x)$ and $\mathbf{q}(x)$ are nonlinear functions with appropriate dimensions.

The goal of the tracking control is to assure that the equilibrium point of the controlled system is asymptotically stable and:

$$\mathbf{y}(t) - \bar{\mathbf{r}}(t) \to 0 \ \text{ when } \ t \to \infty, \tag{9}$$

where $\bar{\mathbf{r}}(t)$ denotes the desired trajectory or the reference signal. To convert the tracking problem in a stability problem, one defines the *desired virtual variables*, represented by the vector $\mathbf{x}_d(t)$, such that, for $\mathbf{x}(t) = \mathbf{x}_d(t)$ and from (8), the condition $\mathbf{y}(t) = \mathbf{q}(\mathbf{x}(t)) = \mathbf{q}(\mathbf{x}_d(t)) = \bar{\mathbf{r}}(t)$ holds.

Defining $\widetilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{x}_d(t)$ as the tracking error and representing the first equation of the plant (8) by the T–S fuzzy model given in (1)–(7), one has:

$$\dot{\widetilde{\mathbf{x}}}(t) = \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}_d(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\Big(\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\Big) - \dot{\mathbf{x}}_d(t). \tag{10}$$

From [43], suppose that the control law $\mathbf{u}(t)$ satisfies the following equation:

$$\sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i\boldsymbol{\tau}_c(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i\mathbf{u}(t) + \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{A}_i\mathbf{x}_d(t) - \dot{\mathbf{x}}_d(t), \tag{11}$$

where $\boldsymbol{\tau}_c(t)$ is the new control law. Then, the error equation (10) can be described by the equations:

$$\begin{aligned}\dot{\widetilde{\mathbf{x}}}(t) &= \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i\boldsymbol{\tau}_c(t) + \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{A}_i(\mathbf{x}(t) - \mathbf{x}_d(t)),\\ \dot{\widetilde{\mathbf{x}}}(t) &= \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{A}_i\widetilde{\mathbf{x}}(t) + \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i\boldsymbol{\tau}_c(t).\end{aligned} \tag{12}$$

The new fuzzy control law $\boldsymbol{\tau}_c(t)$ is determined via parallel distributed compensation (PDC):

$$\boldsymbol{\tau}_c(t) = -\sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{F}_i\widetilde{\mathbf{x}}(t). \tag{13}$$

Replacing (13) into (10), one gets the closed-loop form:

$$\dot{\widetilde{\mathbf{x}}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(\mathbf{x}(t))\alpha_j(\mathbf{x}(t))\left(\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j\right)\widetilde{\mathbf{x}}(t). \tag{14}$$

3.2. Generalized dynamics constraints

The goal of the output tracking is to get the control law $\mathbf{u}(t)$ such that (11) holds and the equilibrium point $\widetilde{\mathbf{x}} = 0$ of the tracking error equation (12) is asymptotically stable in the large. For that, one uses $\mathbf{g}(\mathbf{x}) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{B}_i$, that allows to rewrite equation (11) in a compact form:

$$\mathbf{g}(\mathbf{x})(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) = -\mathbf{A}(\mathbf{x})\mathbf{x}_d(t) + \dot{\mathbf{x}}_d(t), \tag{15}$$

where $\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}_i(t))\mathbf{A}_i \in \Re^{n\times n}$. From (15), the existence of the control law $\mathbf{u}(t)$ depends on the structure of $\mathbf{g}(\mathbf{x})$. Consider that the input matrix $\mathbf{g}(\mathbf{x})$ has full rank and is given by:

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \mathbf{0}_{n-m} \\ \mathbf{B}(\mathbf{x}) \end{bmatrix}, \tag{16}$$

where $\mathbf{0}_{\mathrm{n-m}} \in \Re^{(n-m)\times m}$ denotes a matrix with null elements and $\mathbf{B}(\mathbf{x}) \in \Re^{m\times m}$ is nonsingular. Similarly, $\mathbf{A}(\mathbf{x})$ and $\mathbf{x}_d$ are partitioned as follows:

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} \mathbf{A}_{n-m}(\mathbf{x}) \\ \mathbf{A}_m(\mathbf{x}) \end{bmatrix}, \quad \mathbf{x}_d = \begin{bmatrix} \mathbf{x}_d(t)_{n-m} \\ \mathbf{x}_d(t)_m \end{bmatrix}. \tag{17}$$

Consequently, the condition (15) is described as:

$$\begin{bmatrix} \mathbf{0}_{n-m} \\ \mathbf{B}(\mathbf{x})(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{x}}_d(t)_{n-m} - \mathbf{A}_{n-m}(\mathbf{x})\mathbf{x}_d(t) \\ \dot{\mathbf{x}}_d(t)_m - \mathbf{A}_m(\mathbf{x})\mathbf{x}_d(t) \end{bmatrix}. \tag{18}$$

Then, the virtual variables are determined according to the following conditions:

$$\begin{aligned} \dot{\mathbf{x}}_d(t)_{n-m} &= \mathbf{A}_{n-m}(\mathbf{x})\mathbf{x}_d(t), \\ \dot{\mathbf{x}}_d(t)_m &= \mathbf{A}_m(\mathbf{x})\mathbf{x}_d(t) + \mathbf{B}(\mathbf{x})(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)). \end{aligned} \tag{19}$$

From (13) and (18), the control law proposed in [43] is given by:

$$\mathbf{u}(t) = -\sum_{i=1}^{r} \alpha_i(\mathbf{x}(t))\mathbf{F}_i\widetilde{\mathbf{x}}(t) + \mathbf{B}(\mathbf{x})^{-1}(\dot{\mathbf{x}}_d(t)_m - \mathbf{A}_m(\mathbf{x})\mathbf{x}_d(t)). \tag{20}$$

For many physical systems, the desired virtual variables $\mathbf{x}_d = [x_{1d} \ \cdots \ x_{nd}]^T$ can be exactly determined by (19) [43].

3.3. Tracking control based on observer and state regulator

In tracking design with fuzzy observer [43], the estimation error is required to satisfy:

$$\mathbf{x}(t) - \hat{\mathbf{x}}(t) \to 0 \ \text{ when } \ t \to \infty, \tag{21}$$

where $\hat{\mathbf{x}}(t)$ is the estimated state of $\mathbf{x}(t)$. For that, the nonlinear system (8) is expressed in fuzzy equations, as in (4)–(7), with $\mathbf{q}(\mathbf{x}) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}_i(t))\mathbf{C}_i\mathbf{x}(t)$. Therefore, the generic equation for each observer local model can be written as:

$$\begin{aligned} &\text{Observer Rule } i: \text{IF } x_1(t) \text{ is } \mathcal{M}_1^i \text{ AND } \ldots \text{ AND } x_p(t) \text{ is } \mathcal{M}_p^i, \\ &\qquad \text{THEN} \begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}_i\hat{\mathbf{x}}(t) + \mathbf{B}_i\mathbf{u}(t) + \mathbf{H}_i\left\{\mathbf{y}(t) - \hat{\mathbf{y}}(t)\right\}, \\ \hat{\mathbf{y}}(t) = \mathbf{C}_i\hat{\mathbf{x}}(t). \end{cases} \end{aligned} \tag{22}$$

The vectors $\hat{\mathbf{x}}(t)$, $\hat{\mathbf{y}}(t)$ denote the estimated vectors of $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively, and $\mathbf{H}_i$, $i = 1, 2, \ldots, r$, are the observer gains.

Remark 1. The design of fuzzy observers for plants described by Takagi–Sugeno models assumes that the plant state vector $\mathbf{x}(t)$ is not available, the plant output $\mathbf{y}(t)$ is available and can consider two cases: (i) the premise variables of the plant are available because they depend only on the available output $\mathbf{y}(t)$ such that $\alpha_i(\mathbf{x}(t)) = \alpha_i(\mathbf{y}(t))$, $i = 1, 2, ..., r$; (ii) the premise variables of the plant, $\alpha_i(\mathbf{x}(t))$, $i = 1, 2, ..., r$ are immeasurable because they depend on the plant state vector $\mathbf{x}(t)$ that is not available. In the first case it is possible to obtain the observers equations to estimate the vector $\mathbf{x}(t)$ without estimating the premise variables because $\alpha_i(\mathbf{x}(t)) = \alpha_i(\mathbf{y}(t))$, $i = 1, 2, ..., r$ are available [43]. In the second it is necessary to estimate the premise variables $\alpha_i(\mathbf{x}(t))$, $i = 1, 2, ..., r$, usually using $\alpha_i(\hat{\mathbf{x}}(t))$, $i = 1, 2, ..., r$, where $\hat{\mathbf{x}}(t)$ denotes the estimate vector of $\mathbf{x}(t)$. A design method for observers in the second case, considering that the membership functions satisfy a Lipschitz-like condition, can be found in [43]. Fortunately, in the plant studied in this manuscript, the variables $\alpha_i(\mathbf{x}(t)) = \alpha_i(\mathbf{y}(t))$, $i = 1, 2, ..., r$, are available. Thus, only the first case is studied in

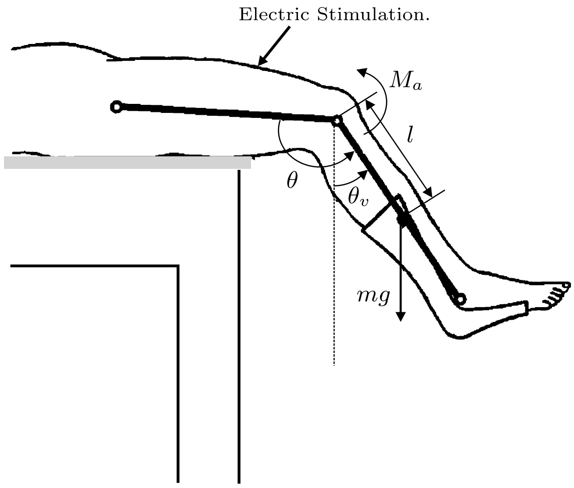

Fig. 1. Shank-Foot Complex and its parameters θ, θ_v and M_a [5].

this manuscript, and then, the observer equations given in (23) and (24) can be easily implemented for estimating the vector $\mathbf{x}(t)$ [43]:

$$\dot{\hat{\mathbf{x}}}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{y}(t)) \left(\mathbf{A}_i \hat{\mathbf{x}}(t) + \mathbf{B}_i \mathbf{u}(t) + \mathbf{H}_i \left\{ \sum_{i=1}^{r} \alpha_i(\mathbf{y}(t)) \mathbf{C}_i \mathbf{x}(t) - \sum_{i=1}^{r} \alpha_i(\mathbf{y}(t)) \mathbf{C}_i \hat{\mathbf{x}}(t) \right\} \right), \tag{23}$$

$$\hat{\mathbf{y}}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{y}(t)) \mathbf{C}_i \hat{\mathbf{x}}(t). \tag{24}$$

In this case, $\mathbf{B}(\mathbf{x})$ and $\mathbf{A}_m(\mathbf{x})$, defined in (17) and above (15) and (16), are available because the variables $\alpha_i(\mathbf{x}(t)) = \alpha_i(\mathbf{y}(t))$, $i = 1, 2, \ldots, r$, are available. Thus, based on the regulator control law (20), the new control law with the observer state is the following [43]:

$$\mathbf{u}(t) = -\sum_{i=1}^{r} \alpha_i(\mathbf{y}(t)) \mathbf{F}_i(\hat{\mathbf{x}}(t) - \mathbf{x}_d(t)) + \mathbf{B}(\mathbf{x})^{-1}(\dot{\mathbf{x}}_d(t)_m - \mathbf{A}_m(\mathbf{x})\mathbf{x}_d(t)). \tag{25}$$

Define the estimation error as $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ [43]. Then, from the Takagi–Sugeno fuzzy model of the plant given in (4) and the observer equations (23) and (24), taking the derivative of $\mathbf{e}(t)$ it follows that:

$$\dot{\mathbf{e}}(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i(\mathbf{y}(t)) \alpha_j(\mathbf{y}(t))(\mathbf{A}_i - \mathbf{H}_j \mathbf{C}_i)\mathbf{e}(t), \tag{26}$$

that is the well-known Takagi–Sugeno observer error equation [46].

4. Dynamic model used in the control of the leg positon of a patient

The mathematical model of the leg used in this manuscript was proposed in [5]. This model relates the applied pulse width with the torque generated on the knee joint. In the modeling [5], the leg was considered as an open kinematic system composed by two rigid segments: the thigh and the shank-foot complex, as shown in Fig. 1. From [5], the system equation is:

$$J\ddot{\theta}_v = -mgl\sin(\theta_v) - M_s - B\dot{\theta} + M_a, \tag{27}$$

where:

- J is the inertial moment of the shank-foot complex;

Table 1
Numeric values of the shank-foot complex parameters [5].

$J = 0.362$ [kg m^2]	$m = 4.37$ [kg]	$l = 23.8$ [cm]
$B = 0.27$ [N m s/rad]	$\lambda = 41.208$ [N m/rad]	$E = 2.024$ [1/rad]
$\omega = 2.918$ [rad]	$\tau = 0.951$ [s]	$G = 42500$ [N m/s]

Table 2
Numeric values of the knee active torque, where $(\theta_v, \dot{\theta}_v, M_a) = (\theta_{v0}, 0, M_{a0})$ is an equilibrium point [5].

θ_{v0}	M_{a0}
30°	4.6069 N m
47°	7.2826 N m
60°	8.7653 N m

- θ_v is the knee angle, between the shank and the vertical axis on sagittal plane;
- $\theta = \theta_v + \pi/2$ is the knee angle, between the shank and the thigh on sagittal plane;
- $\dot{\theta} = \dot{\theta}_v$ is the knee angular velocity;
- m is the mass of shank-foot complex;
- g is the gravitational acceleration;
- l is the distance between the knee and the shank-foot complex mass center;
- B is the viscous friction coefficient;
- M_s is the torque due to the rigidity component;
- M_a is the knee active torque generated by electrical stimulation;
- M_g is the torque due to gravity;
- M_i is the inertial total torque.

The rigidity moment is defined as:

$$M_s = \lambda e^{-E\theta}(\theta - \omega), \tag{28}$$

where λ and E are the coefficients of the exponential terms and ω is the elastic rest angle of the knee. In [5], it was observed that the torque applied to the muscle (M_a) and the electrical stimulation pulse width (P) can be adequately related by the following equation, where G and τ are positive constants:

$$\tau \dot{M}_a + M_a = GP. \tag{29}$$

In [5], the authors suggest methods to experimentally obtain the parameters of interest. In this manuscript, the same parameters in [5] were adopted, as given in Tables 1 and 2, where $(\theta_v, \dot{\theta}_v, M_a) = (\theta_{v0}, 0, M_{a0})$ is an equilibrium point.

Considering (27)–(29), the authors proposed the following state space representation of the system [19,20,23,47,48]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \tilde{f}_{21}(x_1) & -\frac{B}{J} & \frac{1}{J} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{G}{\tau} \end{bmatrix} u, \tag{30}$$

where, by definition, $x_1 = \theta_v$, $x_2 = \dot{\theta}_v$, $x_3 = M_a$ and $u = P$.

The function $\tilde{f}_{21}(x_1)$ is a nonlinear parameter of the system, given by:

$$\tilde{f}_{21}(x_1) = \frac{1}{Jx_1}\left[-mgl\sin(x_1) - \lambda e^{-E\left(x_1 + \frac{\pi}{2}\right)}\left(x_1 + \frac{\pi}{2} - \omega\right)\right]. \tag{31}$$

5. Tracking design for electrical stimulation of paraplegic patients

The advantage of signal tracking is the facility on regulator or observer design without the necessity of getting a new T–S fuzzy model representation for each new operation point or desired output trajectory. This strategy offers a

Table 3
Parameters for the Healthy Group (H) and Paraplegic Group (P).

Subject	J [kg m^2]	m [kg]	l [cm]	λ [N m/rad]	E [1/rad]	w [rad]
H1	0.377	4.05	25.3	1.199	−0.486	2.548
H2	0.358	4.63	23.9	4.679	0.041	2.427
H3	0.399	4.38	24.8	3.657	−0.031	2.71
H4	0.375	3.83	23.7	4.49	0.257	2.701
H5	0.384	4.36	24.3	3.889	−0.079	2.412
P1	0.362	4.37	23.8	41.208	2.024	2.918
P2	0.292	3.42	23.1	3.761	1.317	2.52
P3	0.394	4.76	23.3	15.352	1.644	3.896

Results reported in [5].

new design method for this control problem that is a generalization of the method that considers only a given operation point, designed in [19,20]. From the plant equation that defines the dynamics of the leg movement of the paraplegic patient (given in (30)), a new method is presented, using a change of variables, for the regulator design, and then, for the regulator and observer designs.

5.1. Regulator design with tracking

Note that the plant given in (30) and (31) does not satisfy the condition (18), that must hold in the design method proposed in [43]. This follows firstly observing that the plant (30) and (31) can be rewritten as $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{g}(\mathbf{x})\mathbf{u}(t)$, where $\mathbf{g}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ where defined in (16) and (17). Then, for desired operation points $(x_{10}, x_{20}, x_{30}) = (\theta_{v0}, \dot{\theta}_{v0}, M_{a0}) = (\theta_{v0}, 0, M_{a0}) = (x_{e1}, x_{e2}, x_{e3})$, a necessary condition for (18) is that $\widetilde{f}_{21}(x_1)x_{10} + M_{a0}/J = 0$ for all x_1 in the operation range. In this application, this condition holds only for $x_1 = x_{10}$ and thus the method presented in Section 3 cannot be directly applied to control the plant (30) and (31). A solution for this problem can be obtained considering a new representation of the plant equations (30) and (31), using the definition below:

$$\mathbf{x}_{1n}(t) = \widetilde{f}_{21}(\mathbf{x}_1(t))\mathbf{x}_1(t) = f(\mathbf{x}_1(t)). \tag{32}$$

Considering that $f(\mathbf{x}_1(t))$ is an invertible function, one can write:

$$f(\mathbf{x}_1(t)) = \mathbf{x}_{1n}(t) \iff f^{-1}(\mathbf{x}_{1n}(t)) = \mathbf{x}_1(t). \tag{33}$$

This consideration can be verified by data analysis of two groups of patients (Table 3), five of the healthy group and three of the paraplegic group. Note that in Fig. 2, considering the analyzed groups, it can be observed that the function $f(\mathbf{x}_1(t))$ is an injective function in the operation region ($\mathbf{x}_1(t) \in [0^\circ,\ 70^\circ]$), that assures the existence of $f^{-1}(\mathbf{x}_1(t))$. Thus, we consider that this invertibility condition is an acceptable assumption.

Define a new representation of $f^{-1}(\mathbf{x}_{1n}(t))$ in (33):

$$f^{-1}(\mathbf{x}_{1n}(t)) \stackrel{\triangle}{=} \psi(\mathbf{x}_{1n}(t)). \tag{34}$$

A new representation of (30) can be obtained with the definition (34). Determining the new equations from (30), (32)–(34), defining $\mathbf{x}_{2n}(t) = \mathbf{x}_2(t)$ and $\mathbf{x}_{3n}(t) = \mathbf{x}_3(t)$, one has:

$$\begin{aligned}
\dot{\mathbf{x}}_1(t) &= \frac{\partial\psi(\mathbf{x}_{1n}(t))}{\partial(\mathbf{x}_{1n}(t))}\frac{\mathrm{d}\mathbf{x}_{1n}(t)}{\mathrm{d}t} = \frac{\partial\psi(\mathbf{x}_{1n}(t))}{\partial(\mathbf{x}_{1n}(t))}\dot{\mathbf{x}}_{1n}(t) = \mathbf{x}_{2n}(t) \\
\dot{\mathbf{x}}_{2n}(t) &= \mathbf{x}_{1n}(t) - \frac{B}{J}\mathbf{x}_{2n}(t) + \frac{1}{J}\mathbf{x}_{3n}(t) \\
\dot{\mathbf{x}}_{3n}(t) &= \frac{-1}{\tau}\mathbf{x}_{3n}(t) + \frac{G}{\tau}P(t)
\end{aligned} \tag{35}$$

The following equationing, based on (35), exists if and only if:

$$\frac{\partial\psi(\mathbf{x}_{1n}(t))}{\partial\mathbf{x}_{1n}(t)} \neq 0. \tag{36}$$

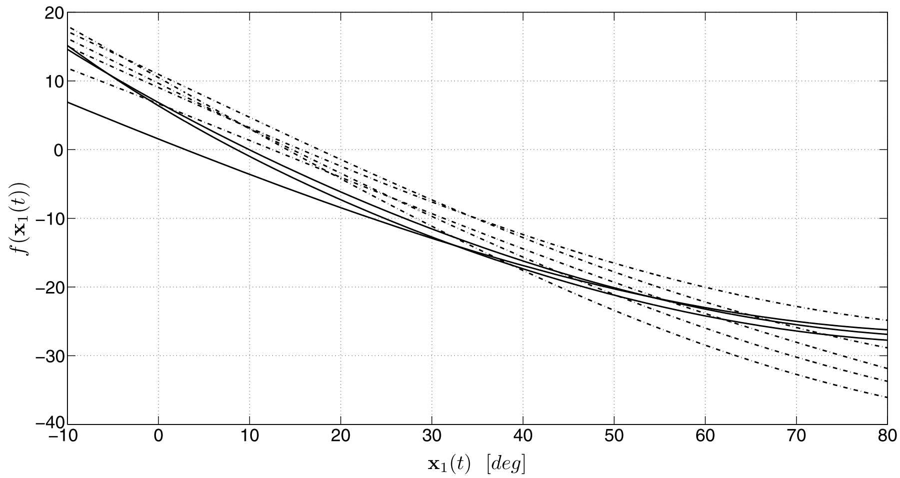

Fig. 2. Function $f(\mathbf{x}_1(t))$: (·— Healthy group) and (— Paraplegic group).

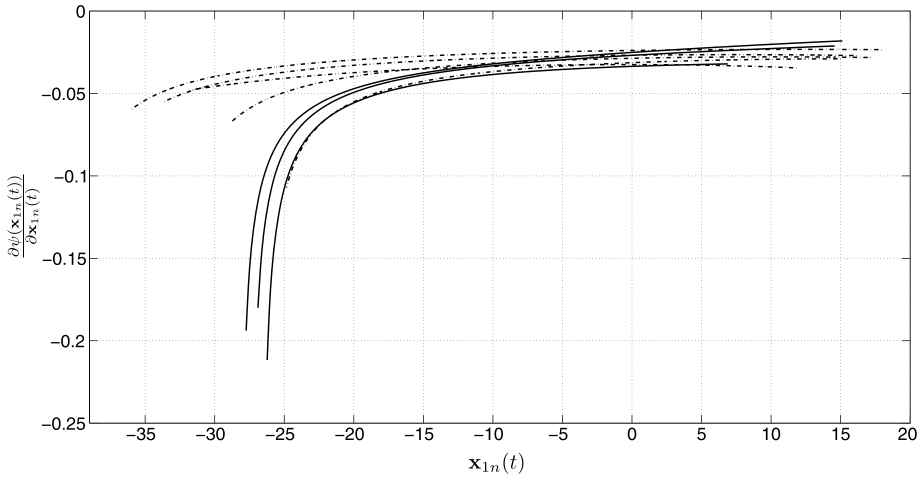

Fig. 3. Verification of (36): (·— Healthy group) and (— Paraplegic group).

Fig. 3 shows that the condition (36) is satisfied, considering the groups studied in [5], described in Table 3. Note that (36) holds in all cases, in the operation region ($\mathbf{x}_{1n}(t) \in [-26.4080,\ 6.3820]$). Thus, we consider that this condition is an admissible assumption.

From (35) and (36) one obtains the new equation:

$$\dot{\mathbf{x}}_{1n}(t) = \frac{1}{\frac{\partial\psi(\mathbf{x}_{1n}(t))}{\partial\mathbf{x}_{1n}(t)}}\mathbf{x}_2(t) = \tilde{f}_{12n}(\mathbf{x}_{1n}(t))\mathbf{x}_2(t) = \tilde{f}_{12n}(\mathbf{x}_{1n}(t))\mathbf{x}_{2n}(t), \tag{37}$$

where:

$$\tilde{f}_{12n}(\mathbf{x}_{1n}(t)) = \frac{1}{\frac{\partial\psi(\mathbf{x}_{1n}(t))}{\partial\mathbf{x}_{1n}(t)}}.$$

The equations (35), (36) and (37) redefine the matrix (30), to represent the new system:

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}}_{1n}(t) \\ \dot{\mathbf{x}}_{2n}(t) \\ \dot{\mathbf{x}}_{3n}(t) \end{bmatrix}}_{\dot{\mathbf{x}}_n(t)} = \underbrace{\begin{bmatrix} 0 & \tilde{f}_{12n}(\mathbf{x}_{1n}(t)) & 0 \\ 1 & \frac{-B}{J} & \frac{1}{J} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}}_{\mathbf{A}(\mathbf{x}_n(t))} \underbrace{\begin{bmatrix} \mathbf{x}_{1n}(t) \\ \mathbf{x}_{2n}(t) \\ \mathbf{x}_{3n}(t) \end{bmatrix}}_{\mathbf{x}_n(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{G}{\tau} \end{bmatrix}}_{\mathbf{g}(\mathbf{x}_n(t))} P. \tag{38}$$

The problem consists of the control input design such that the known equilibrium point $(\theta_{v0}, \dot{\theta}_{v0}, M_{a0})$ of the controlled system is asymptotically stable and thus, from resting conditions, the leg converges to $\theta_v = \theta_{v0}$. Then, consider, for the tracking, that when $t \to \infty$ the desired state vector $\mathbf{x}_{in}(t) \to \mathbf{x}_{ind}$, $i = 1, 2, 3$, where, from (32), $\mathbf{x}_{1nd} = f(\theta_{v0})$, $\mathbf{x}_{2nd} = d(\theta_{v0})/dt = 0$ and $\mathbf{x}_{3nd} = x_3 = M_{a0}$. Write the reference virtual variables as a vector:

$$\mathbf{x}_{nd} = \begin{bmatrix} \mathbf{x}_{1nd} \\ \mathbf{x}_{2nd} \\ \mathbf{x}_{3nd} \end{bmatrix} = \begin{bmatrix} f(\theta_{v0}) \\ 0 \\ M_{a0} \end{bmatrix}. \tag{39}$$

As a natural consequence, $\dot{\mathbf{x}}_{nd} = 0$. Therefore, according to (15):

$$\mathbf{g}(\mathbf{x}_n)(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) = -\mathbf{A}(\mathbf{x}_n)\mathbf{x}_{nd} + \underbrace{\dot{\mathbf{x}}_{nd}}_{=0}. \tag{40}$$

Replacing the terms of the matrices defined in (38) in (40), one gets:

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{G}{\tau} \end{bmatrix}}_{\mathbf{g}(\mathbf{x}_n(t))}(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) = -\underbrace{\begin{bmatrix} 0 & \tilde{f}_{12n}(\mathbf{x}_{1n}(t)) & 0 \\ 1 & \frac{-B}{J} & \frac{1}{J} \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}}_{\mathbf{A}(\mathbf{x}_n(t))}\underbrace{\begin{bmatrix} f(\theta_{v0}) \\ 0 \\ M_{a0} \end{bmatrix}}_{\mathbf{x}_{nd}} \tag{41}$$

where $\mathbf{u}(t) = P(t)$. Applying the distributive property in (41), one finds:

$$\begin{bmatrix} 0 \\ 0 \\ \frac{G}{\tau}(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) \end{bmatrix} = \begin{bmatrix} 0 \\ -f(\theta_{v0}) - \frac{M_{a0}}{J} \\ \frac{M_{a0}}{\tau} \end{bmatrix}. \tag{42}$$

Note that, from (19) and (39), when $\mathbf{x}_n = \mathbf{x}_{nd}$, then:

$$\dot{\mathbf{x}}_{2n} = \dot{\mathbf{x}}_{2nd} = 0 = \mathbf{x}_{1nd} - \frac{B}{J}\mathbf{x}_{2nd} + \frac{1}{J}\mathbf{x}_{3nd} = f(\theta_{v0}) + \frac{M_{a0}}{J}.$$

Then, equations (18) and (42) hold for $t \geq 0$. Hence, from (42), the control law is determined as follows:

$$\begin{aligned} \frac{G}{\tau}(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) &= \frac{M_{a0}}{\tau}, \\ \mathbf{u}(t) &= \frac{M_{a0}}{G} + \boldsymbol{\tau}_c(t), \end{aligned} \tag{43}$$

where, from (13), $\boldsymbol{\tau}_c(t) = -\sum_{i=1}^{r} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{F}_i\widetilde{\mathbf{x}}_n(t)$ and $\widetilde{\mathbf{x}}_n(t) = \mathbf{x}_n(t) - \mathbf{x}_{nd}$.
The vector $\widetilde{\mathbf{x}}_n(t)$ denotes the tracking error vector.

5.2. Tracking results with regulator

In this subsection, the simulation results, obtained through Matlab, are presented. Considering the curve of the nonlinear function $\widetilde{f}_{12n}(\mathbf{x}_{1n}(t))$ defined in (33)–(37) and represented in Fig. 4, the local models $(\mathbf{A}_i, \mathbf{B}_i)$, $i = 1, 2$ which represent the plant (38), as a T–S fuzzy model described in (4), for $r = 2$, can be obtained applying the method proposed in [45]. From Fig. 4, the minimum and the maximum values of $\widetilde{f}_{12n}(\mathbf{x}_{1n}(t))$, for $0 \leq \theta_v \leq 70°$, that is, $-26.4080 \leq x_{1n} \leq 6.3820$, are determined, considering that, during the operation of the system:

$$a_{121} = \max\left\{\tilde{f}_{12n}(\mathbf{x}_{1n}(t))\right\} = -11.8078,$$

$$a_{122} = \min\left\{\tilde{f}_{12n}(\mathbf{x}_{1n}(t))\right\} = -42.6560.$$

Thus, the local models matrices are given by [45]:

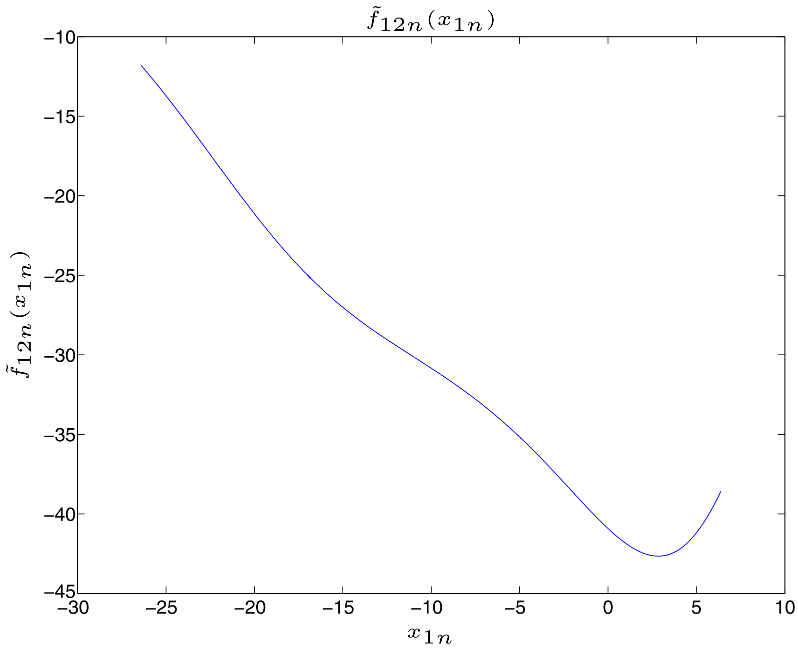

Fig. 4. Curve of $\tilde{f}_{12n}(x_{1n}(t))$.

$$\mathbf{A}_1 = \begin{bmatrix} 0 & -11.8078 & 0 \\ 1 & -0.7458 & 2.7624 \\ 0 & 0 & -1.0515 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} 0 & -42.6560 & 0 \\ 1 & -0.7458 & 2.7624 \\ 0 & 0 & -1.0515 \end{bmatrix}, \tag{44}$$

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 44690 \end{bmatrix}.$$

In the new formulation, the membership functions, for the tracker case, are the following [45]:

$$\alpha_1(\mathbf{x}_{1n}(t)) = \frac{\tilde{f}_{12n}(\mathbf{x}_{1n}(t)) - a_{122}}{a_{121} - a_{122}}, \tag{45}$$

and

$$\alpha_2(\mathbf{x}_{1n}(t)) = \frac{\tilde{f}_{12n}(\mathbf{x}_{1n}(t)) - a_{121}}{a_{122} - a_{121}}. \tag{46}$$

Note that, from (45) and (46):

$$\alpha_1(\mathbf{x}_{1n}(t)) + \alpha_2(\mathbf{x}_{1n}(t)) = 1, \quad \alpha_1(\mathbf{x}_{1n}(t)) \geq 0, \ \ \alpha_2(\mathbf{x}_{1n}(t)) \geq 0 \tag{47}$$

and the nonlinear system (38) can be exactly represented by (44)–(47) and

$$\begin{aligned} \dot{\mathbf{x}}_n(t) &= \left(\sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{A}_i\right)\mathbf{x}_n(t) + \left(\sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{B}_i\right)P(t), \\ &= \mathbf{A}(\boldsymbol{\alpha}(\mathbf{x}_{1n}(t)))\mathbf{x}_n(t) + \mathbf{B}(\boldsymbol{\alpha}(\mathbf{x}_{1n}(t)))P(t). \end{aligned} \tag{48}$$

Considering the regulator design method described in Subsections 3.1 and 3.2 for the nonlinear system (48), from (20), the control law is the following:

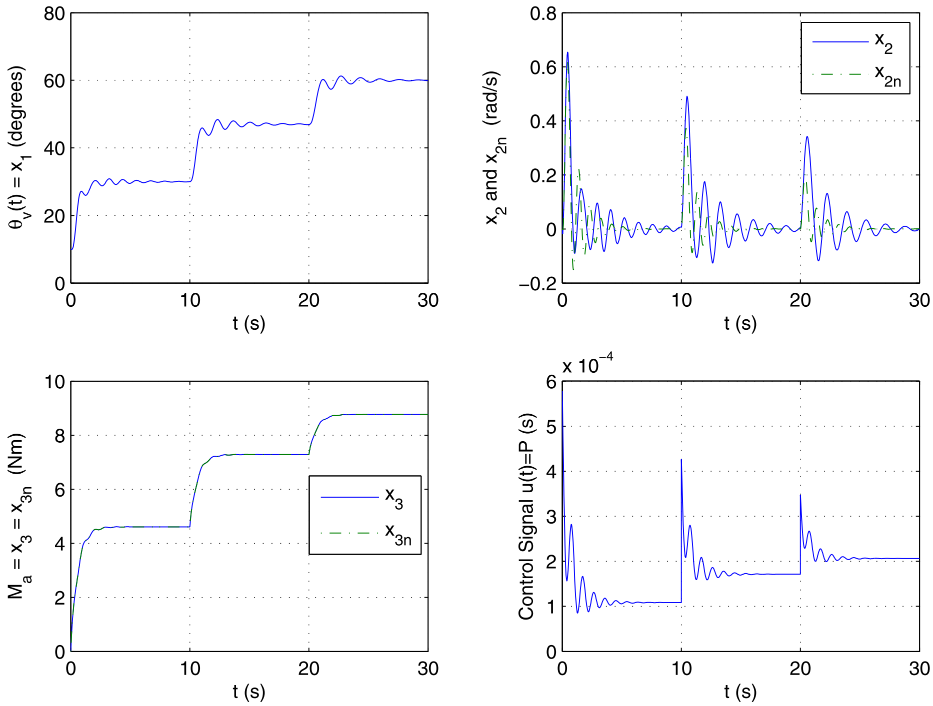

Fig. 5. Simulation of the controlled paraplegic system (30), (49) and (54). The reference angles are 30°, for 0 s $\le t < 10$ s, 47°, for 10 s $\le t < 20$ s, and 60°, for $t \ge 20$ s.

$$P(t) = -\left(\sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{F}_i\right)\tilde{\mathbf{x}}_n(t) + \left(\frac{G}{\tau}\right)^{-1}\left(\underbrace{\dot{\mathbf{x}}_{1nd}}_{0} - \begin{bmatrix} 0 \\ 0 \\ -1.0515 \end{bmatrix}^T \mathbf{x}_{nd}\right), \tag{49}$$

where $\tilde{\mathbf{x}}_n(t) = \mathbf{x}_n(t) - \mathbf{x}_{nd}$, with $\mathbf{x}_{nd}$ defined in (39).

In this design, the specifications are the asymptotic stability of the equilibrium point $\mathbf{x}_n(t) = \mathbf{x}_{nd}$, a decay rate greater than or equal to $\beta_r = 1.6$ and an input constraint $||P(t)||_2 \le \mu + 1.0515(\frac{G}{\tau})^{-1}$, where from Table 1 $1.0515(\frac{G}{\tau})^{-1} = 2.3529 \times 10^{-5}$, and $\mu = 1 \times 10^{-3}$ s, for an initial condition $\mathbf{x}_n(0) = [\,-20.1178\ 0\ 0\,]^T$. For an LMI-based design of the regulator gains, following the method presented in [15,41,42], find matrices $X = X^T > 0$ and M_i, $i = 1, 2, \dots, r$, such that the LMIs (50)–(53) are feasible:

$$\mathbf{X}\mathbf{A}_i^T - \mathbf{M}_i^T\mathbf{B}_i^T + \mathbf{A}_i\mathbf{X} - \mathbf{B}_i\mathbf{M}_i + 2\beta_r\mathbf{X} < 0, \quad i = 1, 2, \dots, r, \tag{50}$$

$$\mathbf{X}\mathbf{A}_i^T - \mathbf{M}_j^T\mathbf{B}_i^T + \mathbf{X}\mathbf{A}_j^T - \mathbf{M}_i^T\mathbf{B}_j^T + \mathbf{A}_i\mathbf{X} + \mathbf{B}_i\mathbf{M}_j + \mathbf{A}_j\mathbf{X} - \mathbf{B}_j\mathbf{M}_i + 4\beta_r\mathbf{X} \le 0, \tag{51}$$

$$i = 1, 2, \dots, r, \quad j = 2, 3, \dots, r \ \text{ and } \ j > i,$$

$$\begin{bmatrix} X & \mathbf{M}_i^T \\ \mathbf{M}_i & \mu^2 I \end{bmatrix} \ge 0, \quad i = 1, 2, \dots, r, \tag{52}$$

$$\begin{bmatrix} 1 & \mathbf{x}_n(0)^T \\ \mathbf{x}_n(0) & \mathbf{X} \end{bmatrix} \ge 0. \tag{53}$$

The LMIs (50)–(51) were feasible and thus, the gains were obtained as follows [15,41,42]:

$$\mathbf{F}_1 = \mathbf{M}_1\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0237 & 0.3929 & 0.2093 \end{bmatrix},$$

$$\mathbf{F}_2 = \mathbf{M}_2\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0301 & 0.2511 & 0.1870 \end{bmatrix}. \tag{54}$$

Fig. 5 shows the simulation results of the paraplegic patient closed-loop system of the plant (30), (or its equivalent representations (38) and (44)–(48)) with the control law (49).

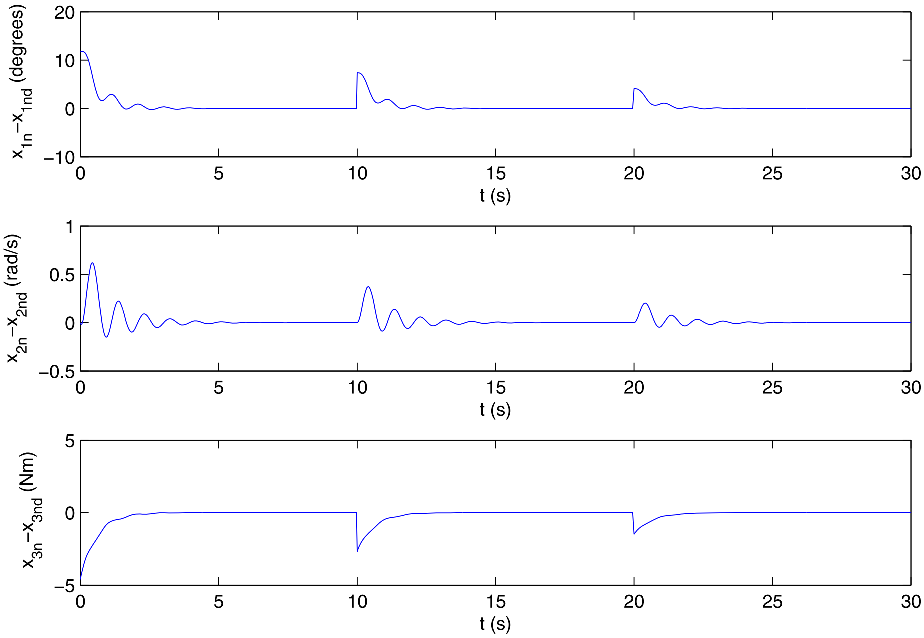

Fig. 6. Signal tracking, for the reference angles 30°, for $0 \text{ s} \le t < 10$ s, 47°, for $10 \text{ s} \le t < 20$ s, and 60°, for $t \ge 20$ s.

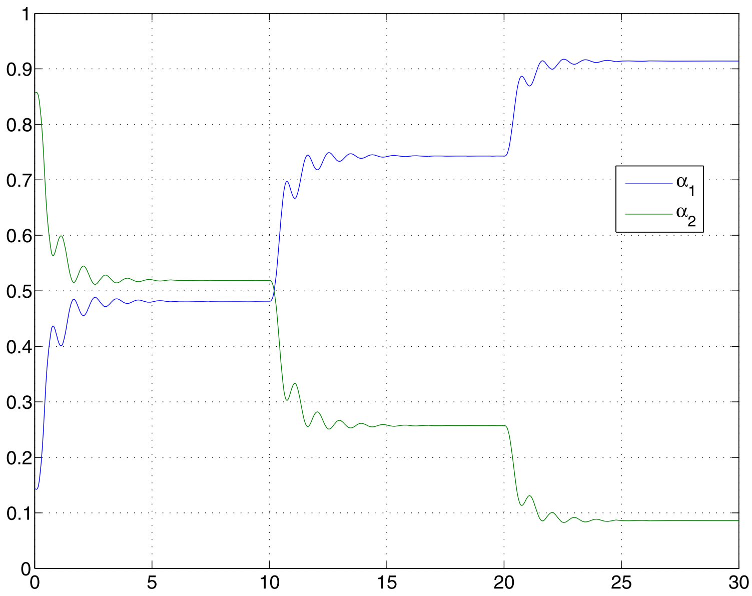

Fig. 7. Membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$ – tracking results with regulator.

Fig. 6 shows the behavior of the signal tracking, using the equation $\widetilde{\mathbf{x}}_n(t) = \mathbf{x}_n(t) - \mathbf{x}_{nd}$. Note that, when $t \longrightarrow \infty$, $\widetilde{\mathbf{x}}_n(t) = 0$.

Fig. 7 shows the membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$.

The results considering tracking are similar to the results that assume a specific design for this operation point. Using the presented tracking, one can easily find the desired responses when the desired position changes, without the need of calculating the new operation point and find a new solution of a set of LMIs, as proposed in [19–23].

5.3. Regulator and observer design with reference tracking

For tracking, consider the representation of the plant described in (38) and (44)–(48), with the output $y(t) = x_{1n}(t)$ given in (6) and the input $u(t) = P(t)$. The observer equations, for the plant (48), given by (23) and (24), remembering that $y(t) = x_{1n}(t)$ is available, can be implemented and are the following:

$$\dot{\hat{\mathbf{x}}}_n(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{x}_{1n}(t)) \left(\mathbf{A}_i \hat{\mathbf{x}}_n(t) + \mathbf{B}_i \mathbf{P}(t) + \mathbf{H}_i \left(\mathbf{y}(t) - \hat{\mathbf{y}}(t) \right) \right),$$
$$\hat{\mathbf{y}}(t) = \mathbf{C}\hat{\mathbf{x}}_n(t), \tag{55}$$

where $\mathbf{C} = [1\ 0\ 0]$.

The control law using the observer state vector, based on (25), is now given by:

$$P(t) = -\left(\sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{F}_i \right) (\hat{\mathbf{x}}_n(t) - \mathbf{x}_{nd}) + \left(\frac{G}{\tau} \right)^{-1} \left(\underbrace{\dot{\mathbf{x}}_{1nd}}_{0} - \begin{bmatrix} 0 \\ 0 \\ -1.0515 \end{bmatrix}^T \mathbf{x}_{nd} \right). \tag{56}$$

Denote the control law with the observer state vector (regulator with observer) given in (56) by $\mathbf{P}_{ro}(t)$ and the control law using only the plant state vector (regulator) presented in (49) by $\mathbf{P}_r(t)$. Then, from (49) and (56), note that:

$$\mathbf{P}_{ro}(t) = \mathbf{P}_r(t) + \sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{F}_i(\mathbf{x}_n(t) - \hat{\mathbf{x}}_n(t)) = \mathbf{P}_r(t) + \sum_{i=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\mathbf{F}_i \mathbf{e}_n(t), \tag{57}$$

where $\mathbf{e}_n(t) = \mathbf{x}_n(t) - \hat{\mathbf{x}}_n(t)$. Note that, for an input $\mathbf{P}(t) = \mathbf{P}_r(t)$, the plant (38) and (44)–(48) can be described from the analysis described in Subsection 3.1, equations (10)–(14):

$$\dot{\tilde{\mathbf{x}}}_n(t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\alpha_j(\mathbf{x}_{1n}(t))(\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j)\tilde{\mathbf{x}}_n(t), \tag{58}$$

where $\tilde{\mathbf{x}}_n(t) = \mathbf{x}_n(t) - \tilde{\mathbf{x}}_{nd}$. Then, for an input $\mathbf{P}_{ro}(t)$, given in (56), from (38), (44)–(48), (57) and (58), it follows that:

$$\dot{\tilde{\mathbf{x}}}_n(t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\alpha_j(\mathbf{x}_{1n}(t))[(\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j)\tilde{\mathbf{x}}_n(t) + \mathbf{B}_i\mathbf{F}_j\mathbf{e}_n(t)]. \tag{59}$$

Consider that, for the plant (30), $\mathbf{x}_1(t)$ is available and $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$ are not available. Then, as from (32) $\mathbf{x}_{1n}(t)$ is available, one can implement the observer described in (55). From the observer equations (55) and the plant equation (48), note that:

$$\dot{\mathbf{e}}_n(t) = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\alpha_j(\mathbf{x}_{1n}(t))(\mathbf{A}_i - \mathbf{H}_i\mathbf{C})\mathbf{e}_n(t), \tag{60}$$

where $\mathbf{e}_n(t) = \mathbf{x}_n(t) - \hat{\mathbf{x}}_n(t)$.

Finally, from (59) and (60), one obtains the equation of the controlled system with observer [46]:

$$\begin{bmatrix} \dot{\tilde{\mathbf{x}}}_n(t) \\ \dot{\mathbf{e}}_n(t) \end{bmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_i(\mathbf{x}_{1n}(t))\alpha_j(\mathbf{x}_{1n}(t)) \begin{bmatrix} \mathbf{A}_i - \mathbf{B}_i\mathbf{F}_j & \mathbf{B}_i\mathbf{F}_j \\ \mathbf{0} & \mathbf{A}_i - \mathbf{H}_i\mathbf{C} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_n(t) \\ \mathbf{e}_n(t) \end{bmatrix}. \tag{61}$$

From this fact, in [46], the authors proved the separation principle for this class of systems, that allows two distinct designs in this case: one for the regulator gains and another for the observer gains.

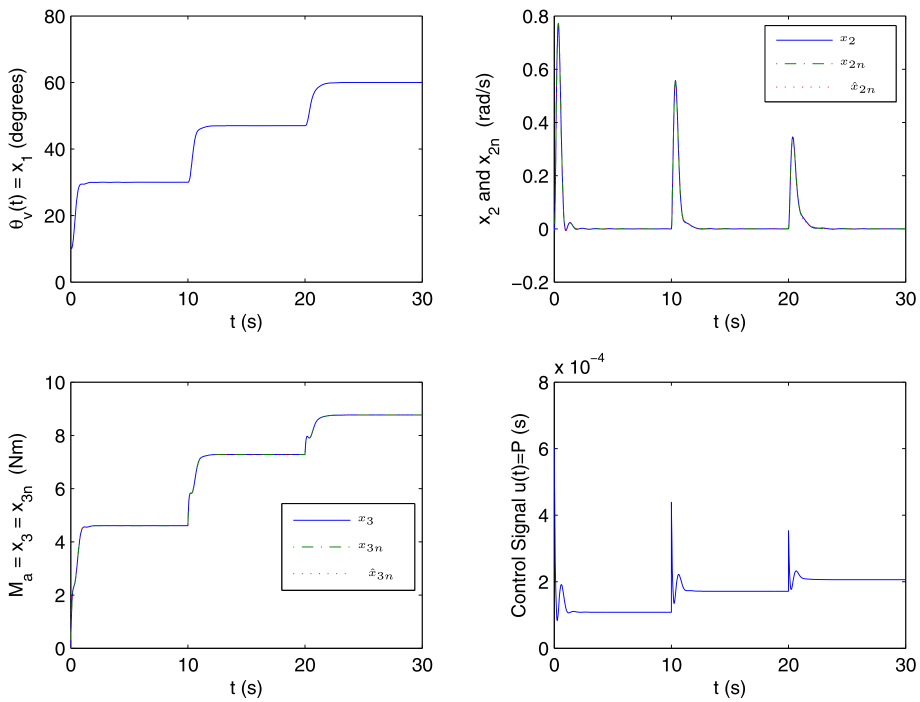

Fig. 8. Dynamic equations simulation for the paraplegic model, for the reference angles 30°, 47° and 60° and initial condition $\mathbf{e}(t) = [0\ \ 0\ \ 0]$.

5.4. Tracking results with regulator and observer

For simulation of (61), consider only $x_1(t) = \theta_v(t)$ available for measurement (therefore $x_{1n}(t)$ is also available). The LMIs (50) and (51), where $\mathbf{X} = \mathbf{X}^T > 0$ are used for designing the regulator gains, and the LMIs (62)–(63), where $\mathbf{P} = \mathbf{P}^T > 0$, for obtaining the observer gains [15,41,42].

$$\mathbf{PA}_i + \mathbf{A}_i^T\mathbf{P} - \mathbf{M}_i\mathbf{C}_i - \mathbf{C}_i^T\mathbf{M}_i^T + 2\beta_o\mathbf{P} < 0, \quad i = 1, 2, \ldots, r. \tag{62}$$

$$\mathbf{A}_i^T\mathbf{P} - \mathbf{C}_j^T\mathbf{M}_i^T + \mathbf{A}_j^T\mathbf{P} - \mathbf{C}_i^T\mathbf{M}_j^T + \mathbf{PA}_i - \mathbf{M}_i\mathbf{C}_j + \mathbf{PA}_j - \mathbf{M}_j\mathbf{C}_i + 4\beta_o\mathbf{P} \le 0, \tag{63}$$
$$i = 1, 2, ..., r, \quad j = 2, 3, ..., r \ \text{ and } j > i.$$

Thus, the LMIs (62) and (63) with $\mathbf{P} = \mathbf{P}^T > 0$, if feasible, guarantee global asymptotic stability of the equilibrium point of $\mathbf{e}_n = \mathbf{0}$ of the observer, with decay rate greater than or equal to β_o.

The design considered as decay rate specifications $\beta_r = 1.6$ in (50) and (51) (regulator), $\beta_o = 8.0$ in (62) and (63) (observer) and an input constraint $\mu = 1 \times 10^{-3}$ in the LMIs (52) and (53), and for initial error vector $\mathbf{e}(t) = [0\ \ 0\ \ 0]^T$. The results are shown in Fig. 8 and the obtained regulator and observer gains were, respectively:

$$\mathbf{F}_1 = \mathbf{M}_1\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0170 & 0.4605 & 0.2184 \end{bmatrix},$$
$$\mathbf{F}_2 = \mathbf{M}_2\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0251 & 0.3373 & 0.2025 \end{bmatrix},$$
$$\mathbf{L}_1 = \mathbf{P}^{-1}\mathbf{M}_1 = 10^3 \times \begin{bmatrix} 0.0571 \\ -0.9149 \\ -3.6903 \end{bmatrix}, \ \mathbf{L}_2 = \mathbf{P}^{-1}\mathbf{M}_2 = 10^4 \times \begin{bmatrix} 0.0196 \\ -0.3526 \\ -1.4325 \end{bmatrix}.$$

Fig. 8 shows the time evolution of equation (61). Note that only $\theta_v(t)$ is available for measurement. Fig. 9 shows the behavior of $\widetilde{\hat{\mathbf{x}}}_n(t) = \mathbf{x}_n(t) - \mathbf{x}_{nd} - \mathbf{e}_n(t)$. When $t \longrightarrow \infty$, $\widetilde{\hat{\mathbf{x}}}_n(t) = 0$. Since the initial error conditions are null and the observer decay rate is five times faster than the regulator decay rate, Fig. 9 is equal to Fig. 6.

Fig. 10 shows the membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$.

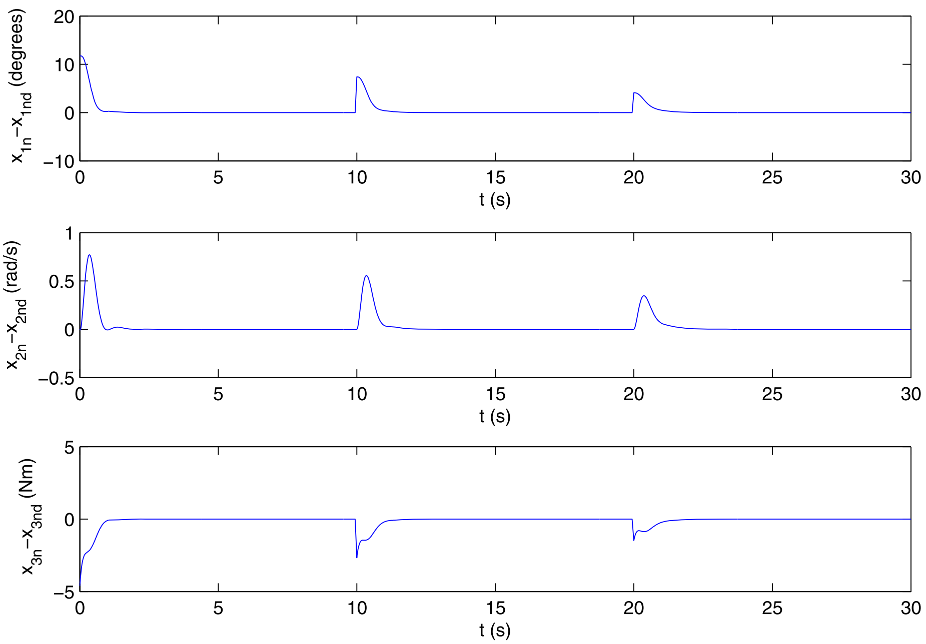

Fig. 9. Signal tracking for the reference angles 30°, 47° and 60°, and initial condition $\mathbf{e}(t) = [0\ \ 0\ \ 0]$.

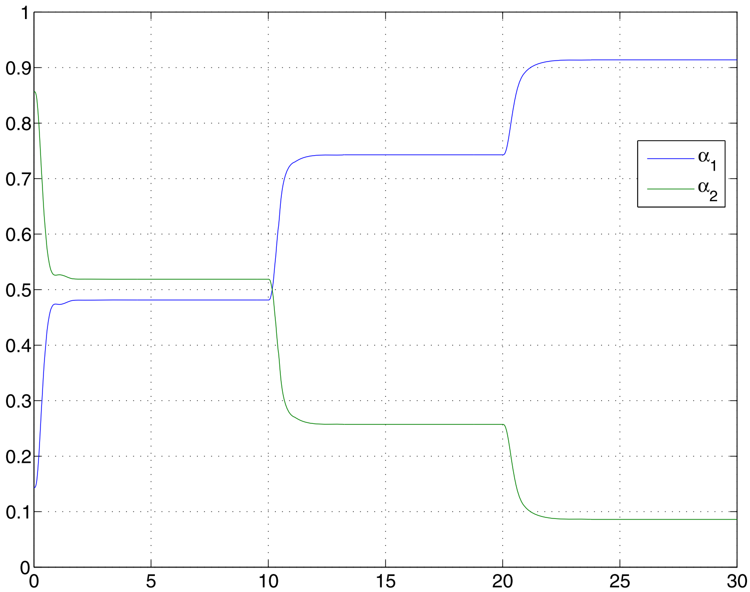

Fig. 10. Membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$ – Tracking results with regulator and observer, with initial condition $\mathbf{e}(t) = [0\ \ 0\ \ 0]$.

For observer tracking, one considers the existence of an initial error between the plant state and the estimated states. In practical applications, it is more real, in view of the difficulty for know the real state parameters of the plant. From (61), the initial error vector is defined as $\mathbf{e}(t) = [-0.8\ \ 0\ \ -0.4]$. One considers that the transient response velocity and input signal constraint are equal to the preview case. The result is shown in Fig. 11. The state variable $\mathbf{x}_{3n}$ is shown with more details in Fig. 12. The regulator and observer gains are, respectively:

$$\mathbf{F}_1 = \mathbf{M}_1\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0170 & 0.4605 & 0.2184 \end{bmatrix},$$
$$\mathbf{F}_2 = \mathbf{M}_2\mathbf{X}^{-1} = 10^{-3} \times \begin{bmatrix} 0.0251 & 0.3373 & 0.2025 \end{bmatrix}.$$

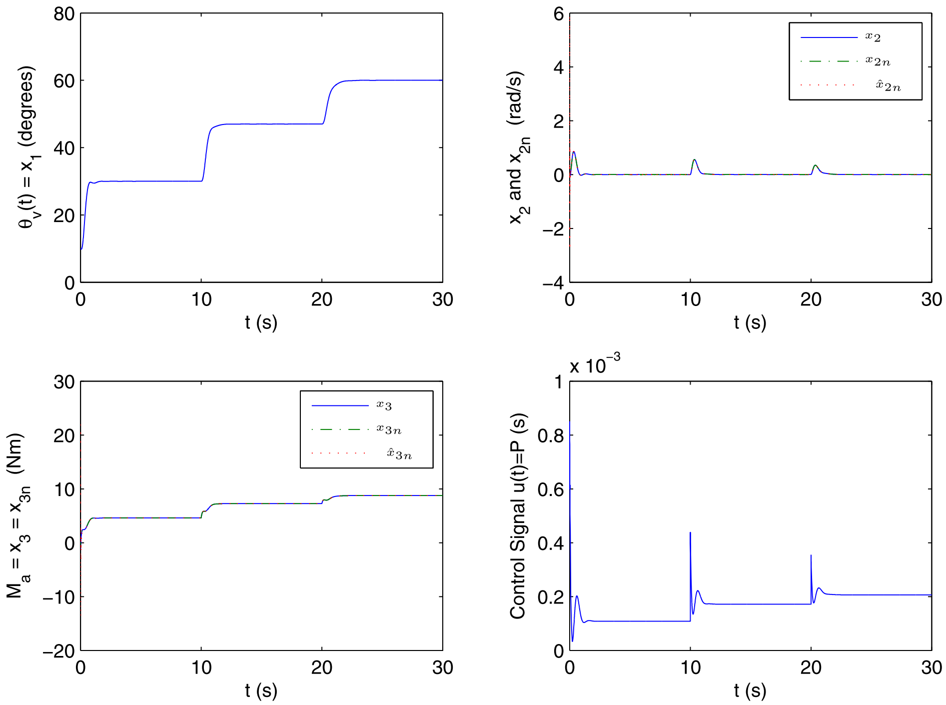

Fig. 11. Dynamic equations simulation for the paraplegic model, for the reference angles 30°, 47° and 60° and initial condition $\mathbf{e}(t) = [-0.8\ 0\ -0.4]$.

$$\mathbf{L}_1 = \mathbf{P}^{-1}\mathbf{M}_1 = 10^3 \times \begin{bmatrix} 0.0571 \\ -0.9149 \\ -3.6903 \end{bmatrix}, \ \mathbf{L}_2 = \mathbf{P}^{-1}\mathbf{M}_2 = 10^4 \times \begin{bmatrix} 0.0196 \\ -0.3526 \\ -1.4325 \end{bmatrix}.$$

Fig. 13 shows the behavior of equation $\widetilde{\hat{\mathbf{x}}}_n(t) = \mathbf{x}_n(t) - \mathbf{x}_{nd} - \mathbf{e}_n(t)$ for the reference angles 30°, 47° and 60°. When $t \longrightarrow \infty$, $\widetilde{\hat{\mathbf{x}}}_n(t) = 0$.

Fig. 14 shows the membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$.

The behavior of the curves attend the control laws that control the system.

6. Tracking control conditions for a class of nonlinear systems

This section presents new conditions regarding the design of the output tracking control method proposed in [43]. The main idea was, based on the procedure presented in Subsection 5.1, to use a suitable nonlinear transformation and represent a class of plants as $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{g}(\mathbf{x})\mathbf{u}$, such that the required tracking conditions (16)–(18), that were proposed in [43], hold. This result allows the application of the state feedback tracking method presented in Subsections 3.1, 3.2, 5.1 and 5.2, and also of the output tracking procedure based on observer and state regulator described in Subsections 3.3, 5.3 and 5.4, for a broader class of nonlinear plants than that considered in [43]. By the authors knowledge there are not general conditions for the representation of nonlinear plants as $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{g}(\mathbf{x})\mathbf{u}$ such that (16)–(18) hold.

Consider that a nonlinear plant can be described by

$$\dot{\mathbf{x}}_p = \mathbf{A}_p(\mathbf{x}_p)\mathbf{x}_p + \mathbf{g}_p(\mathbf{x}_p)\mathbf{u}, \quad \mathbf{y} = \mathbf{q}(\mathbf{x}_p), \tag{64}$$

$$\mathbf{x}_p = \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix}, \qquad \mathbf{A}_p(\mathbf{x}_p) = \begin{bmatrix} \mathbf{A}_{pn-m}(\mathbf{x}_p) \\ \cdots\cdots\cdots \\ \mathbf{A}_{pm}(\mathbf{x}_p) \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{T}(\mathbf{x}_p) & 0 \\ \tilde{\mathbf{R}}(\mathbf{z}) & \mathbf{V}(\mathbf{x}_p) & S \\ \cdots & \cdots & \cdots \\ & \mathbf{A}_{pm}(\mathbf{x}_p) & \end{bmatrix}, \qquad \mathbf{g}_p(\mathbf{x}_p) = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ \mathbf{B}(\mathbf{x}_p) \end{bmatrix} \tag{65}$$

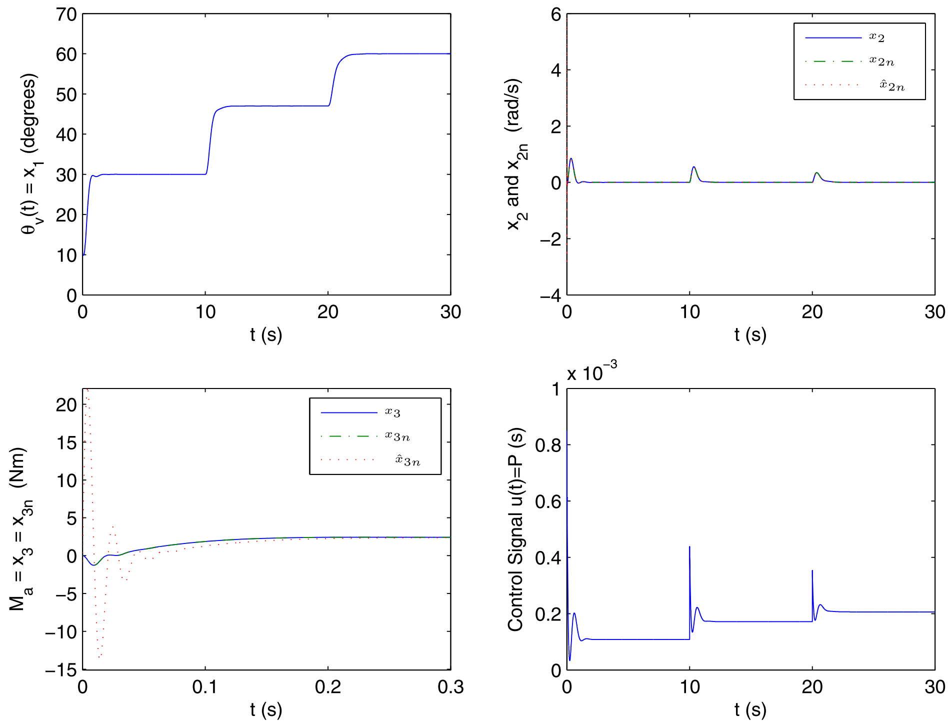

Fig. 12. Dynamic equations simulation for the paraplegic model, for the reference angles 30°, 47° and 60° and initial condition $\mathbf{e}(t) = [-0.8\ \ 0\ \ -0.4]$, with more details at the state variable $\mathbf{x}_{3n}$.

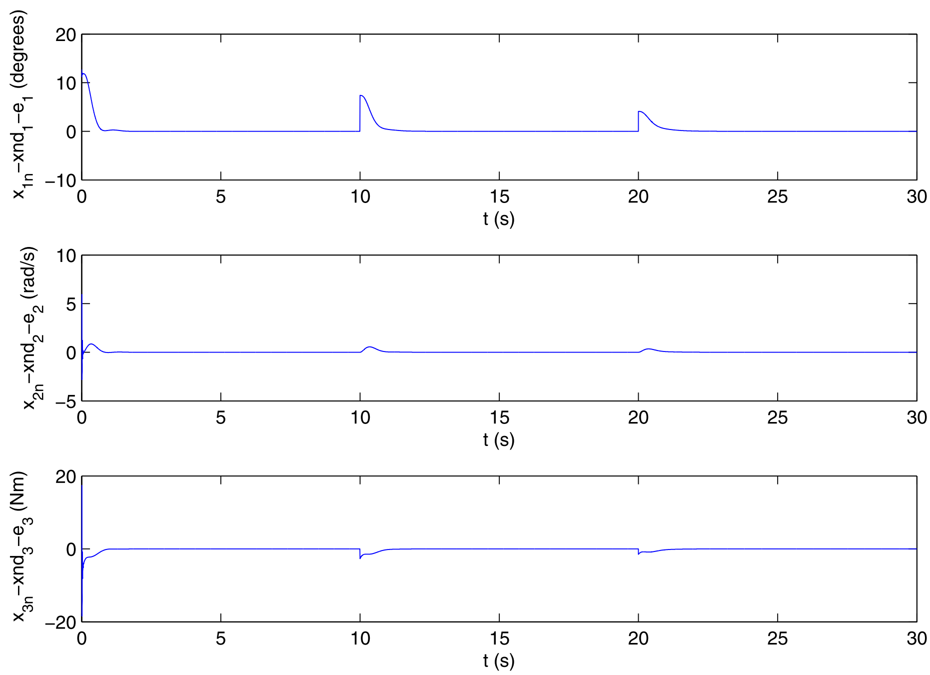

Fig. 13. Simulation of observer for the reference angles 30°, 47° and 60° and initial condition $\mathbf{e}(t) = [-0.8\ \ 0\ \ -0.4]$.

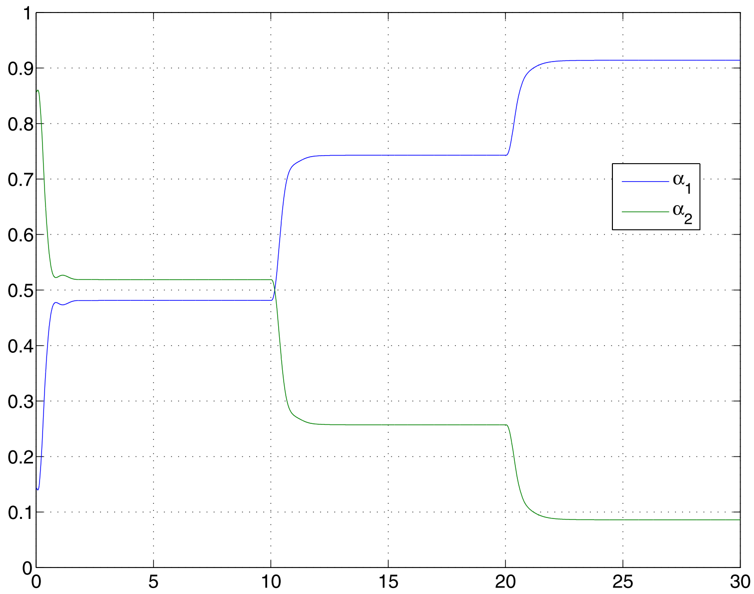

Fig. 14. Membership functions $\alpha_1(\mathbf{x}_{1n}(t))$ and $\alpha_2(\mathbf{x}_{1n}(t))$ – Tracking results with regulator and observer, with initial condition $\mathbf{e}(t) = [-0.8\ \ 0\ \ -0.4]$.

where $\mathbf{x}_p \in \Re^n$, $\mathbf{u}$ and $\mathbf{y} \in \Re^m$, $n-m$ is even, $\mathbf{w}$ and $\mathbf{z} \in \Re^{\frac{(n-m)}{2}}$, $\mathbf{s} \in \Re^m$, $\mathbf{S} \in \Re^{\frac{(n-m)}{2}\times m}$ is a constant matrix, $\mathbf{T}(\mathbf{x}_p)$, $\tilde{\mathbf{R}}(\mathbf{z})$ and $\mathbf{V}(\mathbf{x}_p) \in \Re^{\frac{(n-m)}{2}\times\frac{(n-m)}{2}}$, $\mathbf{A}_{pm}(\mathbf{x}_p) \in \Re^{m\times n}$ and $\mathbf{B}(\mathbf{x}_p) \in \Re^{m\times m}$.

The goal of the output tracking problem described in Subsection 3.1 and proposed in [43] is to assure that the equilibrium point of the controlled system is asymptotically stable and $\mathbf{y}(t) - \bar{\mathbf{r}}(t) \to 0$ when $t \to \infty$, where $\bar{\mathbf{r}}(t)$ denotes the desired reference signal.

Consider that $\bar{\mathbf{r}}(t) = \bar{\mathbf{r}}$ is a constant vector and there exists a constant vector $\mathbf{x}_{pd} = [\mathbf{z}_d^T\ \mathbf{w}_d^T\ \mathbf{s}_d^T]^T$, such that (64) holds and $\mathbf{y}_d = \bar{\mathbf{r}} = \mathbf{g}_p(\mathbf{x}_{pd})$. Therefore, from (64) and (65), note that

$$\begin{aligned}
&\dot{\mathbf{z}}_d = 0 = \mathbf{T}(\mathbf{x}_{pd})\mathbf{w}_d,\\
&\dot{\mathbf{w}}_d = 0 = \tilde{\mathbf{R}}(\mathbf{z}_d)\mathbf{z}_d + \mathbf{V}(\mathbf{x}_{pd})\mathbf{w}_d + \mathbf{S}\mathbf{s}_d, = \mathbf{z}_{nd} + \mathbf{V}(\mathbf{x}_{pd})\mathbf{w}_d + \mathbf{S}\mathbf{s}_d,
\end{aligned} \tag{66}$$

where $\mathbf{z}_{nd} = \tilde{\mathbf{R}}(\mathbf{z}_d)\mathbf{z}_d$.

The next theorem presents sufficient conditions for the representation of the plant (64), (65) with (66) such that the output tracking conditions (16)–(18) hold.

Theorem 1. *Consider the plant (64) and (65), a constant vector $\mathbf{x}_{pd} = [\mathbf{z}_d^T\ \mathbf{w}_d^T\ \mathbf{s}_d^T]^T$ such that the tracking conditions $\mathbf{y}_d = \bar{\mathbf{r}} = \mathbf{g}_p(\mathbf{x}_{pd})$ and (66) hold. Suppose that in an operation region Ω, $\mathbf{T}(\mathbf{x}_p)$ and $\mathbf{B}(\mathbf{x}_p)$ are nonsingular, $\mathbf{R}(\mathbf{z}) = \tilde{\mathbf{R}}(\mathbf{z})\mathbf{z} = [R_1(\mathbf{z})\ R_2(\mathbf{z})\ \cdots\ R_{\frac{n-m}{2}}(\mathbf{z})]^T$ is smooth and the Jacobian matrix*

$$\nabla\mathbf{R}(\mathbf{z}) = \begin{bmatrix} \frac{\partial R_1(\mathbf{z})}{\partial z_1} & \frac{\partial R_1(\mathbf{z})}{\partial z_2} & \cdots & \frac{\partial R_1(\mathbf{z})}{\partial z_{\frac{n-m}{2}}} \\ \vdots & & & \vdots \\ \frac{\partial R_{\frac{n-m}{2}}(\mathbf{z})}{\partial z_1} & \frac{\partial R_{\frac{n-m}{2}}(\mathbf{z})}{\partial z_2} & \cdots & \frac{\partial R_{\frac{n-m}{2}}(\mathbf{z})}{\partial z_{\frac{n-m}{2}}} \end{bmatrix} \tag{67}$$

is nonsingular. Then, in the region Ω, the inverse of the function $R : \Re^{\frac{n-m}{2}} \to \Re^{\frac{n-m}{2}}$ there exists, is smooth and the plant (64) and (65) can be rewritten as

$$\begin{bmatrix} \dot{\mathbf{z}}_n \\ \dot{\mathbf{w}} \\ \dot{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} 0 & \nabla\mathbf{R}(\mathbf{z})\mathbf{T}(\mathbf{x}_p) & 0 \\ \mathbf{I}_m & \mathbf{V}(\mathbf{x}_p) & \mathbf{S} \\ \cdots & \cdots\cdots\cdots\cdots & \cdots \\ & \mathbf{A}_{pm}(\mathbf{x}_p) & \end{bmatrix} \begin{bmatrix} \mathbf{z}_n \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{B}(\mathbf{x}_p) \end{bmatrix}\mathbf{u} = \begin{bmatrix} \mathbf{A}_{n-m}(\mathbf{x}_p) \\ \mathbf{A}_m(\mathbf{x}_p) \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \mathbf{B}(\mathbf{x}_p) \end{bmatrix}\mathbf{u},$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{z}_n \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{R}(\mathbf{z}) \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix}, \qquad \mathbf{x}_p = \begin{bmatrix} \mathbf{z} \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^{-1}(\mathbf{z}_n) \\ \mathbf{w} \\ \mathbf{s} \end{bmatrix}. \tag{68}$$

Furthermore, in the region Ω the output tracking conditions (16)–(18) hold for the system (68).

Proof. The nonlinear transformation $\mathbf{z}_n = \tilde{\mathbf{R}}(\mathbf{z})\mathbf{z} = \mathbf{R}(\mathbf{z})$, in an operation region Ω is smooth and the Jacobian matrix $\nabla\mathbf{R}(\mathbf{z})$ given in (67) is nonsingular. Then $\mathbf{R} : \Re^{\frac{n-m}{2}} \to \Re^{\frac{n-m}{2}}$ is a local diffeomorphism in Ω and thus, its inverse $\mathbf{R}^{-1}$ exists and is smooth [49]. Now, from (65) and (67) it follows that

$$\begin{aligned} \dot{\mathbf{z}}_n &= \nabla\mathbf{R}(\mathbf{z})\dot{\mathbf{z}} = \nabla\mathbf{R}(\mathbf{z})\mathbf{T}(\mathbf{x}_p)\mathbf{w}, \\ \dot{\mathbf{w}} &= \tilde{\mathbf{R}}(\mathbf{z})\mathbf{z} + \mathbf{V}(\mathbf{x}_p)\mathbf{w} + \mathbf{S}\mathbf{s} = \mathbf{z}_n + \mathbf{V}(\mathbf{x}_p)\mathbf{w} + \mathbf{S}\mathbf{s}, \\ \dot{\mathbf{s}} &= \mathbf{A}_{pm}(\mathbf{x}_p)\mathbf{x}_p + \mathbf{B}(\mathbf{x}_p)\mathbf{u}. \end{aligned} \tag{69}$$

Note that (69) can be rewritten as (68).

Considering that $\mathbf{T}(\mathbf{x}_p)$ is nonsingular in Ω, from (66) $\mathbf{T}(\mathbf{x}_{pd})\mathbf{w}_d = 0$ and thus $\mathbf{w}_d = 0$. Therefore, from (66) note that $\mathbf{z}_{nd} + \mathbf{S}\mathbf{s}_d = 0$.

The conditions (16)–(18), applied to the system (68), taking into account that $\mathbf{x}_{pd}$ and $\mathbf{x}_d = [\mathbf{z}_{nd}^T\ \mathbf{w}_d^T\ \mathbf{s}_d^T]^T$ are constant vectors, can be described by

$$\begin{aligned} \begin{bmatrix} 0_{n-m} \\ \mathbf{B}(\mathbf{x}_p)(\mathbf{u}(t) - \boldsymbol{\tau}_c(t)) \end{bmatrix} &= \begin{bmatrix} \dot{\mathbf{x}}_{dn-m} - \mathbf{A}_{n-m}(x)\mathbf{x}_d \\ \dot{\mathbf{x}}_{dm} - \mathbf{A}_m(x)\mathbf{x}_d \end{bmatrix} = - \begin{bmatrix} 0 & \nabla\mathbf{R}(\mathbf{z})\mathbf{T}(\mathbf{x}_p) & 0 \\ \mathbf{I}_m & \mathbf{V}(\mathbf{x}_p) & S \\ \cdots & \cdots\cdots\cdots\cdots & \cdots \\ & \mathbf{A}_{pm}(\mathbf{x}_p) & \end{bmatrix} \begin{bmatrix} \mathbf{z}_{nd} \\ \mathbf{w}_d \\ \mathbf{s}_d \end{bmatrix} \\ &= - \begin{bmatrix} 0 \\ 0 \\ \mathbf{A}_{pm}(\mathbf{x}_p)\mathbf{x}_d \end{bmatrix}. \end{aligned} \tag{70}$$

Hence, considering that the plant state vector $\mathbf{x}_p$ is available, one can design a control law based on (20), given by

$$\begin{aligned} \mathbf{u}(t) &= \boldsymbol{\tau}_c(t) - \mathbf{B}(\mathbf{x}_p)^{-1}\mathbf{A}_{pm}(\mathbf{x}_p)\mathbf{x}_d, \\ \boldsymbol{\tau}_c(t) &= -\sum_{i=1}^{r} \alpha_i(\mathbf{x})\mathbf{F}_i(\mathbf{x} - \mathbf{x}_d), \end{aligned} \tag{71}$$

where in (64) and (65), $(\mathbf{A}_p(\mathbf{x}_p),\ \mathbf{B}_p(x_p)) = \sum_{i=1}^{r} \alpha_i(\mathbf{A}_{pi},\ \mathbf{B}_{pi})$, $\alpha_i \in [0,\ 1]$ for $i \in \{1, 2, ..., r\}$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_r = 1$ is a Takagi–Sugeno representation of the plant. The design of the gains $\mathbf{F}_i$, $i = 1, 2, \ldots, r$ can be done using the same procedure described in Section 5. Now, if $\mathbf{x}_p$ is not available and the plant output $\mathbf{y}$ is available, one can use the output tracking procedure, based on observer and state regulator, described in Subsections 3.3, 5.3 and 5.4. Observe that the results given in Theorem 1 also allow the application of the output tracking design methods proposed in [43], when the premise variables of the plant are available (because they depend only on the available output $\mathbf{y}(t)$) and also when it is necessary to estimate the premise variables (because they depend on the plant state vector $\mathbf{x}(t)$ that is not available) considering that the membership functions satisfy a Lipschitz-like condition (see Remark 1 in Subsection 3.3 and [43] for more details). The proof is concluded. □

Remark 2. Note that the procedure described in Theorem 1 can be applied to the plant (30), for the design (based on [43]) of state feedback tracking controllers (as presented in Subsections 3.1, 3.2, 5.1 and 5.2), and also of output tracking controllers (as described in Subsections 3.3, 5.3 and 5.4). In Subsection 5.3 this plant was rewritten as presented in (38), that has the form given in (68), using the definitions (32)–(34). Observe that from (32)–(37) $\tilde{f}_{12n}(x_{1n}) = [\frac{\partial\psi(x_{1n})}{\partial x_{1n}}]^{-1} = [\frac{\partial f^{-1}(x_{1n})}{\partial x_{1n}}]^{-1} = [\frac{\partial x_1}{\partial x_{1n}}]^{-1}$.

Now, following the method proposed in Theorem 1 for the plant (30), then $z = x_1$, $z_n = R(z) = x_{1n} = \tilde{f}_{21}(x_1)x_1 = f(x_1) = f(z)$, $\frac{\partial R(z)}{\partial z} = \frac{\partial f(z)}{\partial z} = \frac{\partial z_n}{\partial z} = \frac{\partial x_{1n}}{\partial x_1}$. Therefore, note that $\frac{\partial R(z)}{\partial z} = \frac{\partial x_{1n}}{\partial x_1} = [\frac{\partial x_1}{\partial x_n}]^{-1} = \tilde{f}_{12n}(x_{1n})$ and then the procedure used in Subsection 5.3 is equivalent to that proposed in Theorem 1, because (38) and (68) represent the same system. An important fact regarding the method given in Theorem 1, is that it was not necessary to find the inverse of functions for the design of the control law, as in the procedure used in Section 5.

7. Conclusions

The design of the regulator and observer, used to control the position of the leg of a paraplegic patient, met the requirements of stability and performance, using an electrical stimulus applied to the quadriceps muscle. The projects were designed with strict Lyapunov functions, based on Linear Matrix Inequalities (LMIs) and Takagi–Sugeno (T–S) fuzzy models for the representation of the nonlinear plant. Some constraints, such as stability, decay rate, and output constraints were considered. Operating points for the position of the leg were specified for projects controlling nonlinear continuous-time (regulator and/or observer tracking the signal with the regulator and observer). New results were obtained for the project of tracking the position of the leg with the regulator and observer. These results facilitate the designer's vision, because there is no need to design specific drivers for each operating point. The controlled system presented a good performance within a specified range of operation. The proposed method, for tracking the position of a leg, can offer more general feasible designs than the methods proposed in [39] and [40] because it allows the tracking of a broader class of signals than the control strategies presented in [39] and [40]. In [39] and [40], are considered only constant reference signals and all set of the desired operation points for obtaining the maximum and minimum values of the plant nonlinear functions, to get the local models following the method proposed in [30]. The proposed method can be implemented the same as [50] and [51]. By the authors knowledge, in [50], the Takagi–Sugeno fuzzy models method to control the knee joint of paraplegic patients was implemented for the first time. Finally, a more general result based on a suitable nonlinear transformation that can be applied to a broader class of plants for output tracking control design based on [43] is also proposed. The method studied in the manuscript, proposed in [43], was applied in the sensorless control of permanent-magnet synchronous motors and experimental results showed that the proposed scheme is suitable in practical applications [52]. Further researches can explore the results presented in this manuscript for uncertain plants, for instance, using switched control laws for uncertain plants described by Takagi–Sugeno (T–S) fuzzy models with unknown membership functions [53].

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References

[1] P.E. Crago, P.H. Peckham, G.B. Thrope, Modulation of muscle force by recruitment during intramuscular stimulation, IEEE Trans. Biomed. Eng. 27 (12) (1980) 679–684.
[2] P.E. Crago, T. Mortimer, P.H. Peckham, Closed-loop control of force during electrical stimulation of muscle, IEEE Trans. Biomed. Eng. 27 (6) (1980) 306–311.
[3] H.J. Chizeck, L. Kofman, P.E. Crago, G.F. Wilhere, Pulse-train controllers for functional neuromuscular stimulation, IEEE Trans. Biomed. Eng. 14 (5) (1983) 628–639.
[4] R. Riener, T. Fuhr, Patient-driven control of FES-supported standing up: a simulation study, IEEE Trans. Rehabil. Eng. 6 (2) (1998) 113–124.
[5] M. Ferrarin, A. Pedotti, The relationship between electrical stimulus and joint torque: a dynamic model, IEEE Trans. Rehabil. Eng. 8 (3) (2000) 342–352.
[6] J.J. Abbas, H.J. Chizeck, Neural network control of functional neuromuscular stimulation systems: computer simulation studies, IEEE Trans. Biomed. Eng. 42 (11) (1995) 1117–1127.
[7] G.-C. Chang, J.-J. Luh, G.-D. Liao, J.-S. Lai, C.-K. Cheng, B.-L. Kuo, T.-S. Kuo, A neuro-control system for the knee joint position control with quadriceps stimulation, IEEE Trans. Rehabil. Eng. 5 (1) (1997) 2–11.
[8] M. Ferrarin, F. Palazzo, R. Riener, J. Quintern, Model-based control of FES-induced single joint movements, IEEE Trans. Rehabil. Eng. 9 (3) (2001) 245–257.
[9] R. Cardim, M.C.M. Teixeira, E. Assunção, F.A. Faria, Controle de um levitador magnético utilizando modelos fuzzy e derivada de estado da planta, in: VIII Simpósio Brasileiro de Automação Inteligente, Florianópolis, Santa Catarina, Brazil, 2007, in Portuguese.
[10] T.M. Guerra, L. Vermeiren, LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi–Sugeno's form, Automatica 40 (5) (2004) 823–829.
[11] V.F. Montagner, R.C.L.F. Oliveira, V.J.S. Leite, P.L.D. Peres, LMI approach for $\mathcal{H}_\infty$ linear parameter-varying state feedback control, IEE Proc., Control Theory Appl. 152 (2) (2005) 195–201.
[12] V.F. Montagner, R.C.L.F. Oliveira, P.L.D. Peres, Relaxações convexas de convergência garantida para o projeto de controladores para sistemas nebulosos de Takagi–Sugeno, Rev. Control. Autom. 21 (2010) 82–95, in Portuguese.
[13] L.A. Mozelli, R.M. Palhares, F.O. Souza, E.M.A.M. Mendes, Reducing conservativeness in recent stability conditions of TS fuzzy systems, Automatica 45 (6) (2009) 1580–1583.

[14] T. Takagi, M. Sugeno, Fuzzy identification of systems and applications to modeling and control, IEEE Trans. Syst. Man Cybern. 15 (1985) 116–132.
[15] K. Tanaka, T. Ikeda, O.H. Wang, Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs, IEEE Trans. Fuzzy Syst. 6 (2) (1998) 250–265.
[16] M.C.M. Teixeira, E. Assunção, R.G. Avellar, On relaxed LMI-based designs for fuzzy regulators and fuzzy observers, IEEE Trans. Fuzzy Syst. 11 (5) (2003) 613–623.
[17] M.C.M. Teixeira, H.C. Pietrobom, E. Assunção, Novos resultados sobre estabilidade e controle de sistemas não-lineares utilizando modelos fuzzy e LMI, Rev. Control. Autom. 11 (1) (2000) 37–48, in Portuguese.
[18] M.C.M. Teixeira, S.H. Żak, Stabilizing controller design for uncertain non-linear systems using fuzzy models, IEEE Trans. Fuzzy Syst. 7 (2) (1999) 133–142.
[19] M.C.M. Teixeira, G.S. Deaecto, R. Gaino, E. Assunção, A.A. Carvalho, U.C. Farias, Projeto de um controlador fuzzy Takagi–Sugeno para variar o ângulo da articulação do joelho, in: XVI Congresso Brasileiro de Automática, Salvador, Bahia, Brazil, 2006, pp. 2287–2292, in Portuguese.
[20] M.C.M. Teixeira, G.S. Deaecto, R. Gaino, E. Assunção, A.A. Carvalho, U.C. Farias, Design of a fuzzy Takagi–Sugeno controller to vary the joint knee angle of paraplegic patients, in: Lecture Notes in Computer Science, vol. 4234 (3), Springer, Berlin/Heidelberg, 2006, pp. 118–126.
[21] R. Gaino, M.C.M. Teixeira, A.A. Carvalho, E. Assunção, T.I. da Silva, Reguladores e observadores fuzzy Takagi–Sugeno para variar o ângulo da articulação do joelho de um paciente paraplégico, in: VIII Simpósio Brasileiro de Automação Inteligente, Florianópolis, Santa Catarina, Brazil, 2007.
[22] R. Gaino, M.M. Teixeira, R. Cardim, A.A. Carvalho, E. Assunção, M.A.A. Sanches, Controle não-linear fuzzy Takagi–Sugeno do movimento de paraplégicos utilizando acelerômetro, in: XXI Congresso Brasileiro de Engenharia Biomédica (2008), 2008, pp. 1254–1257, in Portuguese.
[23] R. Gaino, M.C.M. Teixeira, A.A. de Carvalho, E. Assunção, R. Cardim, M.A.A. Sanches, M.R. Covacic, Realimentação derivativa e modelo fuzzy Takagi–Sugeno para controle da articulação do joelho de pacientes paraplégicos com o uso de acelerômetros, Braz. J. Biomed. Eng. 27 (2) (2011) 67–78, in Portuguese.
[24] T. Taniguchi, K. Tanaka, H. Ohtake, H.O. Wang, Model construction, rule reduction, and robust compensation for generalized form of Takagi–Sugeno fuzzy systems, IEEE Trans. Fuzzy Syst. 9 (4) (2001) 525–537.
[25] H.C. Pietrobom, Controle de sistemas Não-lineares baseados em LMI utilizando modelos fuzzy, Master thesis, UNESP – Univ. Estadual Paulista, Ilha Solteira, São Paulo, Brazil, 1999, in Portuguese.
[26] E.R.M.D. Machado, Modelagem e controle de sistemas fuzzy Takagi–Sugeno, Ph.D. thesis, UNESP – Univ. Estadual Paulista, Ilha Solteira, São Paulo, Brazil, 2003, in Portuguese.
[27] A.V. Hill, The heat of shortening and the dynamic constant of muscle, Proc. R. Soc. B 126 (1938) 136–195.
[28] A.F. Huxley, Muscle structure and theories of contraction, Prog. Biophys. Biophys. Chem. 7 (1957) 255–318.
[29] A.V. Hill, First and Last Experiments in Muscle Mechanics, Cambridge University Press, Cambridge, England, 1970.
[30] H. Hatze, Myocybernetic Control Models of Skeletal Muscle, University of South Africa Press, Pretoria, South Africa, 1981.
[31] F.E. Zajac, Muscle and tendon: properties, models scaling and application to biomechanics and motor control, Crit. Rev. Biomed. Eng. 17 (4) (1989) 359–411.
[32] W. Herzog, B.M. Nigg, Biomechanics of Musculoskeletal System, Wiley & Sons Ltd., Chichester, England, 1999.
[33] S.J. Dorgan, M.J.O. Malley, A mathematical model for skeletal muscle activated by N let pulse trains, IEEE Trans. Rehabil. Eng. 6 (3) (1998) 286–299.
[34] C.Q. Andrea, J.O.P. Pinto, E. Assunção, M.C.M. Teixeira, L.J. Galotto, Controle ótimo $\mathcal{H}_\infty$ de sistemas não lineares com modelos fuzzy Takagi–Sugeno, Rev. Control. Autom. 19 (2008) 256–269, in Portuguese.
[35] Y.T. Chang, B.S. Chen, A fuzzy approach for robust reference-tracking-control design of nonlinear distributed parameter time-delayed systems and its application, IEEE Trans. Fuzzy Syst. 18 (6) (2010) 1041–1057.
[36] G. Feng, A survey on analysis and design of model-based fuzzy control systems, IEEE Trans. Fuzzy Syst. 14 (5) (2006) 676–697.
[37] K.Y. Lian, J.J. Liou, Output tracking control for fuzzy systems via output feedback design, IEEE Trans. Fuzzy Syst. 14 (5) (2006) 628–639.
[38] J. Yu, J. Gao, Y. Ma, H. Yu, Adaptive fuzzy tracking control for a permanent magnet synchronous motor via backstepping approach, Math. Probl. Eng. 2010 (2010), 13 pages.
[39] M.P.A. Santim, M.C.M. Teixeira, W.A. de Souza, R. Cardim, E. Assunção, Design of a Takagi–Sugeno fuzzy regulator for a set of operation points, Math. Probl. Eng. 2012 (2012) 1–17.
[40] W.A. de Souza, M.C.M. Teixeira, M.P.A. Santim, R. Cardim, E. Assunção, On switched control design of linear time-invariant systems with polytopic uncertainties, Math. Probl. Eng. 2013 (2013) 1–10.
[41] K. Tanaka, H.O. Wang, Fuzzy Control Systems Design and Analysis, A Wiley-Interscience Publication, Canada, 2001.
[42] S. Boyd, L. Ghaoui, E. Feron, V. Balakrishan, Linear Matrix Inequalities in Systems and Control Theory, SIAM Studies in Apllied Mathematics, USA, 1994.
[43] K.-Y. Lian, J.-J. Liou, Output tracking control for fuzzy systems via output feedback design, IEEE Trans. Fuzzy Syst. 14 (5) (2006) 628–639.
[44] P. Gahinet, A. Nemirovski, A.J. Laub, M. Chilali, LMI Control Toolbox – For Use with Matlab, The Mathworks, Inc., 1995.
[45] T. Taniguchi, K. Tanaka, H. Ohatake, H.O. Wang, Model construction, rule reduction, and robust compensation for generalized form of Takagi–Sugeno fuzzy systems, IEEE Trans. Fuzzy Syst. 9 (4) (2001) 525–537.
[46] X.J. Ma, Z.Q. Sun, Y.Y. He, Analysis and design of fuzzy controller and fuzzy observer, IEEE Trans. Fuzzy Syst. 6 (1) (1998) 41–51.
[47] R. Gaino, Controle de movimentos de paraplégico utilizando modelos fuzzy Takagi–Sugeno, Ph.D. thesis, UNESP – Univ. Estadual Paulista, Ilha Solteira, São Paulo, Brazil, 2009, in Portuguese.
[48] M.R. Covacic, R. Gaino, M.C.M. Teixeira, A.A. de Carvalho, E. Assunção, R. Cardim, Sistemas ERP com compensadores dinâmicos para controle da posição angular da perna de pacientes Paraplégicos, in: XVIII Congresso Brasileiro de Automática (2010), Bonito, Mato Grosso do Sul, Brazil, 2010, pp. 1700–1707, in Portuguese.

[49] J.J.E. Slotine, W. Li, Applied Nonlinear Control, 1st edition, Prentice Hall, USA, 1991.

[50] M.A.A. Sanches, Sistema eletrônico para geração e avaliação de movimentos em paraplégicos, Ph.D. thesis, UNESP – Univ. Estadual Paulista, Ilha Solteira, São Paulo, Brazil, 2013, in Portuguese.

[51] M.A.A. Sanches, R. Gaino, R.F. Kozan, M.C.M. Teixeira, A.A. de Carvalho, M.R. Covacic, C.A. Alves, M.F.R. Urban, M.V.N. Junqueira, R. Cardim, E. Assunção, E. Gentilho Junior, Digital controller design considering hardware constraints: application in a paraplegic patient, Braz. J. Biomed. Eng. 30 (3) (2014) 232–241.

[52] K.Y. Lian, C.H. Chiang, H.W. Tu, LMI-based sensorless control of permanent-magnet synchronous motors, IEEE Trans. Ind. Electron. 54 (5) (2007) 2769–2778.

[53] W.A. Souza, M.C.M. Teixeira, R. Cardim, E. Assunção, On switched regulator design of uncertain nonlinear systems using Takagi–Sugeno fuzzy models, IEEE Trans. Fuzzy Syst. 22 (6) (2014) 1720–1727.