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On the Heat Equation with Nonlinearity and Singular Anisotropic Potential on the Boundary

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Abstract This paper concerns with the heat equation in the half-space $\mathbb{R}^n_+$ with nonlinearity and singular potential on the boundary $\partial\mathbb{R}^n_+$. We show a well-posedness result that allows us to consider critical potentials with infinite many singularities and anisotropy. Motivated by potential profiles of interest, the analysis is performed in weak L^p-spaces in which we prove linear estimates for some boundary operators arising from the Duhamel integral formulation in $\mathbb{R}^n_+$. Moreover, we investigate qualitative properties of solutions like self-similarity, positivity and symmetry around the axis $\overrightarrow{Ox_n}$.

Keywords Heat equation · Singular potentials · Nonlinear boundary conditions · Self-similarity · Symmetry · Lorentz spaces

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1 Introduction

Heat equations with singular potentials have attracted the interest of many authors since the work of Baras and Goldstein [8] in the 80's. In a smooth domain $\Omega \subset \mathbb{R}^n$, they studied the Cauchy problem for the linear heat equation

$$u_t - \Delta u - V(x)u = 0, x \in \Omega \text{ and } t > 0, \tag{1.1}$$

with homogeneous Dirichlet boundary condition and singular potential

$$V(x) = \frac{\lambda}{|x|^2}, \tag{1.2}$$

and obtained a threshold value $\lambda_* = \frac{(n-2)^2}{4}$ with $n \geq 3$ for existence of positive L^2-solutions. The potential in Eq. 1.2 is called inverse square (Hardy) potential and is an example of potential arising from negative power laws. This class of potentials appears in a number of physical phenomena (see e.g. [15, 16, 22, 30, 32, 35] and references therein) and can be classified according to the number of singularities (poles), degree of the singularity (order of the poles), dependence on directions (anisotropy), and decay at infinity. One of the most difficult cases is the one of anisotropic critical potentials, namely

$$V(x) = \sum_{i=1}^{l} \frac{v_i\left(\frac{x-x^i}{|x-x^i|}\right)}{|x-x^i|^{\sigma}}, \tag{1.3}$$

where $v_i(z) \in BC(\mathbb{S}^{n-1})$, $x^i \in \overline{\Omega}$, $l \in \mathbb{N} \cup \{\infty\}$, and the parameter σ is the order of the poles $\{x^i\}_{i=1}^{l}$. Here BC stands for bounded continuous functions and $\mathbb{S}^{n-1}$ is the $(n-1)$-sphere. The potential is called isotropic (resp. anisotropic) when the v_i's are independent (resp. dependent) of the directions $\frac{x-x^i}{|x-x^i|}$, that is, they are constant. In the case $l = 1$ (resp. $l > 1$), V is said to be monopolar (resp. multipolar). The criticality means that σ is equal to order of the PDE inside the domain or of the boundary condition, according to the type of problem considered. The critical case introduces further difficulties in the mathematical analysis of the problem because Vu cannot be handled as a lower order term (see [15]). Examples of potentials (1.3) are

$$V(x) = \sum_{i=1}^{l} \frac{\lambda_i}{|x-x^i|^{\sigma}} \text{ and } V(x) = \sum_{i=1}^{l} \frac{(x-x^i).d^i}{|x-x^i|^{\sigma+1}}, \tag{1.4}$$

where $x^i = (x_1^i, x_2^i, ..., x_n^i)$ and $d^i \in \mathbb{R}^n$ are constant vectors. In the theory of Schrodinger operators, the potentials in Eq. 1.4 are called multipolar Hardy potentials and multiple dipole-type potentials, respectively (see [15] and [16]).

In this paper, we consider a nonlinear counterpart for Eq. 1.1 in the half-space with critical singular boundary potential, which reads as

$$\partial_t u = \Delta u \quad \text{in } \Omega,\ t > 0 \tag{1.5}$$

$$\partial_n u = h(u) + V(x')u \quad \text{in } \partial\Omega,\ t > 0 \tag{1.6}$$

$$u(x,0) = u_0(x), \quad \text{in } \Omega, \tag{1.7}$$

where $\Omega = \mathbb{R}^n_+$, $n \geq 3$ and $\partial_n = -\partial_{x_n}$ stands for the normal derivative on $\partial\mathbb{R}^n_+$. For the nonlinear term, we assume that the function $h : \mathbb{R} \to \mathbb{R}$ satisfies $h(0) = 0$ and

$$|h(a) - h(b)| \leq \eta\, |a-b| \left(|a|^{\rho-1} + |b|^{\rho-1}\right), \tag{1.8}$$

where $\rho > 1$ and the constant η is independent of $a, b \in \mathbb{R}$. A classical example of h satisfying these conditions is $h(u) = \pm |u|^{\rho-1} u$.

Our goal is to develop a global-in-time well-posedness theory for Eqs. 1.5–1.7, under smallness conditions on certain weak norms of u_0, V, which allows to consider critical potentials on the boundary with infinite many poles. For that matter, we employ the framework of weak-L^p spaces (i.e., $L^{(p,\infty)}$-spaces) and take $V \in L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)$ and $u_0 \in L^{(n(\rho-1),\infty)}(\mathbb{R}^n_+)$. Since $L^p(\partial\mathbb{R}^n_+)$ contains only trivial homogeneous functions, a motivation naturally appears for considering weak-L^p spaces. In fact, due to Chebyshev's inequality, we have the continuous inclusion $L^p(\partial\mathbb{R}^n_+) \subset L^{(p,\infty)}(\partial\mathbb{R}^n_+)$ and then $L^{(p,\infty)}$ can be regarded as a natural extension of L^p which contains homogeneous functions (and their x^i-translations) of degree $\sigma = -(n-1)/p$. The critical case for Eqs. 1.5–1.7 with potential (1.3) corresponds to $\sigma = 1$ (so, $p = n-1$) and we have that

$$\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)} \leq \sum_{i=1}^{l} \sup_{x'\in\mathbb{S}^{n-2}} \left|v_i(x')\right| \left\| \frac{1}{|x'-x^i|} \right\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)} \leq C \sum_{i=1}^{l} \sup_{x'\in\mathbb{S}^{n-2}} \left|v_i(x')\right|, \tag{1.9}$$

where

$$C = \left\| |x'|^{-1} \right\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)} < \infty. \tag{1.10}$$

Of special interest is when the set $\{x^i\}_{i=1}^{l} \subset \partial\mathbb{R}^n_+$ and so V has a number l of singularities on the boundary which can be infinite provided that the infinite sum in Eq. 1.9 is finite.

We address Eqs. 1.5–1.7 by means of the following equivalent integral formulation

$$u(x,t) = \int_{\mathbb{R}^n_+} G(x,y,t)u_0(y)dy + \int_0^t \int_{\partial\mathbb{R}^n_+} G(x,y',t-s)\left[h(u)+Vu\right](y',s)dy'ds \tag{1.11}$$

where $G(x,y,t)$ is the heat fundamental solution in $\mathbb{R}^n_+$ given by

$$G(x,y,t) = (4\pi t)^{-\frac{n}{2}}\left[e^{-\frac{|x-y|^2}{4t}} + e^{-\frac{|x-y^*|^2}{4t}}\right], \quad x,y \in \overline{\mathbb{R}^n_+},\ t>0, \tag{1.12}$$

with $y^* = (y', -y_n)$ and $y' = (y_1, \cdots, y_{n-1}) \in \partial\mathbb{R}^n_+$. Here solutions for Eq. 1.11 are looked for in $BC((0,\infty); \mathcal{X}_{p,q})$ where $\mathcal{X}_{p,q}$ is a suitable Banach space that can be identified with $L^{(p,\infty)}(\mathbb{R}^n_+) \times L^{(q,\infty)}(\partial\mathbb{R}^n_+)$. The norm in $\mathcal{X}_{p,q}$ provides a $L^{(q,\infty)}$-information for $u|_{\partial\mathbb{R}^n_+}$ without assuming any positive regularity condition on u. Notice that this space is specially useful in order to treat singular boundary terms like Eq. 1.3 with $x \in \partial\mathbb{R}^n_+$. L^r-versions of $\mathcal{X}_{p,q}$ (i.e. $L^{r_1}(\Omega) \times L^{r_2}(\partial\Omega)$) were employed in [36] and [20] in order to study weak solutions for an elliptic and parabolic PDE in bounded domains Ω, respectively. Let us observe that $\left\||x'|^{-1}\right\|_{L^{r_2}(\partial\mathbb{R}^n_+)} = \infty$ for all $1 \leq r_2 \leq \infty$ (compare with Eq. 1.10) which prevents the use of the spaces of [20, 36] for our purposes.

Furthermore, we investigate qualitative properties of solutions like positivity, symmetries (e.g. invariance around the axis $\overrightarrow{Ox_n}$) and self-similarity, under certain conditions on $u_0, V, h(\cdot)$. For the latter, the indexes of $\mathcal{X}_{p,q}$ are chosen so that their norms are invariant by the scaling of Eqs. 1.5–1.6 $u(x,t) \to \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t)$ (see Section 3), namely $p = n(\rho-1)$ and $q = (n-1)(\rho-1)$.

To be more precise, in Theorem 3.1 we prove that Eq. 1.11 is globally well-posed in $\mathcal{X}_{p,q}$ with the above indexes p, q and smallness conditions on the initial data u_0 and the potential norm $\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)}$. After, assuming that $h(\cdot)$, V and u_0 are homogeneous functions of degree ρ, -1 and $-\frac{1}{\rho-1}$, respectively, we obtain the existence of self-similar solutions in Theorem 3.3. Invariance under rotations around the axis $\overrightarrow{Ox_n}$ and positivity of solutions are analyzed in Theorem 3.4.

Hardy and Kato inequalities are tools often used for handling (1.1) (V in the PDE) and Eqs. 1.5–1.6 (V on the boundary) with critical potentials, respectively. They read as

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx \;\leq\; \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\,, \forall\varphi\in C_0^\infty(\mathbb{R}^n) \tag{1.13}$$

$$2\frac{\Gamma(\frac{n}{4})^2}{\Gamma(\frac{n-2}{4})^2}\int_{\partial\mathbb{R}^n_+}\frac{u^2}{|x|}dx \;\leq\; \|\nabla u\|^2_{L^2(\mathbb{R}^n_+)}\,, \forall\varphi\in C_0^\infty(\overline{\mathbb{R}^n_+}), \tag{1.14}$$

respectively, where Γ stands for the gamma function. Our approach in the space $\mathcal{X}_{p,q}$ does not require Eq. 1.13 nor Eq. 1.14. For that, we prove estimates in weak-L^p spaces for some boundary operators linked to Eq. 1.11. In view of Eq. 1.3, these estimates need to be time-independent and thereby we cannot use time-weighted norms *ala kato* (see [28] for this type of norm), making things more difficult-to-treating. This situation leads us to prove estimates that can be seen as extensions of Yamazaki's estimates [43] to boundary operators (see Lemma 4.3). The paper [43] dealt with the heat and Stokes operators inside a half-space, among other smooth domains Ω. Also, we need to show Lemmas 4.1 and 4.2 that seem to have some interest of its own. It is worthy to comment that weak-L^p spaces are examples of shift-invariant Banach spaces of local measure in which global-in-time well-posedness theory of small solutions has been successfully developed for Navier-Stokes equations (see [31] for a review) and, more generally, for parabolic problems with nonlinearities (and possibly other terms) defined inside the domain (see [29]).

The paper of Baras-Goldstein [8] has motivated many works concerning linear and nonlinear heat equations with singular potentials. For Eq. 1.1, we refer the reader to [11, 24, 42] (see also their references) for results on existence, non-existence, decay and self-similar asymptotic behavior of solutions. Versions of Eq. 1.1 with nonlinearities $\pm u^p$ and $\pm|\nabla u|^p$ have been studied in [2–4, 12, 27, 33, 38] where the reader can found results on existence, non-existence, Fujita exponent, self-similarity, bifurcations, and blow-up. Linear and nonlinear elliptic versions of Eq. 1.1 are also considered in the literature (see e.g. [12, 15–19, 40]); as well as the parabolic case, the key tool used in the analysis is Hardy type inequalities, except by [18] and [19]. In these last two references, the authors employed a contraction argument in a sum of weighted spaces and in a space based on Fourier transform, respectively.

In a bounded domain Ω and half-space $\mathbb{R}^n_+$, the nonlinear problem (1.5)–(1.7) with $V\equiv 0$ has been studied by several authors over the past two decades; see, e.g., [5, 6, 23, 26, 37, 39] and their references. In these works, the reader can find many types of existence and asymptotic behavior results in the framework of L^p-spaces. For $V\in L^\infty(\partial\Omega)$ and Ω a bounded smooth domain, results on well-posedness and attractors can be found in [7]. The authors of [13] considered Eqs. 1.5–1.7 with $h(u)\equiv 0$ (linear case) and showed L^p-estimates of solutions, still for $V\in L^\infty(\partial\Omega)$ (see also [14] for the elliptic case). In [25], the authors studied the linear case of Eqs. 1.5–1.7 in a half-space and considered the singular critical potential $V(x')=\frac{\lambda}{|x'|}$. For $u_0\in C_0(\mathbb{R}^n_+)$ (compactly supported data), they obtained a threshold value for existence of positive solutions by using Kato inequality (1.14). This result can be seen as a boundary version of those of Baras and Goldstein [8].

Let us now highlight some points of the present paper. Our results provide a global-in-time well-posedness theory for Eqs. 1.5–1.7 in a space that is larger than L^p-spaces and seems to be new in the study of parabolic problems with nonlinear boundary conditions. Also, the functional setting allows us to consider critical potentials on the boundary with infinite many poles which are not covered by previous results dealing with boundary potentials. As pointed out above, another difference is that our approach relies on boundary estimates on weak-L^p spaces instead of using Hardy and Kato inequalities. Since the smallness condition on u_0 is with respect to the weak norm of such spaces, some initial data with large L^p and H^s-norms can be considered. The data u_0 also can have infinite poles. Results on self-similarity and axial-symmetry are obtained due to the choice of the indexes of $\mathcal{X}_{p,q}$ and the symmetry features of the linear operators arising in the integral formulation (1.11).

The plan of this paper is the following. In the next section we summarize some basic definitions and properties on Lorentz spaces. In Section 3 we define suitable time-functional spaces and state our results which are proved in Section 4. The proofs are organized in five subsections. In the first one we obtain the needed linear estimates. Section 4.2 is devoted to nonlinear estimates linked to the integral formulation (1.11). In Section 4.3, with these estimates in hands, we prove our well-posedness result (Theorem 3.1) by means of a contraction argument. The results on self-similarity and symmetry of solutions are proved in Sections 4.4 and 4.5, respectively.

2 Preliminaries

In this section we fix some notations and summarize basic properties about Lorentz spaces that will be used throughout the paper. For further details, we refer the reader to [9, 10].

For a point $x \in \overline{\mathbb{R}^n_+}$, we write $x = (x', x_n)$ where $x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n \geq 0$. The Lebesgue measure in a measurable $\Omega \subset \mathbb{R}^n$ will be denoted by either $|\cdot|$ or dx. In the case $\Omega = \mathbb{R}^n_+$, one can express $dx = dx'dx_n$ where dx' stands for Lebesgue measure on $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Given a subset $\Omega \subset \mathbb{R}^n$, the distribution function and rearrangement of a measurable function $f : \Omega \to \mathbb{R}$ is defined respectively by

$$L_f(s) = |\{x \in \Omega : |f(x)| > s\}| \text{ and } f^*(t) = \inf\{s > 0 : L_f(s) \leq t\}, t > 0.$$

The *Lorentz space* $L^{(p,r)} = L^{(p,r)}(\Omega) = L^{(p,r)}(\Omega, |\cdot|)$ consists of all measurable functions f in Ω for which

$$\|f\|^*_{L^{(p,r)}(\Omega)} = \begin{cases} \left[\int_0^\infty \left(t^{\frac{1}{p}}[f^*(t)]\right)^r \frac{dt}{t}\right]^{\frac{1}{r}} < \infty, & 0 < p < \infty, 1 \leq r < \infty \\ \sup_{t>0} t^{\frac{1}{p}}[f^*(t)] < \infty, & 0 < p < \infty, \ r = \infty. \end{cases} \tag{2.1}$$

We have that $L^p(\Omega) = L^{(p,p)}(\Omega)$ and $L^{(p,\infty)}$ is also called weak-L^p or Marcinkiewicz space. The quantity (2.1) is not a norm in $L^{(p,r)}$, however it is a complete quasi-norm. Considering

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds,$$

we can endow $L^{(p,r)}$ with the quantity $\|\cdot\|_{L^{(p,r)}}$ obtained from Eq. 2.1 with f^{**} in place of f^*. For $1 < p \leq \infty$, we have that $\|\cdot\|^*_{(p,r)} \leq \|\cdot\|_{(p,r)} \leq \frac{p}{p-1}\|\cdot\|^*_{(p,r)}$ which implies that $\|\cdot\|^*_{(p,r)}$ and $\|\cdot\|_{(p,r)}$ induce the same topology on $L^{(p,r)}$. Moreover, the pair

$(L^{(p,r)}, \|\cdot\|_{L^{(p,r)}})$ is a Banach space. From now on, for $1 < p \leq \infty$ we consider $L^{(p,r)}$ endowed with $\|\cdot\|_{L^{(p,r)}}$, except when explicitly mentioned.

If, for $\lambda > 0$, $\lambda\Omega = \{\lambda x : x \in \Omega\}$ is the *dilation* of the domain Ω, then

$$\|f(\lambda x)\|_{L^{(p,r)}(\Omega)} = \lambda^{-\frac{n}{p}} \|f(x)\|_{L^{(p,r)}(\Omega)}, \tag{2.2}$$

provided that Ω is invariant by dilations, i.e., $\Omega = \lambda\Omega$.

For $1 \leq q_1 \leq p \leq q_2 \leq \infty$ with $1 < p \leq \infty$, the continuous inclusions hold true

$$L^{(p,1)} \subset L^{(p,q_1)} \subset L^p \subset L^{(p,q_2)} \subset L^{(p,\infty)}.$$

The dual space of $L^{(p,r)}$ is $L^{(p',r')}$ for $1 \leq p, r < \infty$. In particular, the dual of $L^{(p,1)}$ is $L^{(p',\infty)}$ for $1 \leq p < \infty$.

Hölder's inequality works well in Lorentz spaces (see [34]). Precisely, if $1 < p_1, p_2, p_3 < \infty$ and $1 \leq r_1, r_2, r_3 \leq \infty$ with $1/p_3 = 1/p_1 + 1/p_2$ and $1/r_3 \leq 1/r_1 + 1/r_2$, then

$$\|fg\|_{L^{(p_3,r_3)}} \leq C\|f\|_{L^{(p_1,r_1)}} \|g\|_{L^{(p_2,r_2)}}, \tag{2.3}$$

where $C > 0$ is a constant independent of f, g.

Finally we recall an interpolation property of Lorentz spaces. For $0 < p_1 < p_2 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $1 \leq r_1, r_2, r \leq \infty$, we have that (see [10, Theorems 5.3.1, 5.3.2])

$$\left(L^{(p_1,r_1)}, L^{(p_2,r_2)}\right)_{\theta,r} = L^{(p,r)}, \tag{2.4}$$

where $(X, Y)_{\theta,r}$ stands for the real interpolation space between X and Y constructed via the $K_{\theta,q}$-method. It is well known that $(\cdot,\cdot)_{\theta,r}$ is an exact interpolation functor of exponent θ on the categories of quasi-normed and normed spaces. When $0 < p_1 \leq 1$, the property (2.4) should be considered with $L^{(p_1,r_1)}$ endowed with the complete quasi-norm $\|\cdot\|^*_{L^{(p_1,r_1)}}$ instead of $\|\cdot\|_{L^{(p_1,r_1)}}$.

3 Functional Setting and Results

Before starting our results, we define suitable function spaces where Eq. 1.11 will be handled. If the potential V is a homogeneous function of degree -1, that is, $V(y) = \lambda V(\lambda y)$ for all $y \in \partial\mathbb{R}^n_+$, then $u_\lambda(x,t) = \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t)$ verifies Eqs. 1.5–1.6, for each fixed $\lambda > 0$, provided that $u(x,t)$ is also a solution. It follows that Eqs. 1.5–1.6 has the following scaling

$$u(x,t) \to u_\lambda(x,t) = \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t),\ \lambda > 0. \tag{3.1}$$

Making $t \to 0^+$ in Eq. 3.1, one obtains

$$u_0(x) \to u_{0,\lambda}(x,0) = \lambda^{\frac{1}{\rho-1}} u_0(\lambda x), \tag{3.2}$$

which gives a scaling for the initial data.

Since the potential V and initial data u_0 are singular, we need to treat Eq. 1.11 in a suitable space of functions without any positive regularity conditions and time decaying. For that matter, let $\mathcal{A}$ be the set of measurable functions $f : \overline{\mathbb{R}^n_+} \to \mathbb{R}$ such that $f|_{\mathbb{R}^n_+}$ and $f|_{\partial\mathbb{R}^n_+}$ are measurable with respect to Lebesgue σ-algebra on $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1} = \partial\mathbb{R}^n_+$, respectively. Consider the equivalence relation in $\mathcal{A}$: $f \sim g$ if and only if $f = g$ a.e. in

$\mathbb{R}^n_+$ and $f|_{\partial\mathbb{R}^n_+} = g|_{\partial\mathbb{R}^n_+}$ a.e. in $\partial\mathbb{R}^n_+$. Given $1 \leq p, q < \infty$, we set $\mathcal{X}_{p,q}$ as the space of all $f \in \mathcal{A}/\sim$ such that

$$\|f\|_{\mathcal{X}_{p,q}} = \|f\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} + \|f|_{\partial\mathbb{R}^n_+}\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)} < \infty.$$

The pair $(\mathcal{X}_{p,q}, \|\cdot\|_{\mathcal{X}_{p,q}})$ is a Banach space and can be isometrically identified with $L^{(p,\infty)}(\mathbb{R}^n_+) \times L^{(q,\infty)}(\partial\mathbb{R}^n_+)$. For $p = n(\rho-1)$ and $q = (n-1)(\rho-1)$, we have from Eq. 2.2 that

$$\|\lambda^{\frac{1}{\rho-1}} f(\lambda x)\|_{\mathcal{X}_{p,q}} = \lambda^{\frac{1}{\rho-1}}\lambda^{-\frac{n}{n(\rho-1)}}\|f\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} + \lambda^{\frac{1}{\rho-1}}\lambda^{-\frac{n-1}{(n-1)(\rho-1)}}\|f|_{\partial\mathbb{R}^n_+}\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)} = \|f\|_{\mathcal{X}_{p,q}},$$

and then $\mathcal{X}_{p,q}$ is invariant by scaling Eq. 3.2.

We shall look for solutions in the Banach space $E = BC((0,\infty); \mathcal{X}_{p,q})$ endowed with the norm

$$\|u\|_E = \sup_{t>0} \|u(\cdot,t)\|_{\mathcal{X}_{p,q}}, \tag{3.3}$$

which is invariant by scaling Eq. 3.1.

3.1 Existence and Self-Similarity

In what follows, we state our well-posedness result.

Theorem 3.1 *Let $n \geq 3$, $\rho > 1$ with $\frac{\rho}{\rho-1} < n-1$, $p = n(\rho-1)$ and $q = (n-1)(\rho-1)$. Let $h : \mathbb{R} \to \mathbb{R}$ verify* Eq. 1.8 *and $h(0) = 0$. Suppose that $V \in L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)$ and $u_0 \in L^{(p,\infty)}(\mathbb{R}^n_+)$.*

(A) *(Existence and uniqueness) There exist $\varepsilon, \delta_1, \delta_2 > 0$ such that if $\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)} < \frac{1}{\delta_1}$ and $\|u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} \leq \frac{\varepsilon}{\delta_2}$ then the integral equation* (1.11) *has a unique global solution $u \in BC((0,\infty); \mathcal{X}_{p,q})$ satisfying $\sup_{t>0}\|u(\cdot,t)\|_{\mathcal{X}_{p,q}} \leq \frac{2\varepsilon}{1-\gamma}$ where $\gamma = \delta_1\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)}$. Moreover, $u(\cdot,t) \rightharpoonup u_0$ in $\mathcal{S}'(\mathbb{R}^n_+)$ as $t \to 0^+$.*

(B) *(Continuous dependence) The solution obtained in item (A) depends continuously on the initial data u_0 and potential V.*

Remark 3.2

(A) The integral solution $u(\cdot,t) \in \mathcal{X}_{p,q}$, for each $t > 0$, even requiring only $u_0 \in L^{(p,\infty)}(\mathbb{R}^n_+)$. It is a kind of "parabolic regularizing effect" in the sense that solutions verifies a property for $t > 0$ that is not necessarily verified by initial data. Here, it comes essentially from the fact that $u_1(x,t) = \int_{\mathbb{R}^n_+} G(x,y,t)u_0(y)dy$ has a trace well-defined on $\partial\mathbb{R}^n_+$ and $u_1|_{\partial\mathbb{R}^n_+} \in L^{(q,\infty)}(\partial\mathbb{R}^n_+)$, for each $t > 0$, even if the data u_0 does not have a trace. Moreover, the estimate (4.4) provides a control on the trace just using the norm of u_0 in $L^{(p,\infty)}(\mathbb{R}^n_+)$.

(B) The time-continuity of the solution $u(\cdot,t) \in \mathcal{X}_{p,q}$ at $t > 0$ comes naturally from the uniform continuity of the kernel $G(x,y,t)$ (1.12) on $\overline{\mathbb{R}^n_+} \times \overline{\mathbb{R}^n_+} \times [\delta,\infty)$, for each fixed $\delta > 0$.

(C) A standard argument shows that solutions of Eq. 1.11 obtained in Theorem 3.1 verifies Eqs. 1.5–1.7 in the sense of distributions.

Since the spaces in which we look for solutions are invariant by scaling Eq. 3.1, it is natural to ask about existence of self-similar solutions. This issue is considered in the next theorem.

Theorem 3.3 *Let u be the global solution obtained in Theorem 3.1 corresponding to the triple $(u_0, V, h(\cdot))$. If u_0, V and $h(\cdot)$ are homogeneous functions of degree $-\frac{1}{\rho-1}$, -1 and ρ, respectively, then u is a self-similar solution, i.e.,*

$$u(x,t) \equiv u_\lambda(x,t) := \lambda^{\frac{1}{\rho-1}} u(\lambda x, \lambda^2 t), \quad \textit{for all } \lambda > 0.$$

3.2 Symmetries and Positivity

In this subsection, we are concerned with symmetry and positivity of solutions. It is easy to see that the fundamental solution (1.12) is positive and invariant by the set $\mathcal{O}_{x_n}$ of all rotations around the axis $\overrightarrow{Ox_n}$. Because of that, it is natural to wonder whether solutions obtained in Theorem 3.1 present positivity and symmetry properties, under certain conditions on the data and potential.

For that matter, let $\mathcal{A}$ be a subset of $\mathcal{O}_{x_n}$. We recall that a function f is symmetric under the action of $\mathcal{A}$ when $f(x) = f(T(x))$ for any $T \in \mathcal{A}$. If $f(x) = -f(T(x))$ for all $T \in \mathcal{A}$, then f is said to be antisymmetric under $\mathcal{A}$.

Theorem 3.4 *Under the hypotheses of Theorem 3.1. Let $\mathcal{U} \subset \mathbb{R}^n_+$ be a positive-measure set and $\mathcal{A}$ a subset of $\mathcal{O}_{x_n}$.*

(A) *Let $h(a) \geq 0$ (resp. ≤ 0) when $a \geq 0$ (resp. ≤ 0). If $u_0 \geq 0$ (resp. ≤ 0) a.e. in $\mathbb{R}^n_+$, $u_0 > 0$ (resp. < 0) in $\mathcal{U}$, and $V \geq 0$ in $\partial\mathbb{R}^n_+$, then u is positive (resp. negative) in $\mathbb{R}^n_+ \times (0,\infty)$.*

(B) *Let $h(a) = -h(-a)$, for all $a \in \mathbb{R}$, and let V be symmetric under the action of $\mathcal{A}|_{\partial\mathbb{R}^n_+}$. For all $t > 0$, the solution $u(\cdot,t)$ is symmetric (resp. antisymmetric), when u_0 is symmetric (resp. antisymmetric) under $\mathcal{A}$.*

Remark 3.5 (Special cases of symmetries) Let $h(a) = -h(-a)$, for all $a \in \mathbb{R}$.

(i) Consider $\mathcal{A} = \mathcal{O}_{x_n}$ and let V be radially symmetric on $\mathbb{R}^{n-1}$. We obtain from item (B) that if u_0 is invariant under rotations around the axis $\overrightarrow{Ox_n}$ then $u(\cdot,t)$ does so, for all $t > 0$.

(ii) Let $\mathcal{A} = \{T_{x_n}\}$ where T_{x_n} is the reflection with respect to $\overrightarrow{Ox_n}$, i.e., $T_{x_n}((x', x_n)) = (-x', x_n)$ for all $x = (x', x_n)$ and $x_n \geq 0$. A function f is said to be $\overrightarrow{Ox_n}$-even (resp. $\overrightarrow{Ox_n}$-odd) when f is symmetric (resp. antisymmetric) under $\{T_{x_n}\}$. If $V(x)$ is an even function then the solution $u(\cdot,t)$ is $\overrightarrow{Ox_n}$-even (resp. $\overrightarrow{Ox_n}$-odd), for all $t > 0$, provided that u_0 is $\overrightarrow{Ox_n}$-even (resp. $\overrightarrow{Ox_n}$-odd).

Remark 3.6 Combining Theorems 3.3 and 3.4, we can obtain solutions that are both self-similar and invariant by rotations around $\overrightarrow{Ox_n}$. For instance, in the case $h(a) = \pm|a|^{\rho-1}a$, just take

$$V(x') = \kappa|x'|^{-1} \text{ and } u_0(x) = \theta\left(\frac{x_n}{|x|}\right)|x|^{-\frac{1}{\rho-1}}$$

where $\theta(z) \in BC(\mathbb{R})$ and κ is a constant.

4 Proofs

This section is devoted to the proofs of the results. We start by estimating in Lorentz spaces some linear operators appearing in the integral formulation (1.11).

4.1 Linear Estimates

Let $f|_0 = f(x', 0)$ stand for the restriction of f to $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. We also denote by $\{E(t)\}_{t\geq 0}$ the heat semigroup in the half-space, namely

$$E(t)f(x) = \int_{\mathbb{R}^n_+} G(x, y, t)f(y)dy \tag{4.1}$$

where $G(x, y, t)$ is the fundamental solution given in Eq. 1.12. For $\delta > 0$ and $1 \leq q_1 \leq q_2 \leq \infty$, let us recall the well-known L^q-estimate for the heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ on $\mathbb{R}^n$:

$$\|(-\Delta_x)^{\frac{\delta}{2}} e^{t\Delta} f\|_{L^{q_2}(\mathbb{R}^n)} \leq Ct^{-\frac{1}{2}\left(\frac{n}{q_1}-\frac{n}{q_2}\right)-\frac{\delta}{2}} \|f\|_{L^{q_1}(\mathbb{R}^n)}, \tag{4.2}$$

where $C > 0$ is a constant independent of f and t, and $(-\Delta_x)^{\frac{\delta}{2}}$ stands for the Riesz potential. For $0 < \delta < n$ and $1 \leq q_1 < q_2 < \infty$ such that $\frac{1}{\delta} < q_1 < \frac{n}{\delta}$ and $\frac{n-1}{q_2} = \frac{n}{q_1} - \delta$, we have the Sobolev trace-type inequality in L^p (see [1, Theorem 2])

$$\|f|_0\|_{L^{q_2}(\partial\mathbb{R}^n_+)} \leq C \left\|(-\Delta_x)^{\frac{\delta}{2}} f\right\|_{L^{q_1}(\mathbb{R}^n)}. \tag{4.3}$$

The next lemma provide a boundary estimate for Eq. 4.1 in the setting of Lorentz spaces.

Lemma 4.1 *Let $1 < d_1 < d_2 < \infty$ and $1 \leq r \leq \infty$. Then there exists a constant $C > 0$ such that*

$$\|[E(t)f]|_0\|_{L^{(d_2,r)}(\partial\mathbb{R}^n_+)} \leq Ct^{-\left(\frac{n}{2d_1}-\frac{n-1}{2d_2}\right)} \|f\|_{L^{(d_1,r)}(\mathbb{R}^n_+)}, \tag{4.4}$$

for all $f \in L^{(d_1,r)}(\mathbb{R}^n_+)$ and $t > 0$.

Proof Consider the extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$

$$\tilde{f}(x) = \begin{cases} f(x', x_n), & x_n > 0 \\ f(x', -x_n), & x_n \leq 0. \end{cases}$$

Now notice that

$$E(t)f(x) = e^{t\Delta}\tilde{f}(x) = (g(\cdot, t) * \tilde{f})(x)$$

where

$$g(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/4t} \tag{4.5}$$

is the heat kernel on the whole space $\mathbb{R}^n$. Therefore, $e^{t\Delta}\tilde{f}(x)$ is an extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ of $E(t)f(x)$ and

$$[E(t)f]|_0 = [e^{t\Delta}\tilde{f}]|_0. \tag{4.6}$$

Let $0 < \delta < 1$, $\frac{1}{\delta} < l < \frac{n}{\delta}$ and $d_2 > r$ be such that $\frac{n-1}{d_2} = \frac{n}{l} - \delta$. It follows from Eqs. 4.6 and 4.3 that

$$\begin{aligned}
\|[E(t)f]|_0\|_{L^{d_2}(\partial\mathbb{R}^n_+)} &\leq C\|(-\Delta_x)^{\frac{\delta}{2}} e^{t\Delta}\tilde{f}\|_{L^l(\mathbb{R}^n)} \\
&\leq Ct^{-\frac{1}{2}\left(\frac{n}{d_1}-\frac{n}{l}\right)-\frac{\delta}{2}} \|\tilde{f}\|_{L^{d_1}(\mathbb{R}^n)} \\
&\leq Ct^{-\frac{1}{2}\left(\frac{n}{d_1}-\frac{n-1}{d_2}\right)} \|f\|_{L^{d_1}(\mathbb{R}^n_+)}.
\end{aligned}$$

Now a real interpolation argument leads us

$$\|[E(t)f]|_0\|_{L^{(d_2,r)}(\partial\mathbb{R}^n_+)} \leq Ct^{-\left(\frac{n}{2d_1}-\frac{n-1}{2d_2}\right)}\|f\|_{L^{(d_1,r)}(\mathbb{R}^n_+)}.$$

□

Let us define the integral operators

$$\mathcal{G}_1(\varphi)(x,t)=\int_{\partial\mathbb{R}^n_+}G(x,y',t)\varphi(y')dy' \text{ and } \mathcal{G}_2(\varphi)(y',t)=\int_{\mathbb{R}^n_+}G(x,y',t)\varphi(x)dx$$

where $G(x,y,t)$ is defined in Eq. 1.12. Notice that the functions $G(x,y',t)$ and $\mathcal{G}_1(\varphi)(x,t)$ are also well-defined for $x=(x',x_n)\in\mathbb{R}^n$. Recall the pointwise estimate for the heat kernel (4.5) on $\mathbb{R}^n$

$$|(-\Delta_x)^{\frac{\delta}{2}}g(x,t)|\leq\frac{C_\delta}{(t+|x|^2)^{\frac{n}{2}+\frac{\delta}{2}}} \quad (\delta\geq 0), \tag{4.7}$$

for all $x\in\mathbb{R}^n$ and $t>0$.

Lemma 4.2 *Let* $1<d_1<d_2<\infty$ *and* $1\leq r\leq\infty$. *Then, there exists* $C>0$ *such that*

$$\|\mathcal{G}_1(\psi)(x',0,t)\|_{L^{(d_2,r)}(\partial\mathbb{R}^n_+,dx')} \leq Ct^{-\left(\frac{n-1}{2d_1}-\frac{n-1}{2d_2}+\frac{1}{2}\right)}\|\psi\|_{L^{(d_1,r)}(\partial\mathbb{R}^n_+,dx')} \tag{4.8}$$

$$\|\mathcal{G}_1(\psi)(x,t)\|_{L^{(d_2,r)}(\mathbb{R}^n_+,dx)} \leq Ct^{-\left(\frac{n-1}{2d_1}-\frac{n}{2d_2}+\frac{1}{2}\right)}\|\psi\|_{L^{(d_1,r)}(\partial\mathbb{R}^n_+,dx')} \tag{4.9}$$

for all $\psi\in L^{(d_1,r)}(\partial\mathbb{R}^n_+)$.

Proof Let $0<\delta<1$, $\frac{1}{\delta}<l<\frac{n}{\delta}$ and $\frac{n-1}{d_2}=\frac{n}{l}-\delta$. Firstly, notice that the trace-type inequality (4.3) yields

$$\|\mathcal{G}_1(\varphi)(x',0,t)\|_{L^{d_2}(\partial\mathbb{R}^n_+)}\leq C\|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x',x_n,t)\|_{L^l(\mathbb{R}^n)}. \tag{4.10}$$

Next we employ Minkowski's inequality for integrals and the pointwise estimate (4.7) to obtain

$$\begin{aligned}
\|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x',x_n,t)\|_{L^l(\mathbb{R},dx_n)} &= \left(\int_{-\infty}^{\infty}|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x',x_n,t)|^l dx_n\right)^{\frac{1}{l}}\\
&\leq C\int_{\partial\mathbb{R}^n_+}\left(\int_{-\infty}^{\infty}\frac{|\varphi(y')|^l}{(t+|x'-y'|^2+x_n^2)^{\frac{nl}{2}+\frac{\delta l}{2}}}dx_n\right)^{\frac{1}{l}}dy'\\
&= 2C\int_{\partial\mathbb{R}^n_+}|\varphi(y')|\left(\int_0^{\infty}\frac{dx_n}{(t+|x'-y'|^2+x_n^2)^{\frac{nl}{2}+\frac{\delta l}{2}}}\right)^{\frac{1}{l}}dy'\\
&= 2C\int_{\partial\mathbb{R}^n_+}|\varphi(y')|(t+|x'-y'|^2)^{-\frac{n}{2}-\frac{\delta}{2}+\frac{1}{2l}}dy' \qquad (4.11)\\
&\leq Ct^{-\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)\theta}\int_{\partial\mathbb{R}^n_+}|x'-y'|^{-2\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)(1-\theta)}|\varphi(y')|dy' \qquad (4.12)
\end{aligned}$$

where Eq. 4.12 is obtained from Eq. 4.11 by using that $(a+b)^{-k} \leq a^{-k\theta}b^{-k(1-\theta)}$ when $0 < \theta < 1$ and $\kappa \geq 0$. Let $d_1 < l$ and $\gamma = (n-1)(\frac{1}{d_1} - \frac{1}{l})$. Let $0 < \theta < 1$ be such that $(n-1) - \gamma = \left(n + \delta - \frac{1}{l}\right)(1-\theta)$. It follows that $\frac{1}{l} = \frac{1}{d_1} - \frac{\gamma}{n-1} > 0$, and Sobolev embedding theorem gives us

$$\left\| \int_{\partial\mathbb{R}^n_+} \frac{1}{|x'-y'|^{(n-1)-\gamma}} |\varphi(y')| dy' \right\|_{L^l(\partial\mathbb{R}^n_+)} \leq C\|\varphi\|_{L^{d_1}(\partial\mathbb{R}^n_+)}. \tag{4.13}$$

Fubini's theorem, Eqs. 4.12 and 4.13 imply that

$$\begin{aligned} \|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x',x_n,t)\|_{L^l(\mathbb{R}^n)} &= \left\| \|(-\Delta_x)^{\frac{\delta}{2}}\mathcal{G}_1(\varphi)(x',x_n,t)\|_{L^l(\mathbb{R},dx_n)} \right\|_{L^l(\mathbb{R}^{n-1},dx')} \\ &\leq Ct^{-\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)\theta} \left\| \int_{\partial\mathbb{R}^n_+} \frac{1}{|x'-y'|^{(n-1)-\gamma}} |\varphi(y')| dy' \right\|_{L^l(\partial\mathbb{R}^n_+,dx')} \\ &\leq Ct^{-\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)\theta} \|\varphi\|_{L^{d_1}(\partial\mathbb{R}^n_+)}. \end{aligned} \tag{4.14}$$

It follows from Eqs. 4.14 and 4.10 that

$$\|\mathcal{G}_1(\varphi)(x',0,t)\|_{L^{d_2}(\partial\mathbb{R}^n_+)} \leq Ct^{-\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)\theta} \|\varphi\|_{L^{d_1}(\partial\mathbb{R}^n_+)} = Ct^{\frac{n-1}{2d_2}-\frac{n-1}{2d_1}-\frac{1}{2}} \|\varphi\|_{L^{d_1}(\partial\mathbb{R}^n_+)}, \tag{4.15}$$

because of the equality

$$\begin{aligned} -\left(\frac{n}{2}+\frac{\delta}{2}-\frac{1}{2l}\right)\theta &= \frac{n-1}{2} - \frac{\gamma}{2} - \frac{1}{2}\left(n+\delta-\frac{1}{l}\right) \\ &= \frac{n-1}{2d_2} - \frac{n-1}{2d_1} - \frac{1}{2}. \end{aligned}$$

Now the estimate (4.8) follows from Eq. 4.15 and real interpolation. The proof of Eq. 4.9 is similar and is left to the reader. □

In the next lemma we obtain refined boundary estimates on the Lorentz space $L^{(d,1)}$ that is the pre-dual one of $L^{(d',\infty)}$. These can be seen as extensions of Yamazaki's estimates (see [43]) to the operators $\mathcal{G}_1$ and $\mathcal{G}_2$.

Lemma 4.3 *Let* $1 < d_1 < d_2 < \infty$. *There exists a constant* $C > 0$ *such that*

$$\int_0^\infty t^{\left(\frac{n-1}{2d_1}-\frac{n-1}{2d_2}\right)-\frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot,0,t)\|_{L^{(d_2,1)}(\partial\mathbb{R}^n_+)} dt \leq C\|\psi\|_{L^{(d_1,1)}(\partial\mathbb{R}^n_+)} \tag{4.16}$$

$$\int_0^\infty t^{\left(\frac{n-1}{2d_1}-\frac{n}{2d_2}\right)-\frac{1}{2}} \|\mathcal{G}_1(\psi)(\cdot,t)\|_{L^{(d_2,1)}(\mathbb{R}^n_+)} dt \leq C\|\psi\|_{L^{(d_1,1)}(\partial\mathbb{R}^n_+)} \tag{4.17}$$

$$\int_0^\infty t^{\left(\frac{n}{2d_1}-\frac{n-1}{2d_2}\right)-1} \|\mathcal{G}_2(\varphi)(\cdot,t)\|_{L^{(d_2,1)}(\partial\mathbb{R}^n_+)} dt \leq C\|\varphi\|_{L^{(d_1,1)}(\mathbb{R}^n_+)}, \tag{4.18}$$

for all $\psi \in L^{(d_1,1)}(\partial\mathbb{R}^n_+)$ *and* $\varphi \in L^{(d_1,1)}(\mathbb{R}^n_+)$.

Proof We start with Eq. 4.18. Let $1 < p_1 < d_1 < p_2 < d_2$ be such that $\frac{1}{d_1} - \frac{1}{p_1} < -\frac{2}{n}$ and $\frac{1}{d_1} - \frac{1}{p_2} < 2$. Noting that $\mathcal{G}_2(\varphi)(y',t) = [E(t)\varphi](y',0)$, Lemma 4.1 yields

$$\|\mathcal{G}_2(\varphi)(\cdot,t)\|_{L^{(d_2,1)}(\partial\mathbb{R}^n_+)} \leq Ct^{-\left(\frac{n}{2p_k}-\frac{n-1}{2d_2}\right)} \|\varphi\|_{L^{(p_k,1)}(\mathbb{R}^n_+)}, \text{ for } k=1,2. \tag{4.19}$$

For $\varphi \in L^{(p_1,\infty)}(\mathbb{R}^n_+) \cap L^{(p_2,\infty)}(\mathbb{R}^n_+)$, we define the following sub-linear operator

$$\mathcal{F}(\varphi)(t) = t^{\frac{n}{2d_1}-\frac{n-1}{2d_2}-1}\|\mathcal{G}_2(\varphi)(\cdot,t)\|_{L^{(d_2,1)}(\partial\mathbb{R}^n_+)}.$$

Since $1 < p_k < d_2$, it follows from Eq. 4.19 that

$$\mathcal{F}(\varphi)(t) \leq Ct^{\left(\frac{n}{2d_1}-\frac{n}{2p_k}\right)-1}\|\varphi\|_{L^{(p_k,1)}(\mathbb{R}^n_+)}.$$

Let $\frac{1}{s_k} = 1-\left(\frac{n}{2d_1}-\frac{n}{2p_k}\right)$ and take $0 < \theta < 1$ such that $\frac{1}{d_1} = \frac{1-\theta}{p_1}+\frac{\theta}{p_2}$. Then $\frac{1-\theta}{s_1}+\frac{\theta}{s_2} = 1$ with $0 < s_1 < 1 < s_2$. Therefore,

$$\begin{aligned}\|\mathcal{F}(\varphi)(t)\|_{L^{(s_k,\infty)}(0,\infty)} &\leq C\left\|t^{-1/s_k}\right\|^*_{L^{(s_k,\infty)}(0,\infty)}\|\varphi\|_{L^{(p_k,1)}(\mathbb{R}^n_+)}\\ &\leq C\|\varphi\|_{L^{(p_k,1)}(\mathbb{R}^n_+)},\end{aligned}$$

and so $\mathcal{F} : L^{(p_k,1)}(\mathbb{R}^n_+) \to L^{(s_k,\infty)}(0,\infty)$ is a bounded sublinear operator, for $k = 1, 2$. Taking

$$m_k = \|\mathcal{F}(\varphi)\|_{L^{(p_k,1)}(\mathbb{R}^n_+)\to L^{(s_k,\infty)}(0,\infty)},$$

and recalling the interpolation properties

$$L^{(d_1,1)} = (L^{(p_1,1)}, L^{(p_2,1)})_{\theta,1} \text{ and } L^1 = (L^{(s_1,\infty)}, L^{(s_2,\infty)})_{\theta,1},$$

we obtain

$$\|\mathcal{F}(\varphi)\|_{L^1(0,\infty)} \leq Cm_1^{1-\theta}m_2^{\theta}\|\varphi\|_{L^{(d_1,1)}(\mathbb{R}^n_+)} \leq C\|\varphi\|_{L^{(d_1,1)}(\mathbb{R}^n_+)},$$

which is exactly Eq. 4.18.

In order to show Eq. 4.16, now we define

$$\mathcal{F}(\psi)(t) = t^{\frac{n-1}{2d_1}-\frac{n-1}{2d_2}-\frac{1}{2}}\|\mathcal{G}_1(\psi)(\cdot,0,t)\|_{L^{(d_2,1)}(\partial\mathbb{R}^n_+)}$$

and obtain by means of Eq. 4.8 that

$$\mathcal{F}(\psi)(t) \leq Ct^{\frac{n-1}{2d_1}-\frac{n-1}{2p_k}-1}\|\psi\|_{L^{(p_k,1)}(\partial\mathbb{R}^n_+)}.$$

Let $\frac{1}{s_k} = 1-\left(\frac{n-1}{2d_1}-\frac{n-1}{2p_k}\right)$ and $0 < \theta < 1$ be such that $\frac{1}{d_1} = \frac{1-\theta}{p_1}+\frac{\theta}{p_2}$. Then $\frac{1-\theta}{s_1}+\frac{\theta}{s_2} = 1$, and one can obtain Eq. 4.16 by proceeding similarly to proof of Eq. 4.18. The proof of Eq. 4.17 follows analogously by considering

$$\mathcal{F}(\psi)(t) = t^{\frac{n-1}{2d_1}-\frac{n}{2d_2}-\frac{1}{2}}\|\mathcal{G}_1(\psi)(\cdot,\cdot,t)\|_{L^{(d_2,1)}(\mathbb{R}^n_+)}$$

and using Eq. 4.9 instead of Eq. 4.8. The details are left to the reader. □

4.2 Nonlinear Estimates

This section is devoted to estimate the operators

$$\mathcal{N}(u)(x,t) = \int_0^t\int_{\partial\mathbb{R}^n_+} G(x,y',t-s)h(u(y',s))dy'ds \tag{4.20}$$

$$\mathcal{T}(u)(x,t) = \int_0^t\int_{\partial\mathbb{R}^n_+} G(x,y',t-s)V(y')u(y',s)dy'ds. \tag{4.21}$$

For that matter, we define (for each fixed $t > 0$)

$$k_t(x, y', s) = \begin{cases} G(x, y', s), & \text{if } 0 < s < t \\ 0, & \text{otherwise} \end{cases}$$

and consider $\mathcal{H}$ the boundary parabolic integral operator

$$\mathcal{H}(f)(x,t) = \int_0^\infty \int_{\partial\mathbb{R}^n_+} k_t(x, y', t-s) f(y', s) dy' ds.$$

For a suitable function φ defined in either $\Omega = \mathbb{R}^n_+$ or $\Omega = \partial\mathbb{R}^n_+$, let us denote

$$\langle \mathcal{H}(f), \varphi \rangle_\Omega = \int_\Omega \mathcal{H}(f)(x,t)\varphi(x)dx.$$

Using Tonelli's theorem, we have that

$$\begin{aligned} \left| \langle \mathcal{H}(f), \varphi \rangle_{\mathbb{R}^n_+} \right| &\le \int_{\mathbb{R}^n_+} \int_0^\infty \left(\int_{\partial\mathbb{R}^n_+} k_t(x, y', t-s) \left| f(y', s) \right| dy' \right) ds \, |\varphi(x)| \, dx \\ &= \int_0^\infty \int_{\partial\mathbb{R}^n_+} \left| f(y', s) \right| \left(\int_{\mathbb{R}^n_+} k_t(x, y', t-s) \, |\varphi(x)| \, dx \right) dy' ds \\ &= \int_0^\infty \langle |f(\cdot, s)| \, , \mathcal{G}_2(|\varphi|)(\cdot, t-s) \rangle_{\partial\mathbb{R}^n_+} ds \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \left| \langle \mathcal{H}f, \varphi \rangle_{\partial\mathbb{R}^n_+} \right| &\le \int_0^\infty \int_{\partial\mathbb{R}^n_+} \left| f(y', s) \right| \mathcal{G}_1(|\varphi|)(y', 0, t-s) dy' ds \\ &= \int_0^\infty \langle |f(\cdot, s)| \, , \mathcal{G}_1(|\varphi|)(\cdot, 0, t-s) \rangle_{\partial\mathbb{R}^n_+} ds, \end{aligned} \tag{4.23}$$

because the kernel of $\mathcal{G}_1(\varphi)(\cdot, 0, t-s)$ and $\mathcal{G}_2(\varphi)(\cdot, t-s)$ is $k_t(x, y', t-s)$.

Lemma 4.4 *Let $n \ge 3$, $\frac{n-1}{n-2} < \rho < \infty$, and $q = (n-1)(\rho-1)$, $p = n(\rho-1)$. There exists a constant $C > 0$ such that*

$$\sup_{t>0} \|\mathcal{H}(f)(\cdot, t)\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} \le C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}^n_+)} \tag{4.24}$$

$$\sup_{t>0} \|\mathcal{H}(f)(\cdot, t)\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)} \le C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho}, \infty)}(\partial\mathbb{R}^n_+)}, \tag{4.25}$$

for all $f \in L^\infty((0,\infty); L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+))$.

Proof Estimate (4.22) and Hölder inequality (2.3) yields

$$\begin{aligned} |\langle \mathcal{H}f, \varphi \rangle_{\mathbb{R}^n_+}| &\le C \int_0^\infty \|f(\cdot, s)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \|\mathcal{G}_2(|\varphi|)(\cdot, t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}^n_+)} ds \\ &\le C \sup_{t>0} \|f(\cdot, t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \int_0^\infty \|\mathcal{G}_2(|\varphi|)(\cdot, t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}^n_+)} ds. \end{aligned} \tag{4.26}$$

Next, notice that $(\frac{q}{\rho})' = \frac{q}{q-\rho} > p'$ and

$$\frac{1}{2}\left(\frac{n}{p'} - \frac{n-1}{(\frac{q}{\rho})'}\right) - 1 = \frac{1}{2}\left(1 - \frac{1}{\rho-1} + \frac{\rho}{\rho-1}\right) - 1 = 0.$$

In view of Eq. 4.26, we can use duality and estimate (4.18) with $d_1 = p'$ and $d_2 = \frac{q}{q-\rho}$ to obtain

$$\begin{aligned}
I_1(t) &= \|\mathcal{H}(f)(\cdot,t)\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} = \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}^n_+)}=1} \left|\langle \mathcal{H}(f), \varphi\rangle_{\mathbb{R}^n_+}\right| \\
&\le C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}^n_+)}=1} \left(\int_0^\infty \|\mathcal{G}_2(|\varphi|)(\cdot,t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}^n_+)} ds\right) \\
&\le C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \sup_{\|\varphi\|_{L^{(p',1)}(\mathbb{R}^n_+)}=1} \|\varphi\|_{L^{(p',1)}(\mathbb{R}^n_+)} \\
&\le C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)},
\end{aligned} \tag{4.27}$$

for a.e. $t > 0$. The estimate (4.24) follows by taking the essential supremum over $(0,\infty)$ in both sides of Eq. 4.27.

Now we deal with Eq. 4.25 which is the boundary part of the norm $\|\cdot\|_{\mathcal{X}_{p,q}}$. We have that $\left(\frac{q}{\rho}\right)' > q'$ and

$$\frac{1}{2}\left(\frac{n-1}{q'} - \frac{1}{(\frac{q}{\rho})'}\right) - \frac{1}{2} = \frac{n-1}{2}\left(\frac{\rho}{q} - \frac{1}{q}\right) - \frac{1}{2} = 0.$$

Proceeding similarly to proof of Eq. 4.27, but using Eq. 4.16 instead of Eq. 4.18, we obtain

$$\begin{aligned}
I_2(t) &= \|\mathcal{H}(f)(\cdot,t)\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)} \\
&\le \sup_{\|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}^n_+)}=1} \int_0^\infty \|f(\cdot,s)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \|\mathcal{G}_1(|\varphi|)(\cdot,0,t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}^n_+)} ds \\
&\le C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \int_0^\infty \|\mathcal{G}_1(|\varphi|)(\cdot,0,t-s)\|_{L^{(\frac{q}{q-\rho},1)}(\partial\mathbb{R}^n_+)} ds \\
&\le C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)} \sup_{\|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}^n_+)}=1} \|\varphi\|_{L^{(q',1)}(\partial\mathbb{R}^n_+)} \\
&= C \sup_{t>0} \|f(\cdot,t)\|_{L^{(\frac{q}{\rho},\infty)}(\partial\mathbb{R}^n_+)},
\end{aligned} \tag{4.28}$$

for a.e. $t \in (0,\infty)$, which is equivalent to Eq. 4.25. □

4.3 Proof of Theorem 3.1

Part (A) Let us write Eq. 1.11 as

$$u = E(t)u_0 + \mathcal{N}(u) + \mathcal{T}(u)$$

where the operators $\mathcal{N}$ and $\mathcal{T}$ are defined in Eqs. 4.20 and 4.21, respectively.

Recall the heat estimate (see e.g. [41, Lemma 3.4])

$$\|E(t)u_0\|_{L^{d_2}(\mathbb{R}^n_+)} \le C t^{-\frac{n}{2}(\frac{1}{d_2}-\frac{1}{d_1})} \|u_0\|_{L^{d_1}(\mathbb{R}^n_+)}, \tag{4.29}$$

for $1 \leq d_1 \leq d_2 \leq \infty$. By using interpolation, Eq. 4.29 leads us to

$$\|E(t)u_0\|_{L^{(d_2,\infty)}(\mathbb{R}^n_+)} \leq Ct^{-\frac{n}{2}(\frac{1}{d_2}-\frac{1}{d_1})}\|u_0\|_{L^{(d_1,\infty)}(\mathbb{R}^n_+)}, \tag{4.30}$$

for $1 < d_1 \leq d_2 < \infty$.

We consider the Banach space $E = BC((0,\infty); \mathcal{X}_{p,q})$ endowed with the norm (3.3). Estimate (4.30) and Lemma 4.1 yield

$$\begin{aligned}\|E(t)u_0\|_E &= \sup_{t>0}\|E(t)u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} + \sup_{t>0}\|E(t)u_0\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)}\\ &\leq C\left(\|u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} + \|u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)}\right)\\ &= \delta_2\|u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} \leq \varepsilon,\end{aligned} \tag{4.31}$$

provided that $\|u_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)} \leq \frac{\varepsilon}{\delta_2}$. In what follows, we estimate the operators $\mathcal{T}$ and $\mathcal{N}$ in order to employ a contraction argument in E. Since $\frac{\rho}{q} = \frac{1}{q} + \frac{\rho-1}{q}$, property (1.8) and Hölder's inequality (2.3) yield

$$\begin{aligned}\|h(u)-h(v)\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}^n_+)} &\leq \eta\|\,|u-v|\,(|u|^{\rho-1}+|v|^{\rho-1})\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}^n_+)}\\ &\leq C\|u-v\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)}(\|u\|^{\rho-1}_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)} + \|v\|^{\rho-1}_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)}).\end{aligned} \tag{4.32}$$

Using Lemma 4.4 and Eq. 4.32, we obtain

$$\begin{aligned}\sup_{t>0}\|\mathcal{N}(u)-\mathcal{N}(v)\|_{\mathcal{X}_{p,q}} &= \sup_{t>0}\|\mathcal{H}(h(u)-h(v))\|_{\mathcal{X}_{p,q}}\\ &\leq C\sup_{t>0}\|h(u)-h(v)\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}^n_+)}\\ &\leq K\|u-v\|_E(\|u\|_E^{\rho-1}+\|v\|_E^{\rho-1}).\end{aligned}$$

Also, noting that $\frac{\rho}{q} = \frac{1}{n-1} + \frac{1}{q}$, we have that

$$\begin{aligned}\|\mathcal{T}(u)-\mathcal{T}(v)\|_E &= \sup_{t>0}\|\mathcal{H}(V(u-v))\|_{\mathcal{X}_{p,q}}\\ &\leq C\sup_{t>0}\|V(u-v)\|_{L^{(q/\rho,\infty)}(\partial\mathbb{R}^n_+)}\\ &\leq \delta_1\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)}\sup_{t>0}\|u(\cdot,t)-v(\cdot,t)\|_{L^{(q,\infty)}(\partial\mathbb{R}^n_+)}\\ &\leq \gamma\|u-v\|_E \text{ with } 0<\gamma<1,\end{aligned}$$

provided that $\gamma = \delta_1\|V\|_{L^{(n-1,\infty)}(\partial\mathbb{R}^n_+)}$. Now consider

$$\Phi(u) = E(t)u_0 + \mathcal{N}(u) + \mathcal{T}(u) \tag{4.33}$$

and the closed ball $B_\varepsilon = \{u \in \mathcal{X}_{p,q}\,;\ \|u\|_{\mathcal{X}_{p,q}} \leq \frac{2\varepsilon}{1-\gamma}\}$ where $\varepsilon > 0$ is chosen in such a way that

$$\left(\frac{2^\rho\varepsilon^{\rho-1}K}{(1-\gamma)^{\rho-1}} + \gamma\right) < 1. \tag{4.34}$$

For all $u, v \in B_\varepsilon$, we obtain that

$$\begin{aligned}
\|\Phi(u)-\Phi(v)\|_E &\leq \|\mathcal{N}(u)-\mathcal{N}(v)\|_E+\|\mathcal{T}(u)-\mathcal{T}(v)\|_E \\
&\leq \|u-v\|_E(K\|u\|_E^{\rho-1}+K\|v\|_E^{\rho-1}+\gamma) \\
&\leq \left(K\frac{2^\rho\varepsilon^{\rho-1}}{(1-\gamma)^{\rho-1}}+\gamma\right)\|u-v\|_{\mathcal{X}_{p,q}}.
\end{aligned} \tag{4.35}$$

Noting that $\Phi(0)=E(t)u_0$, the estimates (4.31) and (4.35) yield

$$\begin{aligned}
\|\Phi(u)\|_E &\leq \|E(t)u_0\|_E+\|\Phi(u)-\Phi(0)\|_E \\
&\leq \varepsilon+(K\|u\|_E^\rho+\gamma\,\|u\|_E) \\
&\leq \varepsilon+\left(K\frac{2^\rho\varepsilon^\rho}{(1-\gamma)^\rho}+\gamma\frac{2\varepsilon}{1-\gamma}\right)\leq\frac{2\varepsilon}{1-\gamma},
\end{aligned}$$

for all $u \in B_\varepsilon$, because of Eq. 4.34. Then the map $\Phi : B_\varepsilon \to B_\varepsilon$ is a contraction and Banach fixed point theorem assures that there is a unique solution $u \in B_\varepsilon$ for Eq. 1.11.

The weak convergence to the initial data as $t \to 0^+$ follows from standard arguments and is left to the reader (see e.g. [21, Lemma 3.8], [29, Lemmas 3.3 and 4.8]).

Part (B) Let $u, \tilde{u} \in B_\varepsilon$ be two solutions obtained in item **(A)** corresponding to pairs (V, u_0) and $(\tilde{V}, \tilde{u}_0)$, respectively. We have that

$$\begin{aligned}
\|u-\tilde{u}\|_E &\leq \|E(t)(u_0-\tilde{u}_0)\|_E+\|\mathcal{N}(u)-\mathcal{N}(\tilde{u})\|_E+\|\mathcal{T}(u)-\mathcal{T}(\tilde{u})\|_E \\
&\leq \delta_2\|u_0-\tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)}+K\|u-\tilde{u}\|_E(\|u\|_E^{\rho-1}+\|\tilde{u}\|_E^{\rho-1}) \\
&\quad +\|\mathcal{H}[(V-\tilde{V})\tilde{u}+V(u-\tilde{u})]\|_E \\
&\leq \delta_2\|u_0-\tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)}+\|u-\tilde{u}\|_E\left(\frac{2^\rho\varepsilon^{\rho-1}K}{(1-\gamma)^{\rho-1}}\right) \\
&\quad +\delta_1\left(\|V-\tilde{V}\|_{L^{(n-1,\infty)}}\|\tilde{u}\|_E+\|V\|_{L^{(n-1,\infty)}}\|u-\tilde{u}\|_E\right) \\
&\leq \delta_2\|u_0-\tilde{u}_0\|_{L^{(p,\infty)}(\mathbb{R}^n_+)}+\left(\frac{2^\rho\varepsilon^{\rho-1}K}{(1-\gamma)^{\rho-1}}+\gamma\right)\|u-\tilde{u}\|_E+\frac{2\delta_1\varepsilon}{1-\gamma}\|V-\tilde{V}\|_{L^{(n-1,\infty)}},
\end{aligned}$$

which gives the desired continuity because of Eq. 4.34.

4.4 Proof of Theorem 3.3

From the fixed point argument in the proof of Theorem 3.1, the solution u is the limit in the space E of the Picard sequence

$$u_1=E(t)u_0,\ u_{k+1}=u_1+\mathcal{N}(u_k)+\mathcal{T}(u_k),\ k\in\mathbb{N}, \tag{4.36}$$

where $\mathcal{N}$ and $\mathcal{T}$ are defined in Eqs. 4.20 and 4.21, respectively. Since $u_0 \in L^{(n(\rho-1),\infty)}(\mathbb{R}^n_+)$ and $V \in L^{(n-1,\infty)}(\mathbb{R}^{n-1})$, we can take u_0 and V as homogeneous functions of degree $-\frac{1}{\rho-1}$ and -1, respectively. Using the kernel property

$$G(x,y,t)=\lambda^n G(\lambda x,\lambda y,\lambda^2 t) \tag{4.37}$$

and homogeneity of u_0, we have that

$$\begin{aligned} u_1(\lambda x, \lambda^2 t) &= \int_{\mathbb{R}^n_+} G(\lambda x, y, \lambda^2 t) u_0(y) dy \\ &= \int_{\mathbb{R}^n_+} \lambda^n G(\lambda x, \lambda y, \lambda^2 t) u_0(\lambda y) dy \\ &= \lambda^{-\frac{1}{\rho-1}} \int_{\mathbb{R}^n_+} G(x, y, t) u_0(y) dy = \lambda^{-\frac{1}{\rho-1}} u_1(x, t), \end{aligned}$$

and then u_1 is invariant by Eq. 3.1. Recalling that $f(\lambda a) = \lambda^\rho f(a)$ and assuming that

$$u_k(x, t) = u_{k,\lambda}(x, t) := \lambda^{\frac{1}{\rho-1}} u_k(\lambda x, \lambda^2 t), \text{ for } k \in \mathbb{N},$$

we obtain

$$\begin{aligned} \mathcal{N}(u_k)(\lambda x, \lambda^2 t) &= \int_0^{\lambda^2 t} \int_{\partial\mathbb{R}^n_+} G(\lambda x, y', \lambda^2 t - s) h(u_k(y', s)) dy' ds \\ &= \lambda^{n-1+2} \int_0^t \int_{\partial\mathbb{R}^n_+} G(\lambda x, \lambda y', \lambda^2 (t-s)) h(\lambda^{-\frac{1}{\rho-1}} \lambda^{\frac{1}{\rho-1}} u_k(\lambda y', \lambda^2 s)) dy' ds \\ &= \lambda^{n+1} \int_0^t \int_{\partial\mathbb{R}^n_+} \lambda^{-n} G(x, y', t-s) \lambda^{-\frac{\rho}{\rho-1}} h(u_k(y', s)) dy' ds \\ &= \lambda^{-\frac{1}{\rho-1}} \mathcal{N}(u_k)(x, t) \end{aligned}$$

and, similarly, $\mathcal{T}(u_k)(\lambda x, \lambda^2 t) = \lambda^{-\frac{1}{\rho-1}} \mathcal{T}(u_k)(x, t)$. It follows that

$$\lambda^{\frac{1}{\rho-1}} u_{k+1}(\lambda x, \lambda^2 t) = u_1(x, t) + \mathcal{N}(u_k) + \mathcal{T}(u_k) = u_{k+1}(x, t)$$

and then, by induction, u_k is invariant by Eq. 3.1 for all $k \in \mathbb{N}$.

Since the norm $\|\cdot\|_E$ is invariant by Eq. 3.1 and $u_k \to u$ in E, it is easy to see that u is also invariant by Eq. 3.1, that is, it is self-similar.

4.5 Proof of Theorem 3.4

Part (A) Let $u_0 \geq 0$ a.e. in $\mathbb{R}^n_+$ and $\mathcal{U} \subset \mathbb{R}^n_+$ be a positive measure set with $u_0 > 0$ in $\mathcal{U}$. It follows from Eq. 1.12 that

$$u_1(x, t) = \int_{\mathbb{R}^n_+} G(x, y, t) u_0(y) dy > 0 \text{ in } \overline{\mathbb{R}^n_+} \times (0, \infty).$$

By using that V is nonnegative in $\mathbb{R}^{n-1}$ and $h(a) \geq 0$ when $a \geq 0$, one can see that $\mathcal{N}(u) + \mathcal{T}(u)$ is nonnegative in $\overline{\mathbb{R}^n_+} \times (0, \infty)$ provided that $u|_{\partial\mathbb{R}^n_+} \geq 0$. Then, an induction argument applied to the sequence (4.36) shows that $u_k > 0$ in $\overline{\mathbb{R}^n_+} \times (0, \infty)$, for all $k \in \mathbb{N}$. Since the convergence in the space E implies convergence in $L^{(p,\infty)}(\mathbb{R}^n_+)$ and in $L^{(q,\infty)}(\partial\mathbb{R}^n_+)$ for each $t > 0$, we have that (up to a subsequence) $u_k(\cdot, t) \to u(\cdot, t)$ a.e. in $(\mathbb{R}^n_+, dx)$ and a.e. in $(\partial\mathbb{R}^n_+, dx')$ for each $t > 0$. It follows that u is a nonnegative function because pointwise convergence preserves nonnegativity. Since $u_1 > 0$, then $u = u_1 + \mathcal{N}(u) + \mathcal{T}(u) \geq u_1 + 0 > 0$ in $\overline{\mathbb{R}^n_+} \times (0, \infty)$, as desired. The proof of the statement concerning negativity is left to the reader.

Part (B) We only will prove the antisymmetric part of the statement, because the symmetric one is analogous. Given a $T \in \mathcal{G}$, we have

$$\begin{aligned}
u_1(T(x), t) &= \int_{\mathbb{R}^n_+} G(T(x), y, t) u_0(y) dy \\
&= \int_{\mathbb{R}^n_+} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|T(x)-y|^2}{4t}} + e^{-\frac{|T(x)-y^*|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}^n_+} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|T((x-T^{-1}(y))|^2}{4t}} + e^{-\frac{|T(x-T^{-1}(y^*))|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}^n_+} \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[e^{-\frac{|x-T^{-1}(y)|^2}{4t}} + e^{-\frac{|x-(T^{-1}(y))^*|^2}{4t}} \right] u_0(y) dy \\
&= \int_{\mathbb{R}^n_+} G(x, T^{-1}(y), t) u(y) dy.
\end{aligned}$$

Making the change of variable $z = T^{-1}(y)$ and using that u_0 is antisymmetric under $\mathcal{G}$, we obtain

$$u_1(T(x), t) = \int_{\mathbb{R}^n_+} G(x, z, t) u_0(T(z)) dz = -\int_{\mathbb{R}^n_+} G(x, z, t) u_0(z) dz = -u_1(x, t).$$

A similar argument shows that

$$\mathcal{L}(\theta)(x, t) = \int_0^t \int_{\partial\mathbb{R}^n_+} G(x, y', t-s) \theta(y', t) dy' ds$$

is antisymmetric when $\theta(\cdot, t)|_{\partial\mathbb{R}^n_+}$ is also, for each $t > 0$. As V is symmetric and $h(a) = -h(-a)$, it follows that

$$\theta(x, t) = h(u(\cdot, t)) + Vu(\cdot, t)$$

is antisymmetric whenever $u(\cdot, t)$ does so. Therefore, by means of an induction argument, one can prove that each element $u_k(\cdot, t)$ of the sequence (4.36) is antisymmetric. Recall from Part (A) that (up a subsequence) $u_k(\cdot, t) \to u(\cdot, t)$ a.e. in $(\mathbb{R}^n_+, dx)$ and in $(\partial\mathbb{R}^n_+, dx')$, for each $t > 0$. Since this convergence preserves antisymmetry, it follows that $u(\cdot, t)$ is also antisymmetric, for each $t > 0$.

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