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# S-matrices and quantum group symmetry of $k$ -deformed sigma models

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## Abstract

Recently, two kinds of integrable deformations of the string world sheet theory in the gauge/gravity correspondence have been constructed (Delduc *et al* 2014 *Phys. Rev. Lett.* **112** 051601; Hollowood *et al* 2014 *J. Phys. A: Math. Theor.* **47** 495402). One class of these, the  $k$  deformations associated to the more general  $q$  deformations but with  $q = e^{i\pi/k}$  a root of unity, has been shown to be related to a particular discrete deformation of the principal chiral models and (semi-)symmetric space sigma models involving a gauged WZW model. We conjecture a form for the exact  $S$ -matrices of the bosonic integrable field theories of this type. The  $S$ -matrices imply that the theories have a hidden infinite dimensional affine quantum group symmetry. We provide some evidence, via quantum inverse scattering techniques, that the theories do indeed possess the finite-dimensional part of this quantum group symmetry.

Keywords: integrable deformations, AdS/CFT, integrable field theory, quantum groups, string theory

## 1. Introduction

One of the most fascinating applications of the quantum inverse scattering method (QISM) to sigma models has been to the exact quantization of the SU(2) principal chiral model (PCM) in the 80s by Faddeev and Reshetikhin (FR) in their seminal paper [1]. The main conceptual idea was to replace the non-ultra-local Poisson bracket of the theory by a new ultra-local one that preserved the PCM equations of motion but under the time flow of a new Hamiltonian. After

doing this, it was possible to construct a magnetic lattice algebra and to apply the QISM allowing for the determination of the exact quantum spectrum and  $S$ -matrix of the  $SU(2)$  PCM in a thermodynamic limit. However, one drawback of this new ultra-localization trick was the lack of a systematic understanding, in the sense that the introduction of the new Poisson bracket (to be quantized) was not rigorously justified and the trick remained unused for many decades precluding the use of the QISM to other sigma models beyond the  $SU(2)$ -PCM considered by FR.

This situation changed only very recently when Delduc, Magro and Vicedo (DMV) in [2], using classical integrability techniques, succeeded in identifying the algebraic structure behind the FR ultra-localization mechanism. This allowed DMV to generalize the procedure to any PCM on a Lie group  $F$  and, more interestingly, to any sigma model on a symmetric space  $F/G$ , and also to the semi-symmetric spaces like  $AdS_5 \times S^5$  that appear in superstring theory and the AdS/CFT correspondence [3]. The generalized FR mechanism was considered by DMV only at classical level and it boils down to a very simple recipe: it is equivalent to replacing in the Poisson bracket the so-called twisting function  $\varphi_\sigma(z)$  associated to the sigma model by the new FR twisting function  $\varphi_{FR}(z)$ , which is a simpler—in some well defined sense—function of the spectral parameter  $z$ . The next step in the implementation of the QISM is the introduction of the lattice algebra regularization of the theory, which is the ultimate goal of the quantization procedure. The lattice Poisson algebras corresponding to the  $F/G$  sigma models and to the  $AdS_5 \times S^5$  superstring were constructed explicitly in [2, 4] and, remarkably, they are deeply related to the Pohlmeyer reduction of the sigma models [5–7]. Unfortunately, the quantization of these lattice Poisson algebras is not known yet and it is still an open problem.

Despite of the fact that the FR mechanism turned out to be a very simple algebraic operation, a key ingredient was still missing, namely, the replacement of the twisting function  $\varphi_\sigma(z) \rightarrow \varphi_{FR}(z)$  was not justified from a physical standpoint. After all, changing the Poisson structure is equivalent to modifying the kinetic term in the action functional and this raises the question as to whether it is possible to find an integrable deformation of the original sigma models that implements in a natural way the FR bracket in some limit of the deformation parameter.

This led to the introduction of the  $\eta$ -deformed sigma models by DMV in [8–10], which are Yang–Baxter type deformations of the form originally introduced by Klimcik in [11] for the PCM. The  $\eta$ -deformed Lagrangian combines the two twisting functions  $\varphi_\sigma(z)$ ,  $\varphi_{FR}(z)$  into a new one  $\varphi_{\epsilon(\eta)}(z)$  leading to a deformed interpolating Poisson bracket of the form  $\{f, g\}_\eta = \{f, g\}_\sigma + \epsilon^2(\eta)\{f, g\}_{FR}$ . Unfortunately, the reality condition of the action and integrability properties restrict the domain of the parameter  $\eta$ . This translates into  $0 \leq \epsilon^2(\eta) < 1$ , which means that the FR bracket is not reachable in parameter space. In other words, it will never be the dominant bracket that has to be quantized within the FR scheme.

An entirely different kind of deformation has been emerging based originally on the observation that the  $S$ -matrix that describes the scattering of excitations on the  $AdS_5 \times S^5$  string world sheet theory admits a quantum group deformation [12–15]. In turn, this was motivated by the observation that the world sheet equations of motion are equivalent, via the Pohlmeyer reduction, to a relativistic generalized sine-Gordon theory [5, 6] and the deformed  $S$ -matrix provides an explicit relation between both theories at the quantum level. This deformation was recently identified in [16] as a deformation of the non-abelian  $T$ -dual of the world sheet sigma model related to the (bosonic) ‘ $\lambda$ -deformations’ of the PCM introduced by Sftsos in [17] and of the symmetric space sigma models constructed by the present authors in [18]. We call these theories the  $k$ -deformed sigma models, rather than  $\lambda$ -deformed, because

the parameter  $q = e^{i\pi/k}$  is fixed, a root of unity, while  $\lambda$  is a coupling constant. In particular, in the bosonic models  $\lambda$  is a running coupling in the quantum theory and, therefore, it is naturally interpreted as the running coupling of the original sigma model. It is only in the classical limit  $k \rightarrow \infty$  that  $\lambda$  is a fixed parameter.

The  $k$ -deformed Lagrangian combines the two twisting functions and Poisson brackets exactly as discussed above but with  $-\infty < \epsilon^2(\lambda) \leq 0$ , corresponding to  $\lambda \in [0, 1]$ , so that in the limit  $\lambda \rightarrow 0$  the action functional indeed possesses the (generalized) FR bracket as its canonical Poisson structure. Here, we will consider the bosonic models of this type.

Finding the lattice algebra regularization of the field theory is by far the most difficult and important step toward the full implementation of the QISM. It is equivalent to the regularization/quantization of the Maillet bracket [19] associated to the field theory under examination, which, for the models of present interest, is non-ultra-local. The quantization of the Maillet bracket is known to be a difficult and longstanding unsolved problem for most of the interesting integrable field theories and for this reason any successful attempt to regularize/quantize it, even in some simplifying limit, has to be considered as a useful enterprise by itself. To date, and in relation to the  $k$ -deformations, the (Maillet) lattice algebra is only known in the limit  $\lambda \rightarrow 0$ , at classical level,  $k = \infty$ , and for any value of the spectral parameter  $z$ . The Poisson lattice algebras coincide precisely with the ones introduced in the DMV papers mentioned above.

In the present work, we find another limit in which we can regularize and quantize the Maillet bracket. It is defined for any value of the parameter  $\lambda \in [0, 1]$  but only for a very specific set of values of the spectral parameter that correspond to the poles of the  $\lambda$ -deformed twisting function  $\varphi_{\epsilon(\lambda)}(z)$ . Remarkably, the lattice algebra turns out to be nothing but the Kac–Moody lattice current algebra introduced in [20, 21]. In particular, this allows us to show in a rather simple way that the  $k$ -deformed sigma models associated to the principal chiral model on a Lie group  $F$  and to the bosonic sigma models on the symmetric spaces  $F/G$ , both possess a finite-dimensional quantum group symmetry  $U_q(\mathfrak{f})$  with a deformation parameter  $q = e^{i\pi/k}$  being a root of unity. This quantum group is the finite-dimensional piece of the infinite-dimensional affine quantum group symmetry that the  $S$ -matrix of these theories are conjectured to enjoy. Furthermore, we also show explicitly how these models naturally implement the FR ultra-localization mechanism as a continuous limiting process, i.e., as  $\lambda \rightarrow 0$  we do have that  $\{f, g\}_\lambda \rightarrow \{f, g\}_{\text{FR}}$ .

It is important to emphasize that in this work we are considering the (bosonic)  $k$ -deformed theories as just  $1+1$  dimensional integrable quantum field theories but not as string world sheet theories. Hence, we do not have the additional complications of a world-sheet theory, like gauge fixing and the Virasoro constraints, to contend with. In particular, in the situation studied here, the coupling  $\lambda$  runs with scale, whereas in the stringy context we would expect  $\lambda$  to be exactly marginal under the renormalization group. For applications of the  $k$ -deformation to string theory see [16, 25–30].

The paper is organized as follows. In section 2, we review the construction of the theories at the Lagrangian level. We then consider the theories at the quantum level and make a series of conjectures for the form of their exact  $S$ -matrices. This motivates the idea that the  $k$ -deformed theory has affine quantum group symmetries. In section 3, we reveal the Poisson–Lie group symmetry of the deformed sigma models, which is the classical precursor of the quantum group. We write down the Maillet  $\mathfrak{r}/\mathfrak{s}$  form of the Poisson brackets of the Lax operator, identify the twisting functions and find their poles. We consider the deformed theories and expand their monodromy matrix at the poles of the twisting function and find the first non-trivial non-local charges suggesting the existence of an affine quantum group. In

section 4, we review the lattice Kac–Moody algebra and the quantization of the monodromy matrix evaluated at the poles of the twisting function, which generates the finite-dimensional quantum group with a root-of-unity  $q$ -deformation parameter. Finally, we draw some conclusions and directions for future work.

## 2. The deformed sigma models

We will consider two classes of bosonic  $k$ -deformed sigma models that we call the  $k$ -principal chiral models ( $k$ -PCMs) and  $k$ -symmetric space sigma models ( $k$ -SSSMs), associated to Lie groups  $F$  and symmetric spaces  $F/G$ , respectively. They are constructed in the following way. For the  $k$ -PCM case [17], one takes the PCM for an  $F$ -valued field  $f$  and then adds to it a WZW model for a second  $F$ -valued field  $\mathcal{F}$ . The original PCM has an  $F_L \times F_R$  global symmetry that acts as  $f \rightarrow U_L f U_R^{-1}$ . An interaction between the PCM and WZW model is generated by gauging the common  $F$ -action

$$f \rightarrow Uf, \quad \mathcal{F} \rightarrow U\mathcal{F}U^{-1}, \quad U \in F, \quad (2.1)$$

and fixing the gauge by setting  $f = I$ . The resulting theory has a global  $F$  symmetry that one can view as arising from the original  $F_R$ -symmetry of the PCM<sup>4</sup>.

For the  $k$ -SSSM case [18], one takes the  $F/G$  sigma model defined in terms of an  $F$ -valued field  $f$  and a gauge field  $B_\mu$  corresponding to the  $G$  right action  $f \rightarrow fU^{-1}$  with  $U \in G$ . The  $F/G$  sigma model also has a global  $F_L$  symmetry that acts as  $f \rightarrow U_L f$  with  $U_L \in F$ . Then, as for the  $k$ -PCM case, one adds to it a WZW model for an  $F$ -valued field  $\mathcal{F}$  and gauges the common  $F$ -action (2.1). Finally, one partially fixes the gauge by setting  $f = I$  and integrates out the gauge field  $B_\mu$ . This leaves the subgroup  $G \subset F$  as the remnant gauge symmetry.

In both cases, the final theory has the form of a gauged WZW model for a group  $F$  where the whole group  $F$ , which acts by the anomaly-free vector action  $\mathcal{F} \rightarrow U\mathcal{F}U^{-1}$ , seems to be gauged. In fact, this is incorrect because the action involves a deformation that breaks the gauge symmetry completely for the  $k$ -PCM case, and breaks it down to the subgroup  $G \subset F$  for the  $k$ -SSSM case<sup>5</sup>:

$$\begin{aligned} S_q = & -\frac{k}{2\pi} \int d^2x \text{Tr} [\mathcal{F}^{-1} \partial_+ \mathcal{F} \mathcal{F}^{-1} \partial_- \mathcal{F} + 2A_+ \partial_- \mathcal{F} \mathcal{F}^{-1} \\ & - 2A_- \mathcal{F}^{-1} \partial_+ \mathcal{F} - 2\mathcal{F}^{-1} A_+ \mathcal{F} A_- + 2A_+ \Omega A_-] \\ & + \frac{k}{12\pi} \int d^3x \epsilon^{abc} \text{Tr} [\mathcal{F}^{-1} \partial_a \mathcal{F} \mathcal{F}^{-1} \partial_b \mathcal{F} \mathcal{F}^{-1} \partial_c \mathcal{F}]. \end{aligned} \quad (2.2)$$

This action has two parameters: the level  $k$ , which is a positive integer that defines the deformation parameter  $q = \exp[i\pi/(c_2(F) + k)]$ , and the coupling constant  $\lambda$  which is related to the coupling constant  $\kappa$  of the original sigma models by means of  $\lambda^{-1} = 1 + \kappa^2/k$ .

The deformation appears via the term that depends on the quantity  $\Omega$  which takes the form<sup>6</sup>

<sup>4</sup> It is worth pointing out, that this simple construction does not describe the  $k$ -deformation of the full superstring world sheet theory worked out in [16].

<sup>5</sup> In the Hamiltonian formulation, the first class constraints associated to  $F$  are transformed into second class constraints for the  $k$ -PCM case. For the  $k$ -SSSM this fate also befalls the first class constraints in the coset part of the Lie algebra  $\mathfrak{f}^{(1)}$  to leave a  $G$  gauge symmetry.

<sup>6</sup> Notice that  $\text{Tr}(A_+ \Omega A_-) = \text{Tr}(\Omega A_+ A_-)$  so that, in these bosonic models,  $\Omega$  coincides with its transpose.

$$\Omega = \begin{cases} \lambda^{-1} & k\text{-PCM} \\ \mathbb{P}^{(0)} + \lambda^{-1}\mathbb{P}^{(1)} & k\text{-SSSM}. \end{cases} \quad (2.3)$$

In the latter case, the  $\mathbb{P}^{(i)}$ ,  $i = 0, 1$ , are the projectors along the  $\mathbb{Z}_2$  decomposition of the Lie algebra  $\mathfrak{f} = \mathfrak{f}^{(0)} \oplus \mathfrak{f}^{(1)}$ , where  $\mathfrak{f}^{(0)}$  is the Lie algebra of  $G$ . Note that the  $\mathbb{Z}_2$  gradation here is one of the defining elements of a symmetric space.

In (2.2), the status of the field  $A_\mu$  is rather subtle. In the  $k$ -PCM case, it is an auxiliary field that can be integrated out since it appears quadratically in the action. In the  $k$ -SSSM case, the component  $A_\mu^{(1)}$  is, similarly, an auxiliary field, but the component  $A_\mu^{(0)}$  is a genuine gauge field that reflects a  $G$ -gauge symmetry of the theory. It can be integrated out, but then one would have to fix the gauge on the field  $\mathcal{F}$ . Alternatively one can fix the gauge by imposing, for example,  $A_+^{(0)} = 0$ .

In the limit  $\lambda \rightarrow 1$ , with  $k \rightarrow \infty$  in such a way that

$$k(\lambda^{-1} - 1) = \kappa^2 \quad (2.4)$$

is fixed, the theories can then be shown to be the non-abelian T-duals of the PCM for the group  $F$ , or of the symmetric space sigma model for  $F/G$ , respectively, with coupling constant  $\kappa$ . Theories which are non-abelian T-duals of each other are related by a canonical transformation and therefore it is a reasonable hypothesis that they can be equivalent at the quantum level. The  $k$ -PCM has a global  $F$  symmetry which acts as  $\mathcal{F} \rightarrow U\mathcal{F}U^{-1}$  which is the remnant of the original  $F_R$  symmetry of the PCM, while the  $k$ -SSSM has no obvious symmetries.

### 2.1. Integrability

The  $k$ -deformed theories are integrable at the classical level<sup>7</sup>. To see this, we can solve for the auxiliary field  $A_\mu$  in the form

$$A_+ = (\Omega - \text{Ad}_{\mathcal{F}^{-1}})^{-1} \mathcal{F}^{-1} \partial_+ \mathcal{F}, \quad A_- = -(\Omega - \text{Ad}_{\mathcal{F}})^{-1} \partial_- \mathcal{F} \mathcal{F}^{-1}. \quad (2.5)$$

The equations-of-motion can then be written in Lax form:

$$[\partial_+ + \mathcal{L}_+(z), \partial_- + \mathcal{L}_-(z)] = 0, \quad (2.6)$$

where, for the  $k$ -PCM

$$\mathcal{L}_\pm(z) = \frac{z}{z \pm 1} \cdot \frac{2}{(1 + \lambda)} A_\pm, \quad (2.7)$$

while for the  $k$ -SSSM

$$\mathcal{L}_\pm(z) = A_\pm^{(0)} + z^{\pm 1} \lambda^{-1/2} A_\pm^{(1)}. \quad (2.8)$$

### 2.2. Quantization and S-matrices

For integrable theories, one way to quantize the theories directly is to intuit a form for the exact  $S$ -matrix by using a combination of the global symmetries, the standard  $S$ -matrix axioms along with factorization and the Yang–Baxter equation that result from integrability. Integrability at the quantum level is thought to follow if there are higher spin conserved, possibly non-local, charges whose conservation is non-anomalous in the quantum theory. The integrability of the PCM is known to be non-anomalous for any group  $F$ , while the absence of

<sup>7</sup> Proved originally for the  $k$ -PCM case in [17] and more generally and in a simpler way in [18].

an anomaly for the  $F/G$  SSSM is not guaranteed in general. For instance, for compact symmetric spaces of definite signature it requires that the subgroup  $G$  is simple [31]<sup>8</sup>. The quantum integrable SSSMs are then associated to either Type II symmetric spaces of the form  $G \times G/G$ , which provide an equivalent way to formulate the PCM for the group  $G$ , or Type I symmetric spaces of the form

$$\begin{aligned} & \text{SU}(N)/\text{SO}(N), & \text{SU}(2N)/\text{Sp}(N), \\ & \text{SU}(2)/\text{U}(1) \equiv \mathbb{C}P^1, & \text{SO}(N)/\text{SO}(N-1) \equiv S^{N-1}, \end{aligned} \quad (2.9)$$

plus a number of possibilities involving the exceptional groups that we shall not consider here.

We make the hypothesis that the  $k$ -deformation of these theories are also integrable at the quantum level. One way to motivate this is to find a suitable ansatz for their  $S$ -matrices which can then be subject to various consistency tests. Note that there are effectively two different kinds of  $k$ -deformations of the PCM because they can also be formulated as a SSSM.

The first issue to consider is how the couplings  $k$  and  $\lambda$  flow in the quantum theory. The integer level  $k$ , of course, cannot have non-trivial RG flow because it multiplies the topological Wess–Zumino term in the action. However, the coupling  $\lambda$  does have a non-trivial RG flow. The loop counting parameter is  $1/k$  and to the one-loop level the beta function is<sup>9</sup>

$$\begin{aligned} k\text{-PCM:} \quad & \mu \frac{d\lambda}{d\mu} = - \frac{2c_2(F)}{k} \left( \frac{\lambda}{1+\lambda} \right)^2, \\ k\text{-SSSM:} \quad & \mu \frac{d\lambda}{d\mu} = - \frac{c_2(F)}{k} \lambda. \end{aligned} \quad (2.10)$$

So in both cases, the coupling  $\lambda$  runs to zero in the UV. In this limit, by integrating out the auxiliary field  $A_\mu$ , for the PCM case, and  $A_\mu^{(1)}$ , for the SSSM case, the theories take the form of a WZW model deformed by a current–current operator. For the PCM this is an ordinary (i.e. non-gauged) WZW model

$$S_\lambda = S_{\text{WZW}}[\mathcal{F}] + \frac{4\pi}{k\lambda} \int d^2x \text{Tr}(\hat{\mathcal{J}}_+ \hat{\mathcal{J}}_-) + \dots, \quad (2.11)$$

where  $\hat{\mathcal{J}}_\pm$  are the standard Kac–Moody currents of the WZW model:

$$\hat{\mathcal{J}}_+ = -\frac{k}{2\pi} \mathcal{F}^{-1} \partial_+ \mathcal{F}, \quad \hat{\mathcal{J}}_- = \frac{k}{2\pi} \partial_- \mathcal{F} \mathcal{F}^{-1}. \quad (2.12)$$

In this case the deformation is marginally relevant, as is also clear from the form of the beta function (2.10).

For the SSSM case, the UV theory is a deformation of the gauged WZW for  $F/G$ ,

$$S_\lambda = S_{\text{gWZW}}[\mathcal{F}, A_\mu^{(0)}] + \frac{4\pi}{k\lambda} \int d^2x \text{Tr}(\hat{\mathcal{J}}_+^{(1)} \hat{\mathcal{J}}_-^{(1)}) + \dots, \quad (2.13)$$

<sup>8</sup> Or requires the addition of supersymmetry.

<sup>9</sup> For the  $k$ -PCM case the beta function can be extracted from the general approaches in [32–34]. For the  $k$ -SSSM case, the beta function has been calculated in [51].

where  $\hat{\mathcal{J}}_{\pm}$  are the Kac–Moody currents of the  $F/G$  gauged WZW model

$$\begin{aligned}\hat{\mathcal{J}}_+ &= -\frac{k}{2\pi}(\mathcal{F}^{-1}\partial_+\mathcal{F} + \mathcal{F}^{-1}A_+^{(0)}\mathcal{F} - A_-^{(0)}), \\ \hat{\mathcal{J}}_- &= \frac{k}{2\pi}(\partial_-\mathcal{F}\mathcal{F}^{-1} - \mathcal{F}A_-^{(0)}\mathcal{F}^{-1} + A_+^{(0)}).\end{aligned}\quad (2.14)$$

Notice that the deformation now is relevant. In the case of a Type II symmetric space, it is worth pointing out that the WZW model involves the product group  $F = G \times G$  and so there are two independent levels  $k_1$  and  $k_2$ .

The integrable deformations of WZW models, non-gauged in (2.11) and gauged in (2.13) have been studied from a CFT perspective in [35].

There is one example of the  $k$ -deformed sigma models that has been studied in some detail and will provide the inspiration for the other cases. This corresponds to the PCM case with  $F = \text{SU}(2)$  [36]. To start with, the PCM models associated to the classical groups have a factorized  $S$ -matrix that can be written schematically in terms of 2 particle to 2-particle matrix elements of the form [37, 38]

$$S^{ab}(\theta) = X^{ab}(\theta)\tilde{S}_{(\infty)}^{ab}(\theta) \otimes \tilde{S}_{(\infty)}^{ab}(\theta), \quad (2.15)$$

where  $\theta = \theta_1 - \theta_2$  is the rapidity difference of the incoming states. The  $X^{ab}(\theta)$  are scalar factors, known also as dressing phases, that are needed to satisfy the  $S$ -matrix constraints of unitarity and crossing symmetry. The states of the theory are labelled by  $a$  and  $b$  which range over  $1, 2, \dots, r$ , the rank of  $F$ . They transform in a product of representations of  $F$ ,  $W_a \otimes W_a$ , where  $W_a$  is the  $a$ th fundamental, or antisymmetric representation, for unitary and symplectic groups, but are not irreducible for the orthogonal cases<sup>10</sup>. The two factors reflect the  $F_L \times F_R$  symmetry of the PCM which act on the group field via the left and right action:  $f \rightarrow U_L f U_R^{-1}$ . In (2.15), the  $\tilde{S}_{(\infty)}^{ab}$  are intertwiners

$$\tilde{S}_{(\infty)}^{ab} : W_a(\theta_1) \otimes W_b(\theta_2) \rightarrow W_b(\theta_2) \otimes W_a(\theta_1). \quad (2.16)$$

These intertwiners are the rational solutions of the Yang–Baxter equation. The significance of the  $\infty$  label will emerge. The  $S$ -matrix is covariant under the  $F_L \times F_R$  symmetry of the theory, but this symmetry extends to an affine Yangian symmetry. The full spectrum of the theory can then be obtained by using the bootstrap equations. The bound states are associated to the antisymmetric representations of  $F$  although for the  $\text{SO}(N)$  groups these representations are sometimes reducible.

For  $F = \text{SU}(2)$ , in [36] it was conjectured a form for the  $S$ -matrix of the  $k$ -deformed theory of the form

$$S(\theta) = X_{(k,\infty)}(\theta)\tilde{S}_{(k)}(\theta) \otimes \tilde{S}_{(\infty)}(\theta), \quad (2.17)$$

where the  $\text{SU}(2)_L$  factor, the one that is gauged in order to construct the deformed theory, has changed from the rational  $\text{SU}(2)$  factor to the quantum group generalization with a quantum group deformation parameter that is a root of unity:  $q^{2(k+2)} = 1$ . It is also important that the quantum group factor  $\tilde{S}_{(k)}(\theta)$  is taken in the so-called ‘interaction-round-a-face’ (IRF) picture, or ‘restricted-solid-on-solid’ (RSOS) picture. The significance of this is that the quantum numbers in this sector are naturally interpreted as kinks. So states of the theory carry a kink charge  $\pm 1$  and also a vector in the two-dimensional representation of  $\text{SU}(2)$ . The detailed form of the  $S$ -matrix is written in [36]. In the present context, the fact that the particle states

<sup>10</sup> If  $V_a$  are the fundamental representations, then  $W_a = \oplus_{j=0}^{a-2j \geq 0} V_{a-2j}$ .



are kinks can be understood as arising from the non-trivial spatial boundary conditions of the WZW models<sup>11</sup>.

Based on the understanding of the  $SU(2)$   $k$ -PCM model above, it is natural to conjecture that the  $k$ -PCM for a generic group  $F$  have a  $S$ -matrix of the schematic form

$$S^{ab}(\theta) = X_{(k,\infty)}^{ab}(\theta) \tilde{S}_{(k)}^{ab}(\theta) \otimes \tilde{S}_{(\infty)}^{ab}(\theta), \quad (2.18)$$

with  $q^{2(k+c_2(F))} = 1$ , where  $c_2(F)$  is the dual Coxeter number of  $F$ . As in the  $SU(2)$  case, the quantum group intertwiner  $\tilde{S}_{(k)}^{ab}(\theta)$  is defined in the IRF or RSOS representation. These  $S$ -matrices in general were first written down in [35] and discussed in detail in [40]. Note that the scalar factors  $X_{(k,\infty)}^{ab}$  here, and below, depend implicitly on the actual group  $F$ . Using TBA techniques, this conjecture has been checked for  $SU(2)$  in [36] and, more recently, for  $SU(N)$  in [51].

The intertwiners  $\tilde{S}_{(k)}^{ab}(\theta)$  are invariant under the action of an affine quantum group  $U_q(\hat{\mathfrak{f}})$  in the following sense. There is an associated co-product  $\Delta$  which describes how the generators act on a tensor product:  $\Delta(u)W_a(\theta_1) \otimes W_b(\theta_2)$ ,  $u \in U_q(\hat{\mathfrak{f}})$ . The theory is invariant under the affine quantum symmetry in the sense that

$$\Delta(u)S^{ab}(\theta_{12}) = S^{ab}(\theta_{12})\Delta(u), \quad u \in U_q(\hat{\mathfrak{f}}). \quad (2.19)$$

So the  $S$ -matrix elements lie in the commutant of  $U_q(\hat{\mathfrak{f}})$  acting on a tensor product representation. In the limit,  $k \rightarrow \infty$ , the quantum group reduces to the Yangian symmetry of the undeformed PCM.

Now we turn to the SSSM. For the Type II cases, there is an obvious form that the  $S$ -matrix should take. One simply generalizes (2.18) to

$$S^{ab}(\theta) = X_{(k_1,k_2)}^{ab}(\theta) \tilde{S}_{(k_1)}^{ab}(\theta) \otimes \tilde{S}_{(k_2)}^{ab}(\theta), \quad (2.20)$$

which means that the symmetry involves a tensor product of quantum groups  $U_{q_1}(\hat{\mathfrak{f}}) \otimes U_{q_2}(\hat{\mathfrak{f}})$ .

For the Type I theories, a bespoke construction is needed in each case based on the  $S$ -matrices that were constructed for the undeformed theories [41]. However, the form of the undeformed  $S$ -matrix leads to a natural conjecture in each case.

$S^{N-1}$  (including  $\mathbb{CP}^1 \equiv S^2$ ): for this series the  $S$ -matrix has the form originally found by Zamolodchikov and Zamolodchikov [42]

$$S(\theta) = X_{(\infty)}(\theta) \tilde{S}_{(\infty)}(\theta)|_{SO(N)}, \quad (2.21)$$

with particles transforming on the  $N$ -dimensional defining representation of  $F = SO(N)$ , which we show explicitly. We conjecture that the  $k$ -deformed theory has an  $S$ -matrix of the form

$$S(\theta) = X_{(k)}(\theta) \tilde{S}_{(k)}(\theta)|_{SO(N)}, \quad (2.22)$$

with  $q^{2(N-2+k)} = 1$ .

$SU(N)/SO(N)$ : for this series the  $S$ -matrix has the form

$$S(\theta) = X_{(\infty)}(\theta) \tilde{S}_{(\infty)}(\theta)|_{SU(N)} \otimes_s \tilde{S}_{(\infty)}(\theta)|_{SU(N)}, \quad (2.23)$$

where  $\otimes_s$  indicates the symmetric product and the particles transform in the symmetric  $N(N+1)/2$ -dimensional representation of  $F = SU(N)$  that acts on both factors. In this case, there are no bound states. There is an analogous set of anti-particles transforming in the conjugate representation [38]. We conjecture that the  $k$ -deformed theory has an  $S$ -matrix of the form

<sup>11</sup> Discussed, for example, in [39].

$$S(\theta) = X_{(k)}(\theta) \tilde{S}_{(k)}(\theta)|_{\text{SU}(N)} \otimes_s \tilde{S}_{(k)}(\theta)|_{\text{SU}(N)}, \quad (2.24)$$

with  $q^{2(N+k)} = 1$ .

$\text{SU}(2N)/\text{Sp}(N)$ : for this series the  $S$ -matrix has the form

$$S^{ab}(\theta) = X_{(\infty)}^{ab}(\theta) \tilde{S}_{(\infty)}(\theta)|_{\text{SU}(2N)} \otimes_a \tilde{S}_{(\infty)}(\theta)|_{\text{SU}(2N)}, \quad (2.25)$$

where  $\otimes_a$  indicates the antisymmetric product. For each  $a$ , the particles transforms in a product of fundamental representations  $V_a \otimes V_a$ , one for each factor. Then the  $\otimes_a$  indicates that these are projected onto the anti-symmetric term of the  $\text{SU}(2N)$  which acts diagonally on both factors. We conjecture that the  $k$ -deformed theory has an  $S$ -matrix of the form

$$S^{ab}(\theta) = X_{(k)}^{ab}(\theta) \tilde{S}_{(k)}(\theta)|_{\text{SU}(2N)} \otimes_a \tilde{S}_{(k)}(\theta)|_{\text{SU}(2N)}, \quad (2.26)$$

with  $q^{2(2N+k)} = 1$ .

### 3. Poisson–Lie group symmetry

In this section, we will look for the classical footprints of a quantum group which appear in the form of global Poisson–Lie group symmetries of the Lagrangian. We will show that the conserved Poisson–Lie charges are given by the monodromies of two (mutually commuting) Kac–Moody currents.

#### 3.1. Non-local symmetries of the $\lambda$ -deformed actions

The action (2.2) is invariant under the discrete  $\mathbb{Z}_2$ -transformation<sup>12</sup>

$$\Pi(\mathcal{F}, A_+, A_-, \lambda, k) = (\mathcal{F}^{-1}, \text{Ad}_{\mathcal{F}^{-1}}(A_+ + \partial_+ \mathcal{F}^{-1}), \Omega A_-, \lambda^{-1}, -k) \quad (3.1)$$

which, in particular, allows one to extend the range of the parameter  $\lambda$  from  $[0, 1]$  to  $[0, \infty)$ . Then, after integrating out the fields  $A_{\pm}$ , the resulting effective action turns out to be invariant under

$$\Pi(\mathcal{F}, \lambda, k) = (\mathcal{F}^{-1}, \lambda^{-1}, -k) \quad (3.2)$$

as pointed out originally in [33], where it was shown that it severely restricts the form of the RG flow equation for  $\lambda$ . This last symmetry will also play an important role in the following.

We now show that the theories have global Poisson–Lie group symmetries. To do this, the natural starting point is to consider left and right infinitesimal  $F$ -transformations separately. Namely,

$$\begin{aligned} \delta_L \mathcal{F} &= \epsilon_L \mathcal{F}, & \delta A_+ &= [\epsilon_L, A_+] - \partial_+ \epsilon_L, & \delta A_- &= 0, \\ \delta_R \mathcal{F} &= \mathcal{F} \epsilon_R, & \delta A_+ &= 0, & \delta A_- &= [\epsilon_R, A_-] - \partial_- \epsilon_R \end{aligned} \quad (3.3)$$

with  $\epsilon_L, \epsilon_R \in \mathfrak{f}$ . We find

$$\begin{aligned} \delta_L S &= -\frac{k}{\pi} \int d^2x \text{Tr}(\epsilon_L [\partial_+ + A_+, \partial_- + \Omega A_-]), \\ \delta_R S &= -\frac{k}{\pi} \int d^2x \text{Tr}(\epsilon_R [\partial_+ + \Omega A_+, \partial_- + A_-]). \end{aligned} \quad (3.4)$$

In the  $k$ -PCM case, the vector action of the group corresponds to taking  $\epsilon_R = -\epsilon$ ,  $\epsilon_L = \epsilon$ , and then adding both variations. This leads to

<sup>12</sup> Since the action is also invariant under  $(\mathcal{F}, A_{\pm}, x) \rightarrow (\mathcal{F}^{-1}, A_{\mp}, -x)$ , this transformation is equivalent to the one written down in [28].

$$\delta_V S = -\frac{k(1-\lambda)}{\pi\lambda} \int d^2x \operatorname{Tr} (\epsilon (\partial_+ A_- + \partial_- A_+)) = -\frac{k(1-\lambda)}{2\pi\lambda} \int d^2x \operatorname{Tr} (\epsilon \partial_\mu A^\mu). \quad (3.5)$$

Then, for global transformations,  $\delta_V S = 0$  provided that the fields  $A_\pm$  vanish at infinity. This shows that the action is invariant off-shell under global vector transformations, as expected, and provides conserved Noether charges. As a consequence, the target space geometry of the deformed PCM is invariant under the vector action of the group  $F$ . In the  $k$ -SSSM case, we get instead

$$\delta_V S = -\frac{k(1-\lambda)}{\pi\lambda} \int d^2x \operatorname{Tr} \{ \epsilon ([\partial_+ + A_+^{(0)}, A_-^{(1)}] + [\partial_- + A_-^{(0)}, A_+^{(1)}]) \}, \quad (3.6)$$

which vanishes off-shell for  $\epsilon$  in the Lie algebra of  $G$ . This exhibits the invariance of the action under  $G$  gauge transformations, but also that the  $k$ -SSSM has no other obvious off-shell (Noether) symmetries. However, since these theories are classically integrable, this is not the end of the story.

Going back to the separate transformations, (3.4), we note that both can be put in the form

$$\delta_{\text{LP}} S \sim \int \epsilon^a \left\{ dJ^a + \frac{1}{2} \tilde{f}^a_{bc} J^b \wedge J^c \right\}, \quad (3.7)$$

for some current  $J$  and some structure constants  $\tilde{f}^a_{bc}$ . On shell, each current is flat and so there exists conserved quantities  $\Gamma$ , the non-abelian moments, such that  $J = \Gamma^{-1} d\Gamma$ . It is well-known that the existence of conserved non-local charges such as these have played an important role proving the integrability of sigma models and finding their exact  $S$ -matrices; for example in [38].

To be more explicit, let us recall that the invariance of the action under generic field transformations provides the equations-of-motion which, in our case, can always be written in the Lax form (2.6). Then, the monodromy matrix<sup>13</sup>

$$m(z) = \text{Pexp} \left[ - \int_{S^1} dx \mathcal{L}(x, z) \right], \quad \mathcal{L} = \mathcal{L}_+ - \mathcal{L}_-, \quad (3.8)$$

is conserved for generic values of  $z$ , which provides an infinite number of, generally non-local, conserved quantities. Remarkably, the left and right variations (3.4) can be written in terms of the Lax pair evaluated at specific values of the spectral parameter  $z$ :

$$\begin{aligned} \delta_L S &= -\frac{k}{\pi} \int d^2x \operatorname{Tr} (\epsilon_L [\partial_+ + \mathcal{L}_+(z_-), \partial_- + \mathcal{L}_-(z_-)]), \\ \delta_R S &= -\frac{k}{\pi} \int d^2x \operatorname{Tr} (\epsilon_R [\partial_+ + \mathcal{L}_+(z_+), \partial_- + \mathcal{L}_-(z_+)]), \end{aligned} \quad (3.9)$$

where in the  $k$ -PCM case

$$z_\pm = \pm \frac{\lambda + 1}{\lambda - 1}, \quad (3.10)$$

and in the  $k$ -SSSM case

$$z_+ = \lambda^{-1/2}, \quad z_- = \lambda^{1/2}. \quad (3.11)$$

<sup>13</sup> We assume here, that the field theories are defined on a spatial circle.

In both, the significance of  $z_{\pm}$  is that

$$\mathcal{L}_+(z_+) = \Omega A_+, \quad \mathcal{L}_-(z_+) = A_-, \quad \mathcal{L}_+(z_-) = A_+, \quad \mathcal{L}_-(z_-) = \Omega A_-. \quad (3.12)$$

The equations (3.9) relate the conservation of  $m(z_+)$  and  $m(z_-)$  to the invariance of the action under left and right transformations, respectively, in the following sense.  $\delta_{L/R} S = 0$  implies that  $\mathcal{L}_{\pm}(z_{\mp})$  is flat, which provides the conserved quantities  $m(z_{\mp})$ .

Notice that the special points  $z_{\pm}$  are uniquely determined by the deformation parameter  $\lambda$  and, furthermore, they are mapped into each other under the discrete transformation (3.2). Below, we will see that these points are precisely the poles of the so-called twisting functions that underlies the Poisson structure of these theories.

The Hamiltonian structure of the  $k$ -PCM and the  $k$ -SSSM can be written in terms of the canonical currents [18]

$$\begin{aligned} \mathcal{J}_+ &= -\frac{k}{2\pi}(\mathcal{F}^{-1}\partial_+ \mathcal{F} + \mathcal{F}^{-1}A_+ \mathcal{F} - A_-), \\ \mathcal{J}_- &= \frac{k}{2\pi}(\partial_- \mathcal{F} \mathcal{F}^{-1} - \mathcal{F} A_- \mathcal{F}^{-1} + A_+), \end{aligned} \quad (3.13)$$

which obey the classical Kac–Moody algebra<sup>14</sup>

$$\{\mathcal{J}_{\pm}^a(x), \mathcal{J}_{\pm}^b(y)\} = f^{abc} \mathcal{J}_{\pm}^c(x) \delta_{xy} \mp \frac{k}{2\pi} \delta^{ab} \delta'_{xy}, \quad \{\mathcal{J}_+^a(x), \mathcal{J}_-^b(y)\} = 0. \quad (3.14)$$

On-shell, we can write these currents as

$$\mathcal{J}_+ = -\frac{k}{2\pi}(\Omega A_+ - A_-), \quad \mathcal{J}_- = -\frac{k}{2\pi}(\Omega A_- - A_+). \quad (3.15)$$

The key observation for the following analysis is that the Kac–Moody currents are related to the Lax connection in a very simple way; namely,

$$\mathcal{L}(z_+) = -\frac{2\pi}{k} \mathcal{J}_+, \quad \mathcal{L}(z_-) = \frac{2\pi}{k} \mathcal{J}_-, \quad (3.16)$$

where  $\mathcal{L} = \mathcal{L}_+ - \mathcal{L}_-$  is the spatial component of the Lax connection. This simple observation will allow us to deduce below that the  $k$ -deformed field theories have a quantum group symmetry. Then, from (3.16) we find that the conserved quantities associated to the global  $F_L \times F_R$  Poisson–Lie group action in both models take a rather universal form

$$m(z_+) = \text{Pexp} \left[ \frac{2\pi}{k} \int_{S^1} dx \mathcal{J}_+(x) \right], \quad m(z_-) = \text{Pexp} \left[ -\frac{2\pi}{k} \int_{S^1} dx \mathcal{J}_-(x) \right]. \quad (3.17)$$

Applying (3.2) to either (3.13) or (3.15) we see that the effect of the discrete transformation is to exchange the two Kac–Moody currents

$$\Pi \mathcal{J}_{\pm} = \mathcal{J}_{\mp}. \quad (3.18)$$

Then,  $m(z_{\pm})$  are mapped into each other under (3.2), so that one of the Poisson–Lie symmetries, say  $F_R$ , can be generated by a combination of  $F_L$  and  $\Pi$ . In section 4 below we turn to the quantum version of the algebra generated by  $m(z_{\pm})$ .

### 3.2. The $v/s$ Maillet forms and twisting functions

In this section, we exploit the Kac–Moody algebraic structure (3.14) of the  $k$ -deformed theories to compute the Poisson brackets of the spatial component of the Lax connection. In

<sup>14</sup> Here, we define  $\delta_{xy} \equiv \delta(x - y)$  and follow the conventions of [18].

particular, we are interested in extracting the twisting function  $\varphi_\epsilon(z)$  induced by the  $\mathfrak{t}/\mathfrak{s}$  form and study its pole structure. We begin with the  $k$ -PCM theories, and then consider the  $k$ -SSSM theories, which require a slightly different treatment.

*k-PCM case:*

We start by writing the spatial component of the  $k$ -PCM Lax connection for generic values of the spectral parameter in the form

$$\mathcal{L}(z) = f_-(z)\mathcal{J}_+ + f_+(z)\mathcal{J}_-, \quad (3.19)$$

with

$$f_\pm(z) = -\frac{z}{1-z^2}(p \pm qz), \quad p = \frac{4\pi\lambda}{k(1-\lambda^2)}, \quad q = \frac{4\pi\lambda}{k(1+\lambda^2)}. \quad (3.20)$$

Then, the Kac–Moody algebra (3.14) provides the Maillet  $\mathfrak{t}/\mathfrak{s}$  representation of the Poisson bracket

$$\begin{aligned} \{\overset{1}{\mathcal{L}}(x, z), \overset{2}{\mathcal{L}}(y, w)\} &= [\overset{1}{\mathfrak{t}}, \overset{1}{\mathcal{L}}(x, z) + \overset{2}{\mathcal{L}}(y, w)]\delta_{xy} \\ &\quad - [\overset{1}{\mathfrak{s}}, \overset{1}{\mathcal{L}}(x, z) - \overset{2}{\mathcal{L}}(y, w)]\delta_{xy} - 2\mathfrak{s}\delta'_{xy}. \end{aligned} \quad (3.21)$$

In this kind of representation, the equation is to be understood as acting on some tensor product  $V \otimes V$ , where  $V$  is some faithful representation of the Lie algebra  $\mathfrak{g}$ , and the index on top show which of the factors the quantity acts on, e.g.  $\overset{1}{\mathcal{L}}(V \otimes V) = (\mathcal{L}V) \otimes V$  and  $\overset{2}{\mathcal{L}}(V \otimes V) = V \otimes (\mathcal{L}V)$ , etc. Quantities like  $\mathfrak{s}$  and  $\mathfrak{t}$  defined below act on both factors.

In (3.21), the  $\mathfrak{t}/\mathfrak{s}$  matrices are given by

$$\mathfrak{t}(z, w) = \left[ \frac{w\varphi_\epsilon(z)^{-1} + z\varphi_\epsilon(w)^{-1}}{z - w} \right] \mathfrak{C}, \quad \mathfrak{s}(z, w) = \left[ \frac{w\varphi_\epsilon(z)^{-1} - z\varphi_\epsilon(w)^{-1}}{z - w} \right] \mathfrak{C}, \quad (3.22)$$

where

$$\varphi_\epsilon(z) = 2p \cdot \frac{1}{z} \cdot \frac{1 - z^2}{q^2 z^2 - p^2} \quad (3.23)$$

is the so-called twisting function and  $\mathfrak{C} = \sum_a T^a \otimes T^a$  is the tensor Casimir. Notice that  $\varphi_\epsilon(z)$  has two poles located precisely at  $z_\pm$

$$z_\pm = \mp \frac{p}{q} = \pm \frac{\lambda + 1}{\lambda - 1}. \quad (3.24)$$

These poles are exactly the evaluation points displayed in (3.10) used to extract the Lie–Poisson group charges out of the monodromy matrix. The extra pole at  $z = 0$  has no physical significance and can be removed by working with  $z \rightarrow 1/z$  instead. In this case, the poles are inverted  $z_\pm \rightarrow z_\pm^{-1}$  and the (3.22) can be rewritten in a more suggestive way

$$\mathfrak{t}(z, w) = -\left[ \frac{\phi_\epsilon(z)^{-1} + \phi_\epsilon(w)^{-1}}{z - w} \right] \mathfrak{C}, \quad \mathfrak{s}(z, w) = -\left[ \frac{\phi_\epsilon(z)^{-1} - \phi_\epsilon(w)^{-1}}{z - w} \right] \mathfrak{C}, \quad (3.25)$$

where<sup>15</sup>

$$\phi_\epsilon(z)^{-1} = -\frac{k}{2\pi}pq[\phi(z)_{\text{PCM}}^{-1} + \epsilon^2\phi(z)_{\text{FR}}^{-1}] \quad (3.26)$$

and

$$\phi(z)_{\text{PCM}} = \frac{1}{z^2} - 1, \quad \phi(z)_{\text{FR}} = 1, \quad \epsilon^2 = -\frac{1}{4\lambda}(1 - \lambda)^2, \quad (3.27)$$

are the twisting functions of the undeformed PCM and the FR model, and  $\epsilon$  is the deformation parameter in the  $\epsilon^2 < 0$  branch.

The  $\mathfrak{r}/\mathfrak{s}$  matrices satisfy a modified classical Yang–Baxter equation, mCYBE, a fact which was proven explicitly in [44]. Notice that in the FR limit  $\lambda \rightarrow 0$ , we have

$$\mathfrak{s}(z, w) = 0 \quad (3.28)$$

and the non-ultra-locality is completely removed. This is the celebrated ultra-localization of the PCM introduced by hand in [1]. Consequently, the  $k$ -PCM can be thought as the Lagrangian implementation of the FR ultra-localization mechanism.

The values of the  $s$  matrix at the poles are given by

$$\mathfrak{s}(z_\pm, z_\pm) = \pm \frac{\pi}{k}, \quad \mathfrak{s}(z_\pm, z_\mp) = 0. \quad (3.29)$$

Using these values in the Maillet bracket (3.21), we recover the Kac–Moody current algebra (3.14) in tensorial form

$$\{\mathcal{J}_\pm(x), \mathcal{J}_\pm(y)\} = \frac{1}{2}[\mathfrak{C}, \mathcal{J}_\pm(x) - \mathcal{J}_\pm(y)]\delta_{xy} \mp \frac{k}{2\pi}\mathfrak{C}\delta'_{xy}. \quad (3.30)$$

The values of  $\mathfrak{r}(z, w)$  at the same poles depends on how we take the limit but are always proportional to the tensor Casimir and hence do not contribute to the Poisson bracket. However, the values at different poles vanish identically.

We end this section, by rewriting (3.21) in a slightly different way which will be useful when we consider the  $k$ -F/G theories. By defining

$$\mathfrak{r}(z, w) = \frac{1}{2}(\mathfrak{R}(z, w) - \mathfrak{R}'(z, w)), \quad \mathfrak{s}(z, w) = -\frac{1}{2}(\mathfrak{R}(z, w) + \mathfrak{R}'(z, w)), \quad (3.31)$$

we can write the Maillet bracket as

$$\{\mathcal{L}^1(x, z), \mathcal{L}^2(y, w)\} = [\mathfrak{R}, \mathcal{L}^1(x, z)]\delta_{xy} - [\mathfrak{R}', \mathcal{L}^2(y, w)]\delta_{xy} + (\mathfrak{R} + \mathfrak{R}')\delta'_{xy}, \quad (3.32)$$

where

$$\mathfrak{R}(z, w) = -\frac{2\phi_\epsilon(w)^{-1}}{z - w}\mathfrak{C}. \quad (3.33)$$

*k*-SSSM case:

Inspired by (3.26), this time we take the following inverse twisting function

$$\varphi_\epsilon(z)^{-1} = c[\varphi(z)_\sigma^{-1} + \epsilon^2\varphi(z)_{\text{FR}}^{-1}], \quad (3.34)$$

<sup>15</sup> In this computation we use  $\epsilon^2 = \pi q/kp^2$ , and  $1 - \epsilon^2 = -\pi/kq$ .

where

$$\varphi(z)_\sigma = \frac{4z^2}{(1-z^2)^2}, \quad \varphi(z)_{\text{FR}} = 1, \quad \epsilon^2 = -\frac{1}{4\lambda}(1-\lambda)^2 \quad (3.35)$$

and  $c$  is a constant yet to be fixed by the evaluation of  $\mathfrak{s}(z, w)$  at the poles of  $\varphi_\epsilon(z)$ .

The deformed twisting function is

$${}^c\varphi_\epsilon(z) = \frac{4}{z^2 + 2(2\epsilon^2 - 1) + z^{-2}} \quad (3.36)$$

which has four poles given by

$$z_+^2 = \lambda^{-1}, \quad z_-^2 = \lambda. \quad (3.37)$$

The two poles corresponding to the positive branch of the square root are the evaluation points (3.11) used to extract the Lie–Poisson group charges out of the monodromy matrix<sup>16</sup>. These poles are important for establishing the quantum group symmetry of the theory, as we shall see below.

The calculation we need to find  $\mathfrak{r}$  and  $\mathfrak{s}$  was already done in [43] for the branch  $0 \leq \epsilon^2 < 1$  and also holds in the present case as the compatibility of the Poisson brackets is independent of the sign of  $\epsilon^2$ . The  $R$ -matrix corresponding to the Poisson bracket in the form (3.32) is<sup>17</sup>

$$\mathfrak{R}(z, w) = -\frac{2}{z^2 - w^2} \left[ \sum_{j=0,1} z^j w^{2-j} \mathfrak{C}^{(j,2-j)} \right] \varphi_\epsilon(w)^{-1}, \quad (3.38)$$

from which follows the  $\mathfrak{r}/\mathfrak{s}$  matrices in our present case

$$\begin{aligned} \mathfrak{r}(z, w) &= -\frac{1}{z^2 - w^2} \sum_{j=0,1} [w^j z^{2-j} \varphi_\epsilon(z)^{-1} + z^j w^{2-j} \varphi_\epsilon(w)^{-1}] \mathfrak{C}^{(j,2-j)}, \\ \mathfrak{s}(z, w) &= -\frac{1}{z^2 - w^2} \sum_{j=0,1} [w^j z^{2-j} \varphi_\epsilon(z)^{-1} - z^j w^{2-j} \varphi_\epsilon(w)^{-1}] \mathfrak{C}^{(j,2-j)}. \end{aligned} \quad (3.39)$$

However, it is important to recall that the expressions for  $\mathfrak{r}/\mathfrak{s}$  written right above follows by using the following modified Lax operator

$$\mathcal{L}(z) = f_-(z) \left( \mathcal{J}_+^{(0)} + \frac{\lambda^{-1/2}}{z} \mathcal{J}_+^{(1)} \right) + f_+(z) \left( \mathcal{J}_-^{(0)} + \frac{\lambda^{1/2}}{z} \mathcal{J}_-^{(1)} \right), \quad (3.40)$$

where

$$f_\pm(z) \equiv -\frac{2\pi}{k} \frac{\lambda}{(1-\lambda^2)} (z^2 - \lambda^{\mp 1}). \quad (3.41)$$

It corresponds to the extension of the Lax operator out of the constraint surface by the only first class constraint present for this case

$$\mathcal{J}_+^{(0)} + \mathcal{J}_-^{(0)}. \quad (3.42)$$

See [18] for details on the Hamiltonian approach. The Maillet bracket then follows as a simple consequence of the underlying Kac–Moody algebra structure.

<sup>16</sup> It would be interesting to find an interpretation for the negative branch poles and their relation with the monodromy matrix.

<sup>17</sup> In the following, we define  $\mathfrak{C}^{(i,j)} = \sum_{T^a \in \mathfrak{t}^{(i)}} \sum_{T^b \in \mathfrak{t}^{(j)}} \eta_{ab} T^a \otimes T^b$  for  $i, j \in \mathbb{Z}_2$ . Only  $\mathfrak{C}^{(0,0)}$  and  $\mathfrak{C}^{(1,1)}$  are non-vanishing and  $\mathfrak{C} = \mathfrak{C}^{(0,0)} + \mathfrak{C}^{(1,1)}$ .

Concentrating on the  $\mathfrak{s}(z, w)$  matrix, after some simplification we find it can be put in the form

$$\mathfrak{s}(z, w) = -\frac{c}{4}[z^2 + w^2 + 2(2\epsilon^2 - 1)]\mathfrak{C}^{(00)} + \frac{c}{4zw}[1 - z^2w^2]\mathfrak{C}^{(11)}. \quad (3.43)$$

Written in this way we can easily evaluate this function at the poles

$$\mathfrak{s}(z_{\pm}, z_{\pm}) = \mp \frac{c}{4\lambda}(1 - \lambda^2)\mathfrak{C}, \quad \mathfrak{s}(z_{\pm}, z_{\mp}) = 0. \quad (3.44)$$

The normalization of the central term in the Kac–Moody algebra imposes the condition

$$-2\mathfrak{s}(z_{\pm}, z_{\pm}) = \mp \frac{2\pi}{k}\mathfrak{C}, \quad (3.45)$$

which fixes the so far unspecified constant  $c$  to be

$$c = -\frac{4\pi\lambda}{k(1 - \lambda^2)}. \quad (3.46)$$

This value for  $c$  can be verified independently by direct use of the extended Lax operator (3.40). In the limit  $\lambda \rightarrow 0$ , we have

$$\mathfrak{s}(z, w) = -\frac{\pi}{k}\mathfrak{C}^{(00)}, \quad (3.47)$$

from which follows that in the coset case the non-ultra-locality can be only partially tamed [2]. As in the  $k$ -PCM case, from the Maillet bracket we recover the Kac–Moody algebra in tensor form.

### 3.3. Expansion of the monodromy around the poles of the twisting function

We work in the de-compatified limit, in which the monodromy matrix takes the general form<sup>18</sup>

$$m = \text{Pexp}\left[-\int_a^b dx \mathcal{L}(x)\right] = \exp\left[\sum_{n=1}^{\infty} (-1)^n F_n\right], \quad (3.48)$$

where for the first three terms we get

$$\begin{aligned} F_1 &= \int_a^b dx \mathcal{L}(x), \\ F_2 &= \frac{1}{4} \int_a^b dx dy [\mathcal{L}(x), \mathcal{L}(y)] \epsilon_{xy}, \\ F_3 &= \frac{1}{12} \int_a^b dx dy dz [\mathcal{L}(x), [\mathcal{L}(y), \mathcal{L}(z)]] \epsilon_{xy} \epsilon_{yz}. \end{aligned}$$

The limit  $a$  and  $b$  will subsequently be taken to  $\pm\infty$ , respectively. Expanding  $\mathcal{L}(x)$  around a chosen point in the complex plane and reorganizing the series in powers of the spectral parameter  $z$ , not to be confused with the integration variable  $z$ , we find the corresponding non-local charges associated to that particular point.

Let us focus first on the  $k$ -PCM theory. When expanded around the poles  $z_{\pm} = \mp p/q$ , we find the following expression for the spatial Lax operator

<sup>18</sup> Take  $g_s = i$  and  $A = \mathcal{L}$  in equation (20) of [48] and re-organize equation (22).



$$\mathcal{L}(x) = j_{(0)\pm}(x) + j_{(1)\pm}(x)\hat{z}_{\pm} + j_{(2)\pm}(x)\hat{z}_{\pm}^2 + j_{(3)\pm}(x)\hat{z}_{\pm}^3 + \mathcal{O}(\hat{z}_{\pm}^4), \quad (3.49)$$

where  $\hat{z}_{\pm} = z - z_{\pm}$  and

$$\begin{aligned} j_{(0)\pm} &= \pm \frac{1}{4\lambda}(1 + \lambda)[(1 - \lambda)I_0 \pm (1 + \lambda)I_1] = \mp \frac{2\pi}{k}\mathcal{J}_{\pm} = \mathcal{L}(z_{\pm}), \\ j_{(1)\pm} &= \frac{1}{8\lambda^2}(1 - \lambda)^2[(1 + \lambda^2)I_0 \pm (1 - \lambda^2)I_1], \\ j_{(2)\pm} &= \pm \frac{1}{16\lambda^3}(1 - \lambda)^3[(1 + \lambda^3)I_0 \pm (1 - \lambda^3)I_1], \\ j_{(3)\pm} &= \frac{1}{32\lambda^4}(1 - \lambda)^4[(1 + \lambda^4)I_0 \pm (1 - \lambda^4)I_1]. \end{aligned} \quad (3.50)$$

The expansion for the Lax operator around the poles (3.49) starts with terms of order  $\hat{z}_{\pm}^0$ . These terms can be removed by judicious gauge transformations,

$$m'(z) = u(b)m(z)u(a)^{-1}, \quad \mathcal{L}' = u(\mathcal{L} + \partial_x)u^{-1}, \quad (3.51)$$

where the gauge transformation takes the form

$$u(x, x_0, z_{\pm}) = \text{Pexp}\left[-\int_{x_0}^x dy \mathcal{L}(z_{\pm})\right] \equiv u(x)_{\pm} \quad (3.52)$$

around each pole, respectively. In this way, (3.48) can be written as follows

$$m(z) = u(b)_{\pm}\text{Pexp}\left[-\int_a^b dx \mathcal{L}'(x)_{\pm}\right]u(a)_{\pm}^{-1}, \quad (3.53)$$

with a transformed Lax operator

$$\mathcal{L}'(x)_{\pm} = j'_{(1)\pm}(x)\hat{z}_{\pm} + j'_{(2)\pm}(x)\hat{z}_{\pm}^2 + j'_{(3)\pm}(x)\hat{z}_{\pm}^3 + \mathcal{O}(\hat{z}_{\pm}^4) \quad (3.54)$$

defined in terms of the non-local dressed currents

$$j'_{(n)\pm}(x) = u(x)_{\pm}^{-1}j_{(n)\pm}(x)u(x)_{\pm}, \quad n \geq 1. \quad (3.55)$$

Taking  $x_0 = a = -\infty$  and  $b = \infty$ , the expansion of the monodromy matrix around the poles  $z_{\pm}$  now takes the form

$$m(z) = m(z_{\pm})\exp\left[\sum_{n=1}^{\infty} q_{(n)\pm}\hat{z}_{\pm}^n\right]. \quad (3.56)$$

As a result of the gauge transformation, all the charges  $q_{(n)\pm}$  are non-local and the first three take the explicit form

$$\begin{aligned} q_{(1)\pm} &= -\int_{-\infty}^{\infty} dx j'_{(1)\pm}(x), \\ q_{(2)\pm} &= \frac{1}{4}\int_{-\infty}^{\infty} dx dy [j'_{(1)\pm}(x), j'_{(1)\pm}(y)]\epsilon_{xy} - \int_{-\infty}^{\infty} dx j'_{(2)\pm}(x), \\ q_{(3)\pm} &= -\frac{1}{12}\int_{-\infty}^{\infty} dx dy dz [j'_{(1)\pm}(x), [j'_{(1)\pm}(y), j'_{(1)\pm}(z)]]\epsilon_{xy}\epsilon_{yz} \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} dx dy [j'_{(1)\pm}(x), j'_{(2)\pm}(y)]\epsilon_{xy} - \int_{-\infty}^{\infty} dx j'_{(3)\pm}(x). \end{aligned} \quad (3.57)$$

Notice that under  $\lambda \rightarrow \lambda^{-1}$ , which corresponds to the transformation (3.2), we do indeed have  $q_{(n)\pm} \rightarrow q_{(n)\mp}$  as claimed. This follows from our earlier observation that we can combine the  $F_L$  group action with (3.2) to generate  $F_R$  and vice versa. If we evaluate these charges at the,

for us, unphysical value of the coupling constant  $\lambda' = -1$ , then the two poles coalesce at  $z_{\pm} = 0$  and the expansion of the monodromy produces charges that coincide with the first three Yangian charges found in [44]

$$q_{(1)\pm}(\lambda') \equiv Q_{(0)}, \quad q_{(2)\pm}(\lambda') \equiv Q_{(1)}, \quad q_{(3)\pm}(\lambda') \equiv Q_{(2)}. \quad (3.58)$$

On the one hand, the dominant term for the expansion around  $z = 0$  is the local Noether charge  $Q_{(0)}$  corresponding to the global vector action of  $F$  and given by (3.5). The Poisson algebra generated by  $Q_{(0)}$  is isomorphic to  $\mathfrak{f}$  and the higher charges  $Q_{(n)}$ ,  $n > 0$  enhance this Poisson algebra to a Yangian, see [44].

On the other hand, the dominant terms for the expansions around  $z_{\pm} = \mp p/q$  are the non-abelian moments (3.17) associated to the global Poisson–Lie group action  $F_L \times F_R$ . They will be shown to generate the finite-dimensional quantum group  $U_q(\mathfrak{f})$  after quantization. It is then natural to expect that higher charges enhance  $U_q(\mathfrak{f})$  to an affine quantum symmetry to match the symmetries of the  $S$ -matrices that we have conjectured to describe these theories in section 2. However, explicit computations of the algebra of higher charges is complicated by the non-trivial dressing factors  $u(x)_{\pm}$ , hence we expect to report about this problem in the near future. Fortunately, some evidence that this enhancement may occur in our case comes from a different kind of deformation of the  $SU(2)$  PCM [45–47], in which a classical  $q$ -deformed Poisson algebra of symmetries induced by the squashing is studied for this particular case. What suggests this is that, technically, the situation in those references is essentially the same as ours, namely, the generators of the  $q$ -deformed symmetry group are extracted from the expansion of the monodromy matrix at the poles of the twisting function.

For the  $k$ -SSSM theories, a similar story holds. We have exactly the same expansion as given by (3.49) but with different expressions for the current components

$$\begin{aligned} j_{(0)\pm} &= \mp \frac{2\pi}{k} \mathcal{J}_{\pm} = \mathcal{L}(z_{\pm}), \\ j_{(1)\pm} &= -\frac{2\pi}{k} \frac{\lambda^{\mp 1/2}}{\lambda^{-1} - \lambda} [2(\mathcal{J}_{+}^{(0)} + \mathcal{J}_{-}^{(0)}) + (1 + \lambda^{\pm 2}) \mathcal{J}_{\pm}^{(1)} + 2\lambda^{\pm 1} \mathcal{J}_{\mp}^{(1)}], \\ j_{(2)\pm} &= -\frac{2\pi}{k} \frac{1}{\lambda^{-1} - \lambda} [\mathcal{J}_{+}^{(0)} + \mathcal{J}_{-}^{(0)} - \lambda^{\pm 1} (\lambda^{\pm 1} \mathcal{J}_{\pm}^{(1)} + \mathcal{J}_{\mp}^{(1)})], \\ j_{(3)\pm} &= -\frac{2\pi}{k} \frac{\lambda^{\pm 3/2}}{\lambda^{-1} - \lambda} [\lambda^{\pm 1} \mathcal{J}_{\pm}^{(1)} + \mathcal{J}_{\mp}^{(1)}], \end{aligned} \quad (3.59)$$

where  $\hat{z}_{\pm} = z - \lambda^{\mp 1/2}$ . An intuitive way to seek for Yangian symmetries is to find a value of  $\lambda$  for which  $j_{(0)\pm} = 0$ , see the first equation of (3.50) with  $\lambda = -1$  in the  $k$ -PCM case. Notice, however, that such a solution does not exist for the  $k$ -SSSM theories, hence we do not expect any Yangian symmetries in this case. This is in agreement with the absence of global Noether charges as noticed above around (3.6).

Then, the expansion of the monodromy matrix around the poles takes the rather universal form (3.56). The model dependence shows up in the higher charges  $q_{(n)\pm}$ ,  $n \geq 1$ . In the  $k$ -PCM they are all Lie-algebra valued while in the  $k$ - $F/G$  they are all coset valued except for the extension associated to the first class constraint.

We now proceed to quantize the monodromies  $m(z_{\pm})$  and find their quantum algebra.

#### 4. Quantum lattice current algebra

In order to quantize a integrable field theory, the technical difficulties in dealing with monodromy matrix are usual addressed by introducing a lattice regulator which does not

break the integrability of the theory. Unfortunately in the current setting, our theories have a non-ultra-local Poisson bracket and it is well known [19] that, even at the classical level, this precludes a direct approach to computing the Poisson bracket of the monodromy matrix for any value of the spectral parameter  $z$ .

In the quantum theory, this means that the lattice current algebra for these field theories for general values of the spectral parameter  $z$  is not even known to exist. Fortunately, we can make progress by working directly at the poles  $z = z_{\pm}$  of the twist function. Our approach in this section draws mainly on the formalism established in [20–22, 49]. Like in these references, our calculations will be restricted to  $\mathfrak{f} = \mathfrak{su}(n)$ <sup>19</sup>.

#### 4.1. The lattice current algebra at the poles

Let us recall that the equations of motion of the  $k$ -deformed theories can be re-cast in Lax form as the zero curvature condition (2.6). This form manifests an infinite-dimensional group of gauge transformations generated by the group elements  $\Gamma(x, z)$

$$\mathcal{L}'_{\pm}(z) = \Gamma(z) \mathcal{L}_{\pm}(z) \Gamma(z)^{-1} - \partial_{\pm} \Gamma(z) \Gamma(z)^{-1}. \quad (4.1)$$

From (3.16) we conclude that, when evaluated at the poles  $z_{\pm}$ , this group becomes the ordinary loop group  $g_{\pm}(x) = \Gamma(x, z_{\pm})$  of gauge transformations of the Kac–Moody currents

$$\mathcal{J}'_{\pm}(x) = g_{\pm}(x) \mathcal{J}_{\pm}(x) g_{\pm}(x)^{-1} \pm \frac{k}{2\pi} \partial_x g_{\pm}(x) g_{\pm}(x)^{-1}, \quad (4.2)$$

which are symmetries of the Kac–Moody algebra (3.30).

Before introducing a lattice regularization of the theory we can infer, via a semi-classical analysis, that the  $k$ -deformation parameter of the quantum group generated by  $m(z_{\pm})$  is indeed a root of unity. This information can be extracted from the Maillet representation (3.21) evaluated at the poles. Introducing the parallel transport

$$u(x, x_0) = \text{Pexp} \left[ - \int_{x_0}^x dy \mathcal{L}(y) \right], \quad (4.3)$$

which obeys

$$\partial_x u(x, x_0) = -\mathcal{L}(x) u(x, x_0), \quad \partial_{x_0} u(x, x_0) = u(x, x_0) \mathcal{L}(x_0), \quad (4.4)$$

and then following Maillet [19], we find in our case

$$\{u^1(x, x_0), u^2(y, y_0)\} = \int_{x_0}^x dz \int_{y_0}^y dz' u^1(x, z) u^2(y, z') \{ \mathcal{L}^1(z), \mathcal{L}^2(z') \} u^1(z, x_0) u^2(z', y_0). \quad (4.5)$$

In the above, for brevity, we have not indicated the spectral parameter dependence but we should keep in mind that  $u$  depend on it. In fact, we are only interested in the result above when it is evaluated at the poles  $z_{\pm}$  in which  $\mathcal{L}(z)$ , as in (3.16), picks out the Kac–Moody currents. In this case we find that

$$\{u^1(x, x_0), u^2(y, y_0)\} = s u^1(x, x_0) u^2(y, y_0) (\epsilon_{xy} + \epsilon_{x_0 y_0}) \mathfrak{C}, \quad (4.6)$$

<sup>19</sup> For the use of lattice Kac–Moody algebras to the deformed PCM of Rajeev [23] in the  $SU(2)$  case, see [24].

where  $s$  is given by

$$\mathfrak{s}(z_{\pm}, z_{\pm}) = s(z_{\pm})\mathfrak{C}, \quad s(z_{\pm}) = \pm \frac{\pi}{k}, \quad (4.7)$$

and  $\epsilon_{xy}$  is the sign function and in what follows we will drop the  $\pm$  indices for simplicity<sup>20</sup>.

The expression (4.6) is the classical exchange algebra we want to quantize semi-classically. We want to emphasize that at the moment we are proceeding in a rather ad hoc fashion in order to provide some inspiration for the lattice algebra defined below. By exploiting the classical commutativity of the  $u$ 's and using the quantization rule<sup>21</sup>  $-i\{*, *\} \rightarrow [*, *]$ , we get

$$\begin{aligned} u^1(x)u^2(y) &= u^2(y)u^1(x)(1 - is\mathfrak{C}\epsilon_{xy} + \dots) \\ &= u^2(y)u^1(x)\mathfrak{R}_{xy}, \end{aligned}$$

where the ellipsis denote sub-leading quantum contributions. From this we infer that, roughly speaking, we should expect something like

$$\mathfrak{R}_{xy} \sim q^{\mathfrak{C}\epsilon_{xy}}, \quad q \sim e^{-is}, \quad q^k \sim 1, \quad (4.8)$$

which is an interesting expression as it relates Maillet's  $s$  matrix, the poles and the  $k$ -deformation parameter. Notice that the effect of the transformation (3.2) is to map  $q \rightarrow q^{-1}$ .

The calculation above should not be considered as a formal one, we use it only as a source of inspiration for the type of quantum algebra we should expect. In particular, the Poisson brackets (4.6) do not obey the Jacobi identity. Indeed, we find

$$\text{Jacobi} = -s^2 u^1(x)u^2(y)u^3(z)[\epsilon_{xy}\epsilon_{yz} + \epsilon_{xz}\epsilon_{zy} + \epsilon_{zx}\epsilon_{xy}]f^{abc}T^a \otimes T^b \otimes T^c, \quad (4.9)$$

which never vanishes for distinct  $x, y, z$ .

After the naïve semi-classical approach, we are now ready to introduce the lattice algebra. The first thing is to discretize the spatial circle  $S^1$  in terms of  $N$  points with coordinates  $x_n = n\Delta$ ,  $n = 1, \dots, N \bmod N$  in the order  $x_1 < x_2 < \dots < x_{N-1} < x_N$ . Then, we define the transport matrix

$$t(x_{n+1}, x_n) = \text{Pexp}\left[-\int_{x_n}^{x_{n+1}} dx \mathcal{L}(x)\right] \quad (4.10)$$

between sites. Evaluating it at the poles  $z_{\pm}$  defines for us the lattice variables

$$J_n^L = \text{Pexp}\left[\frac{2\pi}{k} \int_{x_n}^{x_{n+1}} dx \mathcal{J}_+(x)\right], \quad J_n^R = \text{Pexp}\left[-\frac{2\pi}{k} \int_{x_n}^{x_{n+1}} dx \mathcal{J}_-(x)\right]. \quad (4.11)$$

Choosing a particular ordering, we can define two monodromy matrices on  $S^1$ ,

$$M^L = J_N^L \dots J_1^L, M^R = J_1^R \dots J_N^R. \quad (4.12)$$

However, in order to keep things simpler we will concentrate on the pole  $z_+$  only, drop the index  $L$  and set  $g_+(x) = g(x)$ .

The quantum lattice Kac–Moody algebra at the pole  $z_+$  is then defined by the following algebra

<sup>20</sup> In what follows we fix the ambiguity on the lower limits of integration by setting  $x_0 = y_0$ , which is the normalization condition reproduced in the classical continuum limit of the lattice algebra regularization to be considered below. We will drop explicit dependence on  $x_0$  from the notation.

<sup>21</sup> This kind of naïve replacement is subtle when group-like quantities have quadratic Poisson brackets but this subtlety is not relevant in the semi-classical limit.

$$J_{n+1}^1 \mathfrak{R}^2 J_n = J_n^2 J_{n+1}^1, \quad J_n^1 J_n^2 = \mathfrak{R}^+ J_n^2 J_n^1 \mathfrak{R}^-, \quad (4.13)$$

where  $\mathfrak{R}^+ = \mathfrak{R}^t$  (transpose) and  $\mathfrak{R}^- = \mathfrak{R}^{-1}$  (inverse).

Now, the classical continuum limit is achieved by writing

$$J_n \sim 1 + \frac{2\pi}{k} \Delta \mathcal{J}_+(x), \quad \mathfrak{R} \sim 1 + i\gamma \bar{\mathfrak{r}}, \quad (4.14)$$

where  $\mathfrak{R}$  and  $\bar{\mathfrak{r}}$  are solutions of the quantum/classical YBE such that

$$\bar{\mathfrak{r}} + \bar{\mathfrak{r}}^t = 2\mathfrak{C}, \quad (4.15)$$

where  $\Delta = 1/N \rightarrow 0$  is the lattice spacing and  $x = n/N$ . The first step is to find the classical limit of (4.13). We find

$$\frac{1}{\gamma} \{J_m^1, J_n^2\} = J_m^1 J_n^2 \bar{\mathfrak{r}} \delta_{mn} - \bar{\mathfrak{r}}^t J_m^1 J_n^2 \delta_{mn} - J_m^1 \bar{\mathfrak{r}} J_n^2 \delta_{m,n+1} + J_n^2 \bar{\mathfrak{r}}^t J_m^1 \delta_{m+1,n}. \quad (4.16)$$

The next step is to take the continuum limit. We find that  $\gamma = -s$  and then the Kac–Moody algebra follows. The algebra (4.13) underlies the quantization of the monodromy matrix  $m(z_+)$ .

Recall that classically we have

$$\mathcal{J}_+(x) = \frac{k}{2\pi} \partial_x u(x) u(x)^{-1}, \quad (4.17)$$

where  $u(x)$  is the ‘wave-function’

$$u(x) = \text{Pexp} \left[ \frac{2\pi}{k} \int_{x_0}^x dx \mathcal{J}_+(x) \right]. \quad (4.18)$$

The gauge transformations (4.2) acting on  $\mathcal{J}_+(x)$  are now equivalent to the action of the loop group  $u(x) \rightarrow g(x)u(x)$  on the  $u$ ’s. On the lattice, this classical expressions become<sup>22</sup>

$$u_n = J_n \cdots J_1, \quad u_n = J_n u_{n-1}, \quad u_n \rightarrow g_n u_n, \quad J_n \rightarrow g_n J_n g_{n-1}^{-1}. \quad (4.19)$$

The wave functions also obey the following quantum exchange algebra relations

$$\begin{aligned} u_m^1 u_n^2 &= u_n^2 u_m^1 \mathfrak{R}^+, \quad m > n, \\ u_m^1 u_n^2 &= u_n^2 u_m^1 \mathfrak{R}^-, \quad m < n, \\ u_n^1 u_n^2 &= \mathfrak{R}^+ u_n^2 u_n^1 \mathfrak{R}^-. \end{aligned} \quad (4.20)$$

The last relation means that the classical gauge loop group is promoted to a quantum gauge group with algebra [20]

$$g_n^1 g_n^2 \mathfrak{R}^+ = \mathfrak{R}^+ g_n^2 g_n^1, \quad g_m^1 g_n^2 = g_n^2 g_m^1, \quad m \neq n. \quad (4.21)$$

This quantum group leaves the Kac–Moody lattice algebra (4.13) invariant in the same way its classical counterpart leaves invariant the usual Kac–Moody algebra.

Writing  $\bar{\mathfrak{r}} = -\mathfrak{r} + \mathfrak{C}$ , with  $\mathfrak{r}^t = -\mathfrak{r}$  antisymmetric which explicitly obeys the constraint (4.15), we can find the corrected version of the Poisson bracket (4.6) for the wave-function  $u(x)$  (in the normalization  $x_0 = y_0$ ) directly from the classical/continuum limit of the first two equations of the exchange algebra (4.20), namely,

<sup>22</sup> For any value of  $n$ , the  $u_n$  all start with the term  $J_1$ , which is the normalization condition  $x_0 = y_0$  used before in the semi-classical analysis.

$$\{u^1(x), u^2(y)\} = su^1(x)u^2(y)[\mathfrak{r} + \mathfrak{C}\epsilon_{xy}].$$

This Poisson bracket do obey the Jacobi identity provided the antisymmetric  $\mathfrak{r}$ -matrix is a solution of the mCYBE of the split-type [29]. See the discussion around equation (5.9) of that paper. This decomposition of  $\bar{\mathfrak{r}}$  is consistent with the requirement that it is a solution of the CYBE.

The quantum monodromy matrix is given by  $M = u^N$  and satisfy the quadratic algebra

$$M^1 \mathfrak{R}^2 M^2 (\mathfrak{R}^-)^{-1} = \mathfrak{R}^+ M^2 (\mathfrak{R}^+)^{-1} M^1. \quad (4.22)$$

When we consider the following quantum factorization of the monodromy matrix

$$M = M_-^{-1} M_+, \quad (4.23)$$

the *RTT* quadratic relations given by its factorized components, namely,

$$\mathfrak{R}^+ M_\pm^1 M_\pm^2 = M_\pm^2 M_\pm^1 \mathfrak{R}^+, \quad \mathfrak{R}^+ M_-^1 M_+^2 = M_+^2 M_-^1 \mathfrak{R}^+, \quad (4.24)$$

are compatible with (4.22). Notice that (4.24) is the analogue of equation (13) of [21] in the case of  $sl(2)$ , which is another definition of the associated quantum group  $Fun_q(SL(2)^*)$ . The quantum monodromy matrix  $M(z_+)$  then becomes an element of  $F_q$ . The Poisson–Lie symmetry found in the classical  $k$ -deformed theory is then promoted to a quantum group symmetry.

The continuum limit of (4.20) is

$$u^1(x)u^2(y) = u^2(y)u^1(x)\mathfrak{R}_{xy}, \quad \mathfrak{R}_{xy} = \mathfrak{F} q^{\mathfrak{P}\epsilon_{xy}} \mathfrak{F}^{t-1}, \quad (4.25)$$

where  $\mathfrak{P}$  is the permutation operator and  $\mathfrak{F}$  an invertible matrix. This way of factorizing the usual  $R$  matrix comes from the theory of quasi-Hopf algebras introduced by Drinfeld<sup>23</sup>. It was shown in [21], that this exchange algebra underlies the quantization of the Kac–Moody algebra. Indeed, the quantum version of the classical currents (4.17), namely, the quantum operators

$$j(x) = \partial_x u(x) \cdot u(x)^{-1} \quad (4.26)$$

obey the following algebra

$$[j^1(x), j^2(y)] = -\ln q [\mathfrak{P}, j^1(x) - j^2(y)]\delta_{xy} - 2 \ln q (1 + a) \mathfrak{P}\delta'_{xy}, \quad (4.27)$$

where

$$a = -\frac{n \ln q}{i\pi} \quad (4.28)$$

and  $n$  is the size of the matrices used to represent the currents, e.g.,  $n$  for the fundamental representation of  $\mathfrak{sl}(n)$ , which corresponds to the present case. Taking the ansatz for the deforming parameter

$$q = e^{i\pi/(k+n)}, \quad (4.29)$$

and re-scaling the quantum currents (take  $k \rightarrow k + n$  in (4.17)) it was shown in [21] that the rescaled currents obey the quantum Kac–Moody algebra. Notice that in the semi-classical limit we recover our previous rough result with  $q = e^{-is(z_+)}$ . Using the duality between the Faddeev–Reshetikhin–Takhtajan ( $F_q$ ) and the Drinfeld–Jimbo ( $U_q(\mathfrak{g})$ ) realizations of quantum

<sup>23</sup> For further details we refer, for example, to section 2 of [50] in which the main properties of Drinfeld twists are reviewed.

groups, we infer that the deformed sigma models considered so far are  $q$ -deformed with deformation parameter being a root of unity.

A comment is in order. From the second equation in (3.31) and (4.15) one would naively conclude that

$$\mathfrak{R}(z_+, z_+) = \frac{\pi}{k} \bar{\mathfrak{r}}, \quad (4.30)$$

but the following observation shows this is not the case. The  $z \rightarrow z_+, w \rightarrow z_+$  limit is ambiguous as can be seen from

$$\mathfrak{R}(z, z_+) = 0, \quad \lim_{w \rightarrow z_+} \mathfrak{R}(z_+, w) \sim \mathfrak{C} \quad (4.31)$$

and even if we ignore this and work with the non-zero answer, this identification would imply that the classical limits of (4.22) and (4.24) are trivial, which is inconsistent. Then, the poles of the twisting function are to be treated as punctures in the complex plane  $z$  with a complete separate analysis. In this respect, the quantum  $\mathfrak{R}$ -matrix and its classical counterpart  $\bar{\mathfrak{r}}$  used in (4.13) are artifacts of our quantization scheme not related to the  $\mathfrak{r}/\mathfrak{s}$  operators in the Maillet bracket at all. In the classical/continuum limit, quantities should be independent of the quantization procedure but if, for instance, we consider the symmetry algebras (4.22) and (4.24) they would depend on  $\bar{\mathfrak{r}}$  in this limit, hence the quantized monodromy matrix should remain quantum. Notice that this does not apply to its associated KM algebra because it is independent of  $\bar{\mathfrak{r}}$  in the classical/continuum limit, as claimed in [21].

It is noteworthy that it is the central charge  $k$  of the Kac–Moody algebra instead of parameter  $\lambda$  that sets the deformation of the symmetry group. This is, perhaps, surprising as one might have expected  $q$  to depend on the deformation parameter  $\lambda$  as in the  $\eta$ -deformation [8]. Presumably the reason lies in the fact that the Kac–Moody currents (3.13) obey the Kac–Moody algebra no matter what the deformation parameter  $\lambda$  is and hence their monodromies obey a  $\lambda$ -independent algebra. In this respect, the  $k$ -deformations ( $-\infty < \epsilon^2 \leq 0$ ) are quite different in nature from the  $\eta$ -deformation ( $\epsilon^2 \in [0, 1)$ ). However, there is still the possibility that somehow higher spin charges appearing in the  $z$  expansion of the monodromy matrix around the poles, e.g. (3.56), introduce a  $\lambda$  dependence in a way yet to be understood.

## 5. Conclusions

In this paper, we have conjectured the form of the  $S$ -matrix theories that should describe the  $k$ -deformed principal chiral models and symmetric space sigma models. All of them exhibit affine quantum group symmetries with a deformation parameter  $q$  that is a root of unity. Then, making use of the results of [21], we have provided evidence that the quantum group symmetry appears in a direct quantization that preserves integrability in the form of a finite-dimensional quantum group symmetry, at least for the theories corresponding to  $\mathfrak{f} = \mathfrak{su}(n)$ .

However, there are some important details that remain to be understood. For instance, the proposed  $S$ -matrices exhibit a symmetry under the affine quantum group while it seems difficult to see how to extend the symmetry of the lattice quantization in this way. A proper study of the higher spin non-local charges that appear in the expansion of the monodromy matrix at the poles of the twisting function is certainly interesting from the quantum group theory point of view. In this regard, it would be important to understand the role of these charges in relation to the conjectured affine quantum group symmetry of the  $S$ -matrices for the deformed theories.

It is also not immediately clear how to generalize the approach of [21] to the cases of theories involving the other classical Lie groups. Even more ambitious, would be the generalization to semi-symmetric space sigma models involving Grassmann fields and Lie superalgebras that are needed to apply these ideas to the superstring. In any case, leaving out the technical details, it is worth recalling that the existence of the finite-dimensional quantum group symmetry follows simply from equation (3.16). Namely, from the observation that the spatial component of the Lax operator of these theories evaluated at specific values of the spectral parameter, which are poles of the  $\lambda$ -deformed twisting function, is a Kac–Moody current. Remarkably, the same is true for the  $k$ -deformation of the Green–Schwarz  $\text{AdS}_5 \times S^5$  sigma model (see equation (3.14) of [16]) and, thus, we expect that a similar construction will reveal a finite-dimensional quantum group symmetry with a deformation parameter  $q$  that is a root of unity also in this case.

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## References

- [1] Faddeev L D and Reshetikhin N Y 1986 Integrability Of the principal chiral field model in (1+1)-dimension *Ann. Phys.* **167** 227
- [2] Delduc F, Magro M and Vicedo B 2012 Alleviating the non-ultralocality of coset sigma models through a generalized Faddeev–Reshetikhin procedure *J. High Energy Phys.* **JHEP08(2012)019**
- [3] Delduc F, Magro M and Vicedo B 2012 Alleviating the non-ultralocality of the  $\text{AdS}_5 \times S^5$  superstring *J. High Energy Phys.* **JHEP10(2012)061**
- [4] Delduc F, Magro M and Vicedo B 2012 A lattice poisson algebra for the pohlmeyer reduction of the  $\text{AdS}_5 \times S^5$  superstring *Phys. Lett. B* **713** 347–9
- [5] Grigoriev M and Tseytlin A A 2008 Pohlmeyer reduction of  $\text{AdS}_5 \times S^5$  superstring sigma model *Nucl. Phys. B* **800** 450–501
- [6] Mikhailov A and Schafer-Nameki S 2008 Sine-Gordon-like action for the superstring in  $\text{AdS}_5 \times S^5$  *J. High Energy Phys.* **JHEP05(2008)075**
- [7] Miramontes J L 2008 Pohlmeyer reduction revisited *J. High Energy Phys.* **JHEP10(2008)087**
- [8] Delduc F, Magro M and Vicedo B 2013 On classical  $q$ -deformations of integrable sigma-models *J. High Energy Phys.* **JHEP11(2013)192**
- [9] Delduc F, Magro M and Vicedo B 2014 An integrable deformation of the  $\text{AdS}_5 \times S^5$  superstring action *Phys. Rev. Lett.* **112** 051601 5
- [10] Delduc F, Magro M and Vicedo B 2014 Derivation of the action and symmetries of the  $q$ -deformed  $\text{AdS}_5 \times S^5$  superstring *J. High Energy Phys.* **JHEP10(2014)132**
- [11] Klimcik C 2002 Yang–Baxter sigma models and dS/AdS  $T$  duality *J. High Energy Phys.* **JHEP12(2002)051**
- [12] Hoare B, Hollowood T J and Miramontes J L 2011 A relativistic relative of the magnon  $S$ -matrix *J. High Energy Phys.* **JHEP11(2011)048**
- [13] Hoare B, Hollowood T J and Miramontes J L 2012  $q$ -deformation of the  $\text{AdS}_5 \times S^5$  superstring  $S$ -matrix and its relativistic limit *J. High Energy Phys.* **JHEP03(2012)015**



- [14] Hoare B, Hollowood T J and Miramontes J L 2012 Bound states of the  $q$ -deformed  $\text{AdS}_5 \times S^5$  superstring  $S$ -matrix *J. High Energy Phys.* **JHEP10(2012)076**
- [15] Hoare B, Hollowood T J and Miramontes J L 2013 Restoring unitarity in the  $q$ -deformed world-sheet  $S$ -matrix *J. High Energy Phys.* **JHEP10(2013)050**
- [16] Hollowood T J, Miramontes J L and Schmidt M D 2014 An integrable deformation of the  $\text{AdS}_5 \times S^5$  superstring *J. Phys. A: Math. Theor.* **47** 495402
- [17] Sftos K 2014 Integrable interpolations: from exact CFTs To non-abelian  $T$ -duals *Nucl. Phys. B* **880** 225–46
- [18] Hollowood T J, Miramontes J L and Schmidt D M 2014 Integrable deformations of strings on symmetric spaces *J. High Energy Phys.* **JHEP11(2014)009**
- [19] Maillet J M 1986 New integrable canonical structures in two-dimensional models *Nucl. Phys. B* **269** 54
- [20] Alekseev A, Faddeev L D, Semenov-Tian-Shansky M and Volkov A 1991 The unraveling of the quantum group structure in the WZNW theory (unpublished)
- [21] Alekseev A, Faddeev L D and Semenov-Tian-Shansky M 1992 Hidden quantum groups inside Kac–Moody algebra *Commun. Math. Phys.* **149** 335–45
- [22] Falceto F and Gawedzki K 1993 Lattice Wess–Zumino–Witten model and quantum groups *J. Geom. Phys.* **11** 251–79
- [23] Rajeev S G 1989 Nonabelian bosonization without Wess–Zumino terms: I. New current algebra *Phys. Lett. B* **217** 123
- [24] Stern A and Vitale P 1997 Hidden quantum group symmetry in the chiral model *Nucl. Phys. B* **496** 571–89
- [25] Sftos K and Thompson D C 2014 Spacetimes for  $\lambda$ -deformations *J. High Energy Phys.* **JHEP12(2014)164**
- [26] Demulder S, Sftos K and Thompson D C 2015 Integrable  $\lambda$ -deformations: squashing coset CFTs and  $\text{AdS}_5 \times S^5$  *J. High Energy Phys.* **JHEP1507(2015)019**
- [27] Hoare B, Roiban R and Tseytlin A A 2014 On deformations of  $\text{AdS}_n \times S^n$  supercosets *J. High Energy Phys.* **JHEP06(2014)002**
- [28] Hoare B and Tseytlin A A 2015 On integrable deformations of superstring sigma models related to  $\text{AdS}_n \times S^n$  supercosets *Nucl. Phys. B* **897** 448–78
- [29] Vicedo B 2015 Deformed integrable  $\sigma$ -models, classical  $R$ -matrices and classical exchange algebra on Drinfel’d doubles *J. Phys. A: Math. Theor.* **48** 355203
- [30] Sftos K, Siampos K and Thompson D C 2015 Generalised integrable  $\lambda$ - and  $\eta$ -deformations and their relation *Nucl. Phys. B* **899** 489–512
- [31] Abdalla E, Forger M and Gomes M 1982 On the origin of anomalies in the quantum nonlocal charge for the generalized nonlinear  $\sigma$  models *Nucl. Phys. B* **210** 181
- [32] Sftos K and Siampos K 2014 Gauged WZW-type theories and the all-loop anisotropic non-abelian thirring model *Nucl. Phys. B* **885** 583–99
- [33] Itsios G, Sftos K and Siampos K 2014 The all-loop non-abelian thirring model and its RG flow *Phys. Lett. B* **733** 265
- [34] Tseytlin A A 1994 On a ‘Universal’ class of WZW type conformal models *Nucl. Phys. B* **418** 173
- [35] Ahn C, Bernard D and LeClair A 1990 Fractional supersymmetries in perturbed coset CFTs and integrable soliton theory *Nucl. Phys. B* **346** 409
- [36] Evans J M and Hollowood T J 1995 Integrable theories that are asymptotically CFT *Nucl. Phys. B* **438** 469
- [37] Abdalla E, Abdalla M C B and Lima-Santos A 1984 On the exact solution of the principal chiral model *Phys. Lett. B* **140** 71
- [38] Abdalla E, Abdalla M C B and Lima-Santos A 1984 *Phys. Lett. B* **146** 457
- [39] Abdalla M C B 1985 The exact  $S$  matrices of the principal chiral model with any nonexceptional symmetry group *Phys. Lett. B* **164** 71
- [40] Hollowood T J, Miramontes J L and Schmidt D M 2013 The structure of non-abelian kinks *J. High Energy Phys.* **JHEP10(2013)058**
- [41] Hollowood T J 1994 The analytic structure of trigonometric  $S$  matrices *Nucl. Phys. B* **414** 379
- [42] Abdalla E, Abdalla M C B and Forger M 1988 Exact  $S$  matrices for anomaly free nonlinear  $\sigma$  models on symmetric spaces *Nucl. Phys. B* **297** 374
- [43] Zamolodchikov A B and Zamolodchikov A B 1979 Factorized  $S$  matrices in two-dimensions as the exact solutions of certain relativistic quantum field models *Ann. Phys.* **120** 253

- [43] Vicedo B 2011 The classical  $R$ -matrix Of AdS/CFT and its lie dialgebra structure *Lett. Math. Phys.* **95** 249–74
- [44] Itsios G, Sfetsos K, Siampos K and Torrielli A 2014 The classical Yang–Baxter equation and the associated yangian symmetry of gauged WZW-Type theories *Nucl. Phys. B* **889** 64–86
- [45] Kawaguchi I and Yoshida K 2011 Hybrid classical integrability in squashed sigma models *Phys. Lett. B* **705** 251–4
- [46] Kawaguchi I, Matsumoto T and Yoshida K 2012 The classical origin of quantum affine algebra in squashed sigma models *J. High Energy Phys.* [JHEP04\(2012\)115](#)
- [47] Kawaguchi I, Matsumoto T and Yoshida K 2012 On the classical equivalence of monodromy matrices in squashed sigma model *J. High Energy Phys.* [JHEP06\(2012\)082](#)
- [48] Gardi E, Smillie J M and White C D 2013 The Non-abelian exponentiation theorem for multiple Wilson lines *J. High Energy Phys.* [JHEP06\(2013\)088](#)
- [49] Alekseev A Y, Faddeev L D, Frohlich J and Schomerus V 1998 Representation theory of lattice current algebras *Commun. Math. Phys.* **191** 31–60
- [50] Maillet J M and Sanchez de Santos J 2000 Drinfel’d twists and algebraic Bethe ansatz *L.D. Faddeev’s Seminar on Mathematical Physics (American Mathematical Society Translations: Series 2 vol 201)* (Providence, RI: American Mathematical Society) pp 137–78 (arXiv:[q-alg/9612012](#))
- [51] Appadu C and Hollowood T J 2015 Beta function of  $k$  deformed  $\text{AdS}_5 \times S^5$  string theory *J. High Energy Phys.* [JHEP11\(2015\)095](#)