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Miura and generalized Bäcklund transformation for KdV hierarchy

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Abstract

Using the fact that Miura transformation can be expressed in the form of gauge transformation connecting the KdV and m KdV equations, we discuss the derivation of the Bäcklund transformation and its Miura-gauge transformation connecting both hierarchies.

Keywords: integrable models, solitons, Bäcklund transformation

1. Introduction

Integrable models are characterized by an infinite number of conservation laws which are responsible for the stability of soliton solutions. In fact, these conservation laws may be regarded as hamiltonians generating time evolutions within a multi-time space. Each of these time evolutions, in turn, are associated to a nonlinear equation of motion and henceforth constitute an integrable hierarchy of equations with a common set of conservation laws. It has become clear in the past few years that integrable hierarchies may be systematically constructed from a graded algebraic structure involving affine Kac–Moody algebras. The equations of motion are constructed systematically from a graded algebra embedded within a zero curvature representation. The most well known example are the sine (sinh)-Gordon and m KdV equations, both underlined by the same $\hat{\mathfrak{sl}}(2)$ affine algebra in the principal gradation (see [1] for a review).

Another peculiar feature of integrable models is the existence of Bäcklund transformations which relate two different field configurations of certain nonlinear differential equation. These Bäcklund transformations, among other applications, generate an infinite sequence of soliton solutions from a nonlinear superposition principle (see [2] for review).

Bäcklund transformations have also been employed to describe integrable defects [3] in the sense that two solutions of an integrable model may be interpolated by a defect at certain

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spatial position. The Bäcklund transformation connecting the two field configurations is the key ingredient to preserve the integrability of the system. Under such formulation the canonical energy and momentum are no longer conserved and modifications to ensure its conservation are required in order to take into account the contribution of the defect [4]. Well known (relativistic) integrable models as the sine (sinh)-Gordon, Tzitzeica [3], Lund–Regge [5] and other (non-relativistic) models as nonlinear Schroedinger (NLS), m KdV, etc have been studied within such context [6].

In [7] we have constructed Bäcklund transformation to the m KdV hierarchy by assuming that two field configurations of the same equation of motion were related by a gauge transformation and dubbed from now on Bäcklund-gauge transformation. As a result it was subsequently shown that such Bäcklund-gauge transformation acted in a universal manner in all members of the hierarchy, i.e., the same for all models within the m KdV hierarchy.

In this paper we extend the results of [7] to the KdV hierarchy. We first consider the transformation from the principal to the homogeneous gradation by a global gauge transformation g_1 . Next, by a second, local gauge transformation $g_2(v, \epsilon)$, we realize the Miura transformation relating the fields v of the m KdV to those $J_\epsilon = \epsilon \partial_x v - v^2$, $\epsilon = \pm 1$ of the KdV hierarchies. In fact, the Miura transformation displays a sign ambiguity represented by the ϵ -factor, i.e., solutions of the m KdV hierarchy generate two towers of solutions of the KdV hierarchy labeled by the ϵ -sign.

In section 2 we review the algebraic construction of m KdV hierarchy and construct explicitly the Miura-gauge transformation that maps the equations of the m KdV into equations of the KdV hierarchy. We display the first few positive grade time evolution equations for both hierarchies. In section 3, from the Bäcklund-gauge transformation constructed in [7] we discuss its Miura extension and propose Bäcklund-gauge transformation for the KdV hierarchy. This, however displays a sign ambiguity inherited from the Miura transformation. Each solution, $v(x, t_N)$ of the m KdV hierarchy is mapped into two solutions, $J_\epsilon(x, t_N) = \epsilon \partial_x v - v^2$, $\epsilon = \pm 1$ of the KdV hierarchy. One of the main results of our construction is to show that a pair of Bäcklund solutions of the m KdV hierarchy, v_1, v_2 have to be mapped into a pair of KdV Bäcklund solutions of opposite ϵ -signs, $J_{1+} = \partial_x v_1 - v_1^2$, $J_{2-} = -\partial_x v_2 - v_2^2$ in order to satisfy the KdV Bäcklund equations.

In section 4 we discuss the composition of gauge-Bäcklund transformations for both hierarchies. In fact we start by considering two consecutive gauge-Bäcklund transformations for the m KdV hierarchy, K_1 and K_2 and we show that the product, $K^{II} = K_2 K_1$ generate the so called Type-II Bäcklund transformation proposed in [3]. We follow the same philosophy to construct the Type-II Bäcklund transformation for the KdV hierarchy. First by direct product composition of two Type-I, $\tilde{K}^{II} = \tilde{K}_2 \tilde{K}_1$ and the second by Miura-gauge transforming the Type-II Bäcklund transformation of the m KdV hierarchy. Consistency is shown that the two alternative constructions indeed agree.

2. The algebraic formalism for KdV and m KdV hierarchies

Following the algebraic formalism described in [7] we recall that the nonlinear equations of the m KdV hierarchy can be derived from the zero curvature representation,

$$[\partial_x + A_x, \partial_{t_N} + A_{t_N}] = 0. \quad (2.1)$$

Underlined by an affine $\hat{sl}(2)$ centerless Kac–Moody algebra generated by $h^{(m)} = \lambda^m h$, $E_{\pm\alpha}^{(m)} = \lambda^m E_{\pm\alpha}$, $\lambda \in C$, $m \in Z$ satisfying

$$[h^{(m)}, E_{\pm\alpha}^{(n)}] = \pm 2E_{\pm\alpha}^{(m+n)}, \quad [E_{\alpha}^{(m)}, E_{-\alpha}^{(n)}] = h^{(m+n)}. \quad (2.2)$$

Here $A_x = E^{(1)} + A_0$, $A_{t_N} = D^{(N)} + D^{(N-1)} + \dots + D^{(0)}$, $E^{(1)} = E_{\alpha} + \lambda E_{-\alpha}$ and² $A_0 = v(x, t_N)h$ are constructed according to the principal gradation, $Q_p = 2\lambda \frac{d}{d\lambda} + 1/2h$ which decomposes the affine $\hat{sl}(2) = \sum_i \mathcal{G}_i$ algebra into graded subspaces according to powers of the spectral parameter λ ,

$$[Q_p, \mathcal{G}_a] = a\mathcal{G}_a, \quad [\mathcal{G}_a, \mathcal{G}_b] \in \mathcal{G}_{a+b}, \quad a, b \in \mathbb{Z}, \quad (2.3)$$

where the subspaces $\mathcal{G}_{2m}$ and $\mathcal{G}_{2m+1}$ contains the following generators

$$\begin{aligned} \mathcal{G}_{2m} &= \{h^{(m)} = \lambda^m h\}, \\ \mathcal{G}_{2m+1} &= \{\lambda^m (E_{\alpha} + \lambda E_{-\alpha}), \lambda^m (E_{\alpha} - \lambda E_{-\alpha})\}. \end{aligned} \quad (2.4)$$

Here $D^{(i)} \in \mathcal{G}_i$. The zero curvature (2.1) decomposes according to the graded structure as

$$\begin{aligned} [E^{(1)}, D^{(N)}] &= 0 \\ [E^{(1)}, D^{(N-1)}] + [A_0, D^{(N)}] + \partial_x D^{(N)} &= 0 \\ &\vdots \\ [A_0, D^{(0)}] + \partial_x D^{(0)} - \partial_{t_N} A_0 &= 0, \end{aligned} \quad (2.5)$$

which allows solving for $D^{(i)}$, $i = 0, \dots, N$ and the last equation in (2.5) yields the time evolution for fields A_0 . We should point out that $D^{(i)}$ are constructed systematically for each value of N and so is A_{t_N} . The first few explicit equations are (see for instance [7, 9]),

$$4\partial_{t_3} v - \partial_x (\partial_x^2 v - 2v^3) = 0 \quad m\text{KdV}, \quad (2.6)$$

$$16\partial_{t_5} v - \partial_x (\partial_x^4 v - 10v^2 (\partial_x^2 v) - 10v (\partial_x v)^2 + 6v^5) = 0, \quad (2.7)$$

$$\begin{aligned} 64\partial_{t_7} v - \partial_x (\partial_x^6 v - 70(\partial_x v)^2 (\partial_x^2 v) - 42v (\partial_x^2 v)^2 - 56v (\partial_x v) (\partial_x^3 v)) \\ + \partial_x (14v^2 \partial_x^4 v - 140v^3 (\partial_x v)^2 - 70v^4 (\partial_x^2 v) + 20v^7) = 0 \\ \dots \text{etc.} \end{aligned} \quad (2.8)$$

Consider now the *global* gauge transformation generated by

$$g_1 = \begin{pmatrix} \zeta & 1 \\ \zeta & -1 \end{pmatrix}, \quad \zeta^2 = \lambda \quad (2.9)$$

which transforms $A_{x,m\text{KdV}}^{\text{princ}} = E^{(1)} + v(x, t_N)h = \begin{pmatrix} v & 1 \\ \lambda & -v \end{pmatrix}$, into

$$A_{x,m\text{KdV}}^{\text{hom}} = g_1 (A_{x,m\text{KdV}}^{\text{princ}}) g_1^{-1} = g_1 (E^{(1)} + v(x, t_N)h) g_1^{-1} = \begin{pmatrix} \zeta & v \\ v & -\zeta \end{pmatrix}, \quad (2.10)$$

i.e., transforms the principal into homogeneous gradation, $Q_{\text{hom}} = \zeta \frac{d}{d\zeta}$.

A subsequent *local* Miura-gauge transformation [10, 11]

$$g_2(v, \epsilon) = \begin{pmatrix} 1 & \epsilon \\ -\epsilon v & -v + 2\epsilon\zeta \end{pmatrix}, \quad (2.11)$$

² $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

transforms $A_{x,mKdV} \rightarrow A_{x,KdV}$. i.e.,

$$A_{x,KdV} = g_2(v, \epsilon) A_{x,mKdV}^{\text{hom}} g_2^{-1}(v, \epsilon) - \partial_x g_2(v, \epsilon) g_2^{-1}(v, \epsilon) = \begin{pmatrix} \zeta & -1 \\ J & -\zeta \end{pmatrix}. \quad (2.12)$$

And realizes the Miura transformation,

$$J = \epsilon \partial_x v - v^2, \quad \epsilon^2 = 1. \quad (2.13)$$

We should emphasize that for each solution $v(x, t_N)$ of the evolution equations for the $mKdV$ hierarchy, the Miura transformation (2.13) generates two towers of solutions, $J_\epsilon(x, t_N)$, $\epsilon = \pm 1$, of the KdV hierarchy [11]. The zero curvature under the homogeneous gradation³

$$[\partial_x + A_{x,KdV}, \partial_{t_N} + A_{t_N,KdV}] = 0, \quad (2.14)$$

with $A_{t_N,KdV} = \tilde{D}^{(N)} + \tilde{D}^{(N-1)} + \dots + \tilde{D}^{(0)}$, $\tilde{D}^{(j)} \in \tilde{\mathcal{G}}^j$ yields the KdV hierarchy equations of motion. For instance

$$A_{t_3,KdV} = \begin{bmatrix} \zeta^3 + \frac{1}{2}\zeta J + \frac{1}{4}\partial_x J & -\zeta^2 - \frac{1}{2}J \\ \zeta^2 J + \frac{1}{2}\zeta \partial_x J + \frac{1}{4}\partial_x^2 J + \frac{1}{2}J^2 & -\zeta^3 - \frac{1}{2}\zeta J - \frac{1}{4}\partial_x J \end{bmatrix} \quad (2.15)$$

yields the KdV equation

$$4\partial_{t_3} J - \partial_x^3 J - 6J\partial_x J = (\epsilon \partial_x - 2v)[4\partial_{t_3} v - \partial_x(\partial_x^2 v - 2v^3)] = 0. \quad (2.16)$$

Similarly from A_{t_5} and for A_{t_7} , given in the [appendix](#), we find the Sawada–Kotera equation [13]

$$16\partial_{t_5} J - \partial_x^5 J - 20\partial_x J \partial_x^2 J - 10J \partial_x^3 J - 30J^2 \partial_x J \\ = (\epsilon \partial_x - 2v)[16\partial_{t_5} v - \partial_x(\partial_x^4 v - 10v^2 \partial_x^2 v - 10v(\partial_x v)^2 + 6v^5)] = 0, \quad (2.17)$$

and⁴

$$64\partial_{t_7} J - \partial_x^7 J - 70\partial_x^2 J \partial_x^3 J - 42\partial_x J \partial_x^4 J - 70(\partial_x J)^3 - 14J \partial_x^5 J \\ - 280J \partial_x J \partial_x^2 J - 70J^2 \partial_x^3 J - 140J^3 \partial_x J \\ = (\epsilon \partial_x - 2v)(64\partial_{t_7} v - \partial_x(\partial_x^6 v - 70(\partial_x v)^2 \partial_x^2 v - 42v(\partial_x^2 v)^2 - 56v\partial_x v \partial_x^3 v \\ - 14v^2 \partial_x^4 v + 140v^3(\partial_x v)^2 + 70v^4 \partial_x^2 v - 20v^7)) = 0 \quad (2.18)$$

respectively. Equations (2.16)–(2.18) are displayed as explicit examples as illustration of the formalism. Higher flows (time evolutions) can be systematically constructed for generic N from the same formalism.

3. Bäcklund transformation

3.1. $mKdV$

In this section we start by noticing that the zero curvature representation (2.1) and (2.14) are invariant under gauge transformations of the type

³ Notice that under the homogeneous gradation the decomposition of the affine Lie algebra $\hat{sl}(2) = \sum_a \tilde{\mathcal{G}}_a$, $a \in \mathbb{Z}$, $\tilde{\mathcal{G}}_a = \{\zeta^a h, \zeta^a E_a, \zeta^a E_{-a}\}$.

⁴ In general, we find $KdV(J) = (\epsilon \partial_x - 2v)mKdV(v)$.

$$A_\mu(\phi, \partial_x \phi, \dots) \rightarrow \tilde{A}_\mu = K^{-1} A_\mu K + K^{-1} \partial_\mu K, \quad (3.1)$$

where A_μ stands for either A_{t_N} or A_x .

The key ingredient of this section is to consider two field configurations ϕ_1 and ϕ_2 embedded in $A_\mu(\phi_1)$ and $A_\mu(\phi_2)$ satisfying the zero curvature representation and assume that they are related by a Bäcklund-gauge transformation generated by $K(\phi_1, \phi_2)$ preserving the equations of motion (e.g. zero curvature (2.1) or (2.14)), i.e.,

$$K(\phi_1, \phi_2) A_\mu(\phi_1) = A_\mu(\phi_2) K(\phi_1, \phi_2) + \partial_\mu K(\phi_1, \phi_2). \quad (3.2)$$

If we now consider the Lax operator $L = \partial_x + A_x$ for m KdV case within the principal gradation,

$$A_{x,m\text{KdV}} = E^{(1)} + A_0 = \begin{bmatrix} \partial_x \phi(x, t_N) & 1 \\ \lambda & -\partial_x \phi(x, t_N) \end{bmatrix} \quad (3.3)$$

is common to all members of the hierarchy defined by (2.1). We find that

$$K(\phi_1, \phi_2) A_{x,m\text{KdV}}(\phi_1) = A_{x,m\text{KdV}}(\phi_2) K(\phi_1, \phi_2) + \partial_x K(\phi_1, \phi_2), \quad (3.4)$$

where the Bäcklund-gauge generator $K(\phi_1, \phi_2)$ is given by [7, 9]

$$K(\phi_1, \phi_2) = \begin{bmatrix} 1 & -\frac{\beta}{2\lambda} e^{-(\phi_1 + \phi_2)} \\ -\frac{\beta}{2} e^{(\phi_1 + \phi_2)} & 1 \end{bmatrix} \quad (3.5)$$

and β is the Bäcklund parameter. Equation (3.4) is satisfied provided

$$\partial_x(\phi_1 - \phi_2) = -\beta \sinh(\phi_1 + \phi_2). \quad (3.6)$$

For the sinh-Gordon (s-g) model, the equations of motion $\partial_t \partial_x \phi_a = 2 \sinh 2\phi_a$, $a = 1, 2$ are satisfied if we further introduce the time component of the Bäcklund transformation,

$$\partial_t(\phi_1 + \phi_2) = \frac{4}{\beta} \sinh(\phi_2 - \phi_1). \quad (3.7)$$

Equation (3.7) is compatible with (3.2) for $A_\mu = A_{t_N}$ with

$$A_{t,s-g} = \begin{bmatrix} 0 & \lambda^{-1} e^{-2\phi} \\ e^{2\phi} & 0 \end{bmatrix}. \quad (3.8)$$

For higher graded time evolutions the time component of the Bäcklund transformation can be derived from the appropriated time component of the two-dimensional gauge potential. Several explicit examples within the positive and negative graded m KdV sub-hierarchies were discussed in [7]. We now give a general argument that the Bäcklund Transformation derived from the gauge transformation (3.4) for arbitrary N provides equations compatible with the equation of motion. Consider the zero curvature representation for certain field configuration, namely ϕ_1 , i.e.,

$$[\partial_x + A_x(\phi_1), \partial_{t_N} + A_{t_N}(\phi_1)] = 0. \quad (3.9)$$

Under the gauge transformation,

$$\begin{aligned} & K(\phi_1, \phi_2) [\partial_x + A_x(\phi_1), \partial_{t_N} + A_{t_N}(\phi_1)] K(\phi_1, \phi_2)^{-1} \\ &= [K(\partial_x + A_x(\phi_1)) K^{-1}, K(\partial_{t_N} + A_{t_N}(\phi_1)) K^{-1}] \\ &= [\partial_x + A_x(\phi_2), \partial_{t_N} + A_{t_N}(\phi_2)] = 0, \end{aligned} \quad (3.10)$$

where the last equality comes from our assumption (3.4).

The gauge transformation of the first entry in the zero curvature representation implies the x -component of the Bäcklund transformation (3.6). Since the zero curvature (3.9) and (3.10) implies that both ϕ_1 and ϕ_2 satisfy the same equation of motion, the gauge transformation (3.4) for $A_\mu = A_{t_N}$ of the second entry in (3.10) generates the time component of BT which, by construction has to be consistent with the equations of motion with respect to time t_N .

3.2. KdV

In order to extend the same philosophy to the KdV hierarchy recall the fact that the two-dimensional gauge potential $A_{x,\text{KdV}}$ can be obtained by Miura-gauge transformation from the homogeneous $m\text{KdV}$ gauge potentials $A_{m\text{KdV}}^{\text{hom}}$ as in (2.12), i.e.,

$$A_{x,\text{KdV}}(J) = g_2(v, \epsilon) g_1(A_{x,m\text{KdV}}(v)) g_1^{-1} g_2^{-1}(v, \epsilon) - \partial_x g_2(v, \epsilon) g_2^{-1}(v, \epsilon), \quad (3.11)$$

where $v = \partial_x \phi(x, t_N)$. By assuming (3.2) for the KdV hierarchy, i.e.,

$$\tilde{K}(J_1, J_2) A_{x,\text{KdV}}(J_1) = A_{x,\text{KdV}}(J_2) \tilde{K}(J_1, J_2) + \partial_x \tilde{K}(J_1, J_2). \quad (3.12)$$

the Bäcklund-gauge transformation for the KdV hierarchy $\tilde{K}(J_1, J_2)$ constructed in terms of $K(\phi_1, \phi_2)$ can be written as

$$\tilde{K} = g_2(v_2, \epsilon_2) (g_1 K(\phi_1, \phi_2) g_1^{-1}) g_2(v_1, \epsilon_1)^{-1}. \quad (3.13)$$

At this stage we should recall that for each solution of the $m\text{KdV}$ hierarchy v , the Miura transformation (2.13) generates two solutions, $J_{\epsilon_i} = \epsilon_i \partial_x v_i - v_i^2$, $\epsilon = \pm 1$ satisfying the associated equation of motion of the KdV hierarchy. This is precisely why we assume ϵ_1 and ϵ_2 in equation (3.13) independent. In terms of $m\text{KdV}$ variables $v_i = \partial_x \phi_i$, $\tilde{K}$ is given for the particular case where $\epsilon_1 = -\epsilon_2 \equiv \epsilon$ and denote⁵ $\tilde{K}(J_1, J_2) = \tilde{K}(\epsilon_1 = -\epsilon_2 \equiv \epsilon)$,

$$\begin{aligned} \tilde{K}(J_1, J_2)_{11} &= 1 - \frac{v_1 \epsilon}{\zeta} - \frac{\beta}{4\zeta} (1 - \epsilon) e^{-p} - \frac{\beta}{4\zeta} (1 + \epsilon) e^p \\ \tilde{K}(J_1, J_2)_{12} &= -\frac{1}{\zeta} \\ \tilde{K}(J_1, J_2)_{21} &= \frac{\beta}{4\zeta} (-v_1(1 + \epsilon) + v_2(1 - \epsilon)) e^{-p} \\ &\quad + \frac{\beta}{4\zeta} (v_1(1 - \epsilon) - v_2(1 + \epsilon)) e^p - \frac{v_1 v_2}{\zeta} \\ \tilde{K}(J_1, J_2)_{22} &= -1 - \frac{v_2 \epsilon}{\zeta} - \frac{\beta}{4\zeta} (1 + \epsilon) e^{-p} - \frac{\beta}{4\zeta} (1 - \epsilon) e^p, \end{aligned} \quad (3.14)$$

where $p = \phi_1 + \phi_2$. Substituting in equation (3.12) we find the following equations:

- Matrix element 11:

$$\zeta^{-1} : J_1 - \epsilon \partial_x v_1 - \frac{1}{2} \beta v_1 (e^p - e^{-p}) + v_1 v_2 = 0. \quad (3.15)$$

⁵ Notice that $\tilde{K}$ is given in terms of $m\text{KdV}$ variables v_1, v_2 and we need to rewrite it in terms of KdV variables J_1, J_2 . This requires solving Riccati equation $v = v(J)$ (2.13).

- Matrix element 12:

$$\zeta^{-1} : v_1 - v_2 + \frac{\beta}{2}(e^p - e^{-p}) = 0. \quad (3.16)$$

- Matrix element 21:

$$\begin{aligned} \zeta^0 : J_1 + J_2 + \frac{\beta v_1}{2}(1 + \epsilon)e^{-p} - \frac{\beta v_1}{2}(1 - \epsilon)e^p \\ - \frac{\beta v_2}{2}(1 - \epsilon)e^{-p} + \frac{\beta v_2}{2}(1 + \epsilon)e^p + 2v_1v_2 = 0, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \zeta^{-1} : \epsilon(J_1v_2 - J_2v_1) - v_1\partial_x v_2 - v_2\partial_x v_1 \\ - \frac{\epsilon\beta}{2}v_1v_2(e^p - e^{-p}) = 0. \end{aligned} \quad (3.18)$$

- Matrix element 22:

$$\zeta^{-1} : J_2 - \epsilon\partial_x v_2 - \frac{\beta v_2}{2}(e^p - e^{-p}) = 0. \quad (3.19)$$

Using the mixed Miura transformation, i.e., $\epsilon_2 = -\epsilon_1 \equiv \epsilon$,

$$J_1 = \epsilon\partial_x v_1 - v_1^2, \quad J_2 = -\epsilon\partial_x v_2 - v_2^2 \quad (3.20)$$

together with the m KdV Bäcklund transformation (3.6)

$$v_1 - v_2 = -\frac{\beta}{2}(e^p - e^{-p}), \quad (3.21)$$

we find that equations (3.15), (3.16), (3.18) and (3.19) are identically satisfied. Defining the new variable Q and taking into account the Bäcklund equation (3.21) we find the following equality

$$\frac{1}{2}Q = \epsilon v_1 + \frac{\beta}{4}(1 - \epsilon)e^{-p} + \frac{\beta}{4}(1 + \epsilon)e^p \quad (3.22)$$

$$= \epsilon v_2 + \frac{\beta}{4}(1 + \epsilon)e^{-p} + \frac{\beta}{4}(1 - \epsilon)e^p, \quad (3.23)$$

Eliminating v_1 and v_2 from equations (3.22) and (3.23) we find

$$v_1 = \frac{\epsilon}{2}Q - \frac{\beta}{4}(\epsilon - 1)e^{-p} - \frac{\beta}{4}(\epsilon + 1)e^p, \quad (3.24)$$

$$v_2 = \frac{\epsilon}{2}Q - \frac{\beta}{4}(\epsilon + 1)e^{-p} - \frac{\beta}{4}(\epsilon - 1)e^p \quad (3.25)$$

and henceforth

$$\begin{aligned} \frac{\beta}{4}(-v_1(1 + \epsilon) + v_2(1 - \epsilon))e^{-p} \\ + \frac{\beta}{4}(v_1(1 - \epsilon) - v_2(1 + \epsilon))e^p - v_1v_2 = \frac{\beta^2}{4} - \frac{Q^2}{4}. \end{aligned} \quad (3.26)$$

Equation (3.17) then becomes

$$J_1 + J_2 = \frac{\beta^2}{2} - \frac{Q^2}{2}. \quad (3.27)$$

From (3.22) and (3.23) we find that

$$Q = \epsilon(v_1 + v_2) + \frac{\beta}{2}(e^p + e^{-p}). \quad (3.28)$$

Acting with ∂_x in (3.28) and using (3.20) and (3.21),

$$\begin{aligned} \partial_x Q &= \epsilon \partial_x(v_1 + v_2) + \frac{\beta}{2}(v_1 + v_2)(e^p - e^{-p}) \\ &= \epsilon \partial_x(v_1 + v_2) - (v_1 - v_2)(v_1 + v_2) \\ &= J_1 - J_2 \\ &= \partial_x(\omega_1 - \omega_2), \end{aligned} \quad (3.29)$$

where we have used $J_i \equiv \partial_x w_i$, $i = 1, 2$. It therefore follows that

$$Q = w_1 - w_2 \quad (3.30)$$

and the Bäcklund transformation for the spatial component of the KdV equation becomes,

$$J_1 + J_2 = \partial_x P = \frac{\beta^2}{2} - \frac{(w_1 - w_2)^2}{2}, \quad P = w_1 + w_2. \quad (3.31)$$

Which is in agreement with the Bäcklund transformation proposed in [12] and with [14].

In the new variable Q defined in (3.22) and (3.23) we rewrite the gauge-Bäcklund transformation $\tilde{K}(J_1, J_2)$ in (3.14) as

$$\tilde{K}(J_1, J_2, \beta) = -\frac{1}{\zeta} \begin{pmatrix} -\zeta + \frac{1}{2}Q & 1 \\ \frac{-\beta^2}{4} + \frac{1}{4}Q^2 & \zeta + \frac{1}{2}Q \end{pmatrix}. \quad (3.32)$$

Other cases with $\epsilon_1 = \epsilon_2 = \pm 1$ lead to trivial Bäcklund transformations in the sense that (3.12) for $\tilde{K}(\pm 1, \pm 1)$ is trivially satisfied for m KdV Bäcklund and Miura transformations (3.6) and (2.13). There is no new equation relating the two KdV fields J_1 and J_2 . From now on we shall only consider $\tilde{K}(+1, -1) \equiv \tilde{K}$ given in (3.32) and Miura transformation given by (3.20).

We now discuss the extension of the Bäcklund transformation to the time component of the KdV hierarchy. Notice that in the zero curvature representation the spatial component of the two-dimensional gauge potential A_x is the same for all flows and therefore universal among the different evolution equations. They differ from the time component A_{t_N} written according to the algebraic graded structure and parametrized by the integer N .

$$A_{t_N, \text{KdV}} = \tilde{\mathcal{D}}^{(N)} + \tilde{\mathcal{D}}^{(N-1)} + \dots + \tilde{\mathcal{D}}^{(0)}, \quad \tilde{\mathcal{D}}^{(j)} \in \tilde{\mathcal{G}}^j. \quad (3.33)$$

The Bäcklund-gauge transformation (3.32) acting on the potentials $A_{t_3, \text{KdV}}$, $A_{t_5, \text{KdV}}$ and $A_{t_7, \text{KdV}}$ given by equations (A.2)–(A.4) of the appendix leads to the following Bäcklund equations respectively

$$4\partial_{t_3} P = -Q\partial_x^2 Q + \frac{1}{2}((\partial_x Q)^2 + 3(\partial_x P)^2), \quad (3.34)$$

$$\begin{aligned} 16\partial_{t_5} P &= -Q\partial_x^4 Q + \partial_x Q\partial_x^3 Q + 5\partial_x P\partial_x^3 P \\ &+ \frac{1}{2}(5(\partial_x^2 P)^2 - (\partial_x^2 Q)^2) + \frac{5}{2}\partial_x P((\partial_x P)^2 + 3(\partial_x Q)^2), \end{aligned} \quad (3.35)$$

$$\begin{aligned}
64\partial_{t_7}P = & -Q\partial_x^6Q + \partial_xQ\partial_x^5Q + 7\partial_xP\partial_x^5P - \partial_x^2Q\partial_x^4Q + 14\partial_x^2P\partial_x^4P \\
& + 35\partial_xQ\partial_x^2Q\partial_x^2P + 35\partial_xP\partial_xQ\partial_x^3Q + \frac{21}{2}(\partial_x^3P)^2 + \frac{1}{2}(\partial_x^3Q)^2 \\
& + \frac{35}{2}\partial_xP((\partial_x^2P)^2 + (\partial_x^2Q)^2) + \frac{35}{2}\partial_x^3P((\partial_xP)^2 + (\partial_xQ)^2) \\
& + \frac{105}{4}(\partial_xP)^2(\partial_xQ)^2 + \frac{35}{8}(\partial_xP)^4 + \frac{35}{8}(\partial_xQ)^2,
\end{aligned} \tag{3.36}$$

where $\partial_P = J_1 + J_2$. Equations (3.31) and (3.34) coincide with the Bäcklund transformation proposed in [12] for the KdV equation. Equations (3.31) and (3.35) correspond to those derived for the Sawada–Kotera equation in [14]⁶. In the appendix we have checked the consistency between the spatial, (3.31) and time components (3.34)–(3.36) of the Bäcklund transformations for $N = 3, 5$ and 7 . By direct calculation, using software Mathematica, we indeed recover the evolution equations (2.16)–(2.18). We would like to point out that our method is systematic and provides the Bäcklund transformations for arbitrary time evolution in terms of its time component 2D gauge potential $A_{t_N, \text{KdV}}$ in terms of graded subspaces $\tilde{D}^{(i)}$, $i = 0, \dots, N$. The examples given above for t_3 , t_5 and t_7 just illustrate the potential of the formalism.

3.3. Examples

- Vacuum—one soliton solution Consider $\phi_1 = 0$ and $\phi_2 = \ln\left(\frac{1+\rho}{1-\rho}\right)$, $\rho = e^{2kx+2k^N t_N}$, $N = 3, 5, 7$ two solutions of the m KdV hierarchy. The mixed Miura transformation yields

$$J_+^1 = \partial_x^2\phi_1 - (\partial_x\phi_1)^2 = 0, \quad J_-^2 = -\partial_x^2\phi_2 - (\partial_x\phi_2)^2 \tag{3.37}$$

Integrating to obtain $J = \partial_x w$ we find

$$w_1 = 0, \quad w_2 = -\frac{4k}{1+\rho} + 2k \tag{3.38}$$

Type-I Bäcklund transformation $\partial_x(w_1 + w_2) = \frac{\beta^2}{2} - \frac{1}{2}(w_1 - w_2)^2$ is satisfied by (3.38) for $\beta = \pm 2k$

- Scattering of two one-soliton solutions
Consider the one-soliton of the m KdV hierarchy given by

$$\phi_i = \ln\left(\frac{1 + R_i\rho}{1 - R_i\rho}\right), \quad i = 1, 2 \quad \rho = e^{2kx+2k^N t_N}, \quad N = 3, 5, 7 \dots \tag{3.39}$$

Miura transformation generates two one-soliton solutions of the KdV hierarchy, namely

$$J_+^1 = \partial_x^2\phi_1 - (\partial_x\phi_1)^2; \tag{3.40}$$

$$J_-^2 = -\partial_x^2\phi_2 - (\partial_x\phi_2)^2; \tag{3.41}$$

leading to

$$w_1 = -\frac{4k}{1 + R_1\rho} + 2k, \quad w_2 = -\frac{4k}{1 - R_2\rho} + 2k. \tag{3.42}$$

The Type-I Bäcklund transformation is satisfied for $R_1 = R_2$. Notice that although

⁶ Notice that there are typos in equation (45.11) of [14].

$R_1 = R_2$, J_1 and J_2 correspond to different solutions due to opposite ϵ -sings in the Miura transformation.

- One-soliton into two-soliton solution

Taking ϕ_1 given by the one-soliton solution (3.39) and ϕ_2 by

$$\phi_2 = \ln \left(\frac{1 + \delta(\rho_1 - \rho_2) - \rho_1 \rho_2}{1 - \delta(\rho_1 - \rho_2) - \rho_1 \rho_2} \right), \quad \delta = \frac{k_1 + k_2}{k_1 - k_2} \quad (3.43)$$

leading to

$$w_2 = -\frac{2(k_1^2 - k_2^2)(1 + \rho_1)(1 + \rho_2)}{k_1 - k_2 - (k_1 + k_2)(\rho_1 - \rho_2) - (k_1 - k_2)\rho_1 \rho_2}, \quad (3.44)$$

where $\rho_i = e^{2k_i x + 2k_i^N t_N}$, $i = 1, 2$ satisfy the Type-I Bäcklund transformation for $\beta = \pm 2k_2$.

All these verifications were made in the software Mathematica.

4. Fusing and Type-II Bäcklund transformation

In this section we shall consider the composition of two gauge-Bäcklund transformations leading to the Type-II Bäcklund transformation. Let us consider a situation in which we start with a Bäcklund relation transforming solution v_1 into another solution v_0 . A second subsequent Bäcklund relation transforms v_0 into v_2 . Such algebraic relation for the m KdV hierarchy is described by

$$K^{\text{II}}(v_1, v_0, v_2) = K(v_2, v_0)K(v_0, v_1), \quad (4.1)$$

where $K(v_i, v_j)$ is given in (3.5) with $\beta = \beta_{ij}$. It leads to

$$K^{\text{II}}(v_1, v_0, v_2) = \begin{bmatrix} 1 + \frac{\beta_{10}\beta_{02}}{4\lambda}e^q & \frac{e^{-\phi_0}}{2\lambda}(\beta_{01}e^{-\phi_1} + \beta_{02}e^{-\phi_2}) \\ -\frac{1}{2}e^{\phi_0}(\beta_{01}e^{\phi_1} + \beta_{02}e^{\phi_2}) & 1 + \frac{\beta_{10}\beta_{02}}{4\lambda}e^{-q} \end{bmatrix}, \quad (4.2)$$

where $q = \phi_1 - \phi_2$ and $\sigma^2 = -\frac{4}{\beta_{10}\beta_{02}}$. Inserting the following identity

$$(\beta_{01}e^{\phi_1} + \beta_{02}e^{\phi_2})(\beta_{01}e^{-\phi_1} + \beta_{02}e^{-\phi_2}) = \beta_{01}\beta_{02}(\eta + e^q + e^{-q}), \quad (4.3)$$

where $\eta = \frac{\beta_{10}^2 + \beta_{02}^2}{\beta_{10}\beta_{02}}$. Defining $\Lambda = -\phi_0 - \ln(\beta_{02}e^{-\phi_1} + \beta_{01}e^{-\phi_2}) - \ln \frac{\sigma}{4}$, equation (4.2) becomes

$$K^{\text{II}}(v_1, v_0, v_2) = \begin{bmatrix} 1 - \frac{1}{\sigma^2\lambda}e^q & \frac{e^{\Lambda-p}}{2\lambda\sigma}(e^q + e^{-q} + \eta) \\ -\frac{2}{\sigma}e^{p-\Lambda} & 1 - \frac{1}{\lambda\sigma^2}e^{-q} \end{bmatrix}. \quad (4.4)$$

Equation (3.4) with $K^{\text{II}}(v_1, v_0, v_2)$ leads to the following Bäcklund equations

$$\partial_x q = -\frac{1}{2\sigma}e^{\Lambda-p}(e^q + e^{-q} + \eta) - \frac{2}{\sigma}e^{p-\Lambda}, \quad (4.5)$$

$$\partial_x \Lambda = \frac{1}{2\sigma}e^{\Lambda-p}(e^q - e^{-q}). \quad (4.6)$$

Equations (4.5) and (4.6) coincide with the x -component of the Type-II Bäcklund transformation proposed for the sine-gordon model in [3]. Considering now the time component of the 2D gauge potential for $t = t_3$, (i.e., for the m KdV equation),

$$A_{t_3} = \lambda E_\alpha + \lambda^2 E_{-\alpha} + v \lambda h + \frac{1}{2}(\partial_x v - v^2) E_\alpha - \frac{1}{2}(\partial_x v + v^2) \lambda E_{-\alpha} + \frac{1}{4}(\partial_x^2 v - 2v^3) h \quad (4.7)$$

we find from equation (3.2),

$$16\sigma^3 \partial_{t_3} q = e^{\Lambda-p}(e^q + e^{-q} + \eta)[2\sigma^2(\partial_x^2 p + \partial_x^2 q) + \sigma^2(\partial_x p + \partial_x q)^2 - 8e^q] + 4e^{p-\Lambda}[-2\sigma^2(\partial_x^2 p + \partial_x^2 q) + \sigma^2(\partial_x p + \partial_x q)^2 - 8e^{-q}] + 16\sigma \partial_x p(e^q + e^{-q} + \eta) \quad (4.8)$$

together with

$$4\sigma \partial_b \Lambda = (v_1^2 + \partial_x v_1) e^{\Lambda-q-p} - (v_2^2 + \partial_x v_2) e^{\Lambda+q-p}, \quad (4.9)$$

which is compatible with equations of motion for the m KdV model. These Type-II Bäcklund equations (4.5)–(4.9) coincide with those derived in detail in [7] where $x \rightarrow x_+$, $t \rightarrow x_-$ and was extended to all positive higher graded equation within the m KdV hierarchy⁷. In the case of the KdV hierarchy

$$\tilde{K}^{\text{typeII}}(J_1, J_0, J_2) = \tilde{K}(J_2, J_0, \beta_{02}) \tilde{K}(J_0, J_1, \beta_{01}), \quad (4.10)$$

where

$$\tilde{K}(J_j, J_i, \beta_{ij}) = -\frac{1}{\zeta} \begin{bmatrix} -\zeta + \frac{1}{2} Q_{ij} & 1 \\ -\frac{\beta_{ij}^2}{4} + \frac{1}{4} Q_{ij}^2 & \zeta + \frac{1}{2} Q_{ij} \end{bmatrix}.$$

Such transformation can be interpreted as an extended Bäcklund transformation dubbed Type-II Bäcklund transformation (see [3]). Explicitly we find directly from (4.10)

$$\begin{aligned} [\tilde{K}^{\text{II}}(J_1, J_0, J_2)]_{11} &= 1 - \frac{1}{2\zeta} Q - \frac{(\beta_+ + \beta_-)}{2\zeta^2} + \frac{Q}{8\zeta^2} (Q + P - 2\Omega) \\ [\tilde{K}^{\text{II}}(J_1, J_0, J_2)]_{12} &= \frac{1}{2\zeta^2} Q \\ [\tilde{K}^{\text{II}}(J_1, J_0, J_2)]_{22} &= 1 + \frac{1}{2\zeta} Q - \frac{(\beta_+ - \beta_-)}{2\zeta^2} + \frac{Q}{8\zeta^2} (Q - P + 2\Omega) \\ [\tilde{K}^{\text{II}}(J_1, J_0, J_2)]_{21} &= -\frac{\beta_+}{4\zeta^2} Q + \frac{\beta_-}{4\zeta^2} (P - 2\Omega) + \frac{Q}{8\zeta^2} \left(-\Omega^2 + \Omega P - \frac{P^2}{4} + \frac{Q^2}{4} \right) \\ &\quad - \frac{\beta_-}{\zeta} + \frac{Q}{4\zeta} (P - 2\Omega), \end{aligned} \quad (4.11)$$

where $Q = Q_{10} + Q_{02} = w_1 - w_2$, $P = Q_{10} - Q_{02} + 2\Omega = w_1 + w_2$, $\Omega = w_0$, $4\beta_\pm = \beta_{01}^2 \pm \beta_{02}^2$ and $Q_{ij} = w_i - w_j$.

Acting with $\tilde{K}^{\text{typeII}}(J_1, J_0, J_2)$ in (3.12) we find the Bäcklund transformation for the KdV equation, i.e.,

⁷ Observe that the Type-II Bäcklund transformation via gauge transformation was constructed in [8] where a solution presented there was chosen to reproduce the Bäcklund transformation proposed in [3]. Here we choose a gauge transformation solution of [8] that reproduces [9].

$$\begin{aligned}\partial_x Q &= 2\beta_- - \frac{1}{2}PQ + \Omega Q, \\ \partial_x(2\Omega + P) &= 2\beta_+ - \frac{1}{4}P^2 - \frac{1}{4}Q^2 - \Omega^2 + \Omega P.\end{aligned}\quad (4.12)$$

Similarly for the time component gauge potential (A.2) we find

$$\begin{aligned}\partial_{t_3} Q &= \frac{1}{2}\partial_x P \partial_x Q + \frac{1}{2}\partial_x \Omega \partial_x Q + \frac{1}{4}Q \partial_x^2 \Omega + \frac{1}{4}\Omega \partial_x^2 Q - \frac{P}{8}\partial_x^2 Q - \frac{Q}{8}\partial_x^2 P \\ \partial_{t_3}(2\Omega + P) &= \frac{1}{4}(\partial_x P)^2 + \frac{1}{4}(\partial_x Q)^2 + \frac{1}{2}\partial_x P \partial_x \Omega + (\partial_x \Omega)^2 - \frac{P}{8}\partial_x^2 P - \frac{Q}{8}\partial_x^2 Q \\ &\quad + \frac{1}{4}P \partial_x^2 \Omega + \frac{1}{4}\Omega \partial_x^2 P - \frac{1}{2}\Omega \partial_x^2 \Omega.\end{aligned}\quad (4.13)$$

Equations (4.12) and (4.13) are compatible and lead to the equations of motion (2.16).

Alternatively in terms of the m KdV Bäcklund transformation (3.13), equation (4.10) can be obtained by gauge-Miura transformation, i.e.,

$$\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2) = g_2(v_2, \epsilon_2)g_1(K(\phi_2, \phi_0)\mathbb{I}K(\phi_0, \phi_1))g_1^{-1}g_2(v_1, \epsilon_1)^{-1}$$

where we may introduce the identity element, $\mathbb{I} = g_1^{-1}g_2(v_0, \epsilon_0)^{-1}g_2(v_0, \epsilon_0)g_1$ depending upon an arbitrary ϵ -sign, say, ϵ_0 . As argued when establishing (3.13), we are considering transitions with opposite ϵ -signs such that $\epsilon_1 = -\epsilon_0 = \epsilon$ and $\epsilon_0 = -\epsilon_2 = -\epsilon$. It therefore follows that

$$\begin{aligned}\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2) &= g_2(v_2, \epsilon)g_1[K(\phi_2, \phi_0)K(\phi_0, \phi_1)]g_1^{-1}g_2(v_1, \epsilon)^{-1} \\ &= g_2(v_2, \epsilon)g_1K^{\text{II}}(\phi_2, \phi_1)g_1^{-1}g_2(v_1, \epsilon)^{-1}.\end{aligned}\quad (4.14)$$

The equation (4.14) yields

$$\begin{aligned}[\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2)]_{11} &= 1 + \frac{(1+\epsilon)}{4\sigma\zeta}e^{\Lambda-p}(e^q + e^{-q} + \eta) - \frac{(1-\epsilon)}{\sigma\zeta}e^{p-\Lambda} \\ &\quad - \frac{(1-\epsilon)}{2\sigma^2\zeta^2}e^{-q} - \frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^q - \frac{(1-\epsilon)}{\sigma\zeta^2}v_1e^{p-\Lambda} \\ &\quad - \frac{(1+\epsilon)}{4\sigma\zeta^2}v_1e^{\Lambda-p}(e^q + e^{-q} + \eta), \\ [\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2)]_{12} &= \frac{(1-\epsilon)}{\sigma\zeta^2}e^{p-\Lambda} - \frac{(1+\epsilon)}{4\sigma\zeta^2}e^{\Lambda-p}(e^q + e^{-q} + \eta), \\ [\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2)]_{21} &= -\frac{\epsilon}{\sigma^2\zeta}(e^q - e^{-q}) - \frac{(1-\epsilon)}{\sigma\zeta}(v_1 + v_2)e^{p-\Lambda} \\ &\quad - \frac{(1+\epsilon)}{4\sigma\zeta}(v_1 + v_2)e^{\Lambda-p}(e^q + e^{-q} + \eta) \\ &\quad + \frac{(v_1 + v_2)}{2\sigma^2\zeta^2}(e^q - e^{-q}) - \frac{\epsilon(v_1 - v_2)}{2\sigma^2\zeta^2}(e^q + e^{-q}) \\ &\quad + \frac{(1+\epsilon)}{4\sigma\zeta^2}v_1v_2e^{\Lambda-p}(e^q + e^{-q} + \eta) - \frac{(1-\epsilon)}{\sigma\zeta^2}v_1v_2e^{p-\Lambda}, \\ [\tilde{K}^{\text{TypeII}}(J_1, J_0, J_2)]_{22} &= 1 - \frac{(1+\epsilon)}{4\sigma\zeta}e^{\Lambda-p}(e^q + e^{-q} + \eta) + \frac{(1-\epsilon)}{\sigma\zeta}e^{p-\Lambda} \\ &\quad - \frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^{-q} - \frac{(1-\epsilon)}{2\sigma^2\zeta^2}e^q + \frac{(1-\epsilon)}{\sigma\zeta^2}v_2e^{p-\Lambda} \\ &\quad + \frac{(1+\epsilon)}{4\sigma\zeta^2}v_2e^{\Lambda-p}(e^q + e^{-q} + \eta).\end{aligned}\quad (4.15)$$

Comparing the matrix elements of (4.10) with (4.14) we find the following relations between the m KdV and KdV variables:

- matrix element 11

$$\zeta^{-1} : -\frac{1}{2}Q = \frac{(1+\epsilon)}{4\sigma}e^{\Lambda-p}(e^q + e^{-q} + \eta) - \frac{(1-\epsilon)}{\sigma}e^{p-\Lambda}, \quad (4.16)$$

$$\begin{aligned} \zeta^{-2} : & -\frac{(\beta_+ + \beta_-)}{2} + \frac{Q}{8}(Q + P - 2\Omega) = -\frac{(1-\epsilon)}{2\sigma^2}e^{-q} - \frac{(1+\epsilon)}{2\sigma^2\zeta^2}e^q \\ & -\frac{(1-\epsilon)}{\sigma\zeta^2}v_1e^{p-\Lambda} - \frac{(1+\epsilon)}{4\sigma\zeta^2}v_1e^{\Lambda-p}(e^q + e^{-q} + \eta) \end{aligned} \quad (4.17)$$

- matrix element 21:

$$\begin{aligned} \zeta^{-1} : & -\beta_- + \frac{Q}{4}(P - 2\Omega) = -\frac{\epsilon}{\sigma^2}(e^q - e^{-q}) - \frac{(1-\epsilon)}{\sigma}(v_1 + v_2)e^{p-\Lambda} \\ & -\frac{(1+\epsilon)}{4\sigma\zeta}(v_1 + v_2)e^{\Lambda-p}(e^q + e^{-q} + \eta), \end{aligned} \quad (4.18)$$

$$\begin{aligned} \zeta^{-2} : & -\frac{\beta_+}{4}Q + \frac{\beta_-}{4}(P - 2\Omega) + \frac{Q}{8}(-\Omega^2 + \Omega P - \frac{P^2}{4} + \frac{Q^2}{4}) \\ & = +\frac{(v_1 + v_2)}{2\sigma^2}(e^q - e^{-q}) - \frac{\epsilon(v_1 - v_2)}{2\sigma^2}(e^q + e^{-q}) \\ & + \frac{(1+\epsilon)}{4\sigma}v_1v_2e^{\Lambda-p}(e^q + e^{-q} + \eta) - \frac{(1-\epsilon)}{\sigma}v_1v_2e^{p-\Lambda} \end{aligned} \quad (4.19)$$

- matrix element 22:

$$\begin{aligned} \zeta^{-2} : & -\frac{(\beta_+ - \beta_-)}{2} + \frac{Q}{8}(Q - P + 2\Omega) = -\frac{(1+\epsilon)}{2\sigma^2}e^{-q} - \frac{(1-\epsilon)}{2\sigma^2}e^q \\ & + \frac{(1-\epsilon)}{\sigma}v_2e^{p-\Lambda} + \frac{(1+\epsilon)}{4\sigma}v_2e^{\Lambda-p}(e^q + e^{-q} + \eta). \end{aligned} \quad (4.20)$$

The element 12 and the element 22 with ζ^{-1} gives us the same result of (4.16). Eliminating the m KdV variables p, q and Λ we recover the Type-II Bäcklund transformation for the KdV hierarchy (4.12) as shown in the [appendix](#).

4.1. Examples and solutions

- Vacuum—1-soliton—vacuum

The first example is to consider vacuum to 1-soliton and back to vacuum again given by the following configuration,

$$w_1 = 0, \quad w_0 = \Omega = -\frac{4k}{1 + \rho(x, t_N)} + 2k, \quad w_2 = 0 \quad (4.21)$$

with $\rho(x, t_N) = e^{2kx + 2k^N t_N}$. It is straightforward to check that equations (4.12) and (4.13) are satisfied for $\beta_- = 0$ and $\beta_+ = 2k^2$.

- 1-soliton—2 soliton—1-soliton

Consider now a configuration of 1-soliton transforming into a 2-solitons solution and

back to 1-soliton. It is described by

$$w_i = -\frac{4k_i}{1 + \rho_i(x, t_N)} + 2k_i, \quad i = 1, 2 \quad \rho_i = e^{2k_i x + 2k_i^N t_N}, \quad (4.22)$$

$$\Omega = w_0 = -\frac{2(k_1^2 - k_2^2)(1 + \rho_1)(1 + \rho_2)}{k_1 - k_2 - (k_1 + k_2)(\rho_1 - \rho_2) - (k_1 - k_2)\rho_1\rho_2}. \quad (4.23)$$

Equations (4.12) and (4.13) are satisfied for $\beta_- = k_2^2 - k_1^2$ and $\beta_+ = k_1^2 + k_2^2$.

- Vacuum—1-soliton—2-soliton

Consider the solution of equation (4.12) and (4.13)

$$w_1 = 0, \quad w_0 = \Omega = -\frac{4k_1}{1 + \rho_1(x, t_N)} + 2k_1 \quad (4.24)$$

and

$$w_2 = -\frac{2(k_1^2 - k_2^2)(1 + \rho_1)(1 + \rho_2)}{k_1 - k_2 - (k_1 + k_2)(\rho_1 - \rho_2) - (k_1 - k_2)\rho_1\rho_2}, \quad (4.25)$$

where $\rho_i = e^{2k_i x + 2k_i^N t_N}$ equations (4.12) and (4.13) are satisfied for $\beta_- = k_1^2 - k_2^2$ and $\beta_+ = k_1^2 + k_2^2$.

5. Conclusions

This paper follows the line of reasoning of [7] where we have constructed Bäcklund transformation for the entire mKdV hierarchy from an universal gauge transformation. Such Bäcklund-gauge transformation relates two different field configurations, preserves the zero curvature and henceforth its corresponding evolution equations.

The main result of this paper is the extension of such construction to the KdV hierarchy by proposing a Miura-gauge transformation denoted by the product $g_2 g_1$ given in (2.9) and (2.11) mapping the mKdV into the KdV hierarchy (see (2.12)). A subtle point is that such Miura mapping allows a sign ambiguity such that each solution of the mKdV hierarchy defines two solutions for its KdV counterpart. The Bäcklund-gauge transformation for the KdV hierarchy is constructed by Miura-gauge transforming the Bäcklund transformation of the mKdV system as shown in (3.13). An interesting fact is that the Bäcklund transformation for the KdV hierarchy is solved by mixed Miura solutions generated by the mKdV Bäcklund solutions. A few simple explicit examples illustrate our conjecture. A more general evidence of the mixed Miura solutions is shown to agree with the Bäcklund transformation proposed in [12, 14] for the first two KdV flows.

The composition law of two subsequent Bäcklund-gauge transformations leading to Type-II Bäcklund transformation (see (4.1)) introduced in [3] in the context of sine-Gordon and Tzitzeica models was extended to the KdV hierarchy. Within our construction, we have employed the direct fusion of two KdV Bäcklund-gauge transformations in (4.10) and alternatively, the Miura transformation of mKdV Type-II Bäcklund transformation as shown in (4.14). These two approaches generate relations between the mKdV and KdV variables which were shown in the appendix A.2, to be consistent.

Finally we should mention that the idea of an universality of the Bäcklund-gauge transformation is most probably valid for other hierarchies such as the AKNS and higher rank Toda theories. It would be interesting to see how such examples can be worked out technically.

It should be interesting to develop the concept of integrable hierarchies for discrete cases and investigate whether the arguments involving Bäcklund-gauge transformation employed in this paper can be extended. The relation between the integrable discrete m KdV [15] and its Miura transformation to discrete KdV equations should be understood under the algebraic formalism.

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Note added in proof. After this paper was finished we were informed about the existence of the book ‘Introduction to Soliton Theory’ by Dendyuan Chen, Science Press, Beijing, (in chinese) that may contain some overlapping results.

Appendix

A.1. Zero curvature for KdV hierarchy

Here we write down the time component of the two-dimensional gauge potential generating the first three flows for the KdV hierarchy. Let

$$A_{t_N, \text{KdV}} = \mathcal{D}'^{(N)} + \mathcal{D}'^{(N-1)} + \dots + \mathcal{D}'^{(0)}, \quad \mathcal{D}'^{(j)} \in \mathcal{G}'^j, \quad (\text{A.1})$$

where we find by solving the zero curvature representation (2.14) in the homogeneous gradation. For $N = 3$ we have

$$\begin{aligned} \mathcal{D}'^{(3)} &= \zeta^3 h, \\ \mathcal{D}'^{(2)} &= -\zeta^2 E_\alpha + J \zeta^2 E_{-\alpha}, \\ \mathcal{D}'^{(1)} &= \frac{1}{2} \partial_x J \zeta E_{-\alpha} + \frac{1}{2} J \zeta h, \\ \mathcal{D}'^{(0)} &= -\frac{1}{2} J E_\alpha + \left(\frac{1}{4} \partial_x^2 J + \frac{1}{2} J^2 \right) E_{-\alpha} + \frac{1}{4} \partial_x J h, \end{aligned} \quad (\text{A.2})$$

and $N = 5$

$$\begin{aligned} \mathcal{D}'^{(5)} &= \zeta^5 h, \\ \mathcal{D}'^{(4)} &= -\zeta^4 E_\alpha + J \zeta^4 E_{-\alpha}, \\ \mathcal{D}'^{(3)} &= \frac{1}{2} \partial_x J \zeta^3 E_{-\alpha} + \frac{1}{2} J \zeta^3 h, \\ \mathcal{D}'^{(2)} &= -\frac{1}{2} J \zeta^2 E_\alpha + \left(\frac{1}{4} \partial_x^2 J + \frac{1}{2} J^2 \right) \zeta^2 E_{-\alpha} \zeta^2 + \frac{1}{4} \partial_x J \zeta^2 h, \\ \mathcal{D}'^{(1)} &= \left(\frac{1}{8} \partial_x^3 J + \frac{3}{4} J \partial_x J \right) \zeta E_{-\alpha} + \left(\frac{1}{8} \partial_x^2 J + \frac{3}{8} J^2 \right) \zeta h \\ \mathcal{D}'^{(0)} &= \left(-\frac{1}{8} \partial_x^2 J - \frac{3}{8} J^2 \right) E_\alpha + \left(\frac{1}{16} \partial_x^3 J + \frac{3}{8} J \partial_x J \right) h \\ &\quad + \left(\frac{1}{16} \partial_x^4 J + \frac{3}{8} (\partial_x J)^2 + \frac{1}{2} J \partial_x^2 J + \frac{3}{8} J^3 \right) E_{-\alpha} \end{aligned} \quad (\text{A.3})$$

and for $N = 7$

$$\begin{aligned}
\mathcal{D}'^{(7)} &= \zeta^7 h, \\
\mathcal{D}'^{(6)} &= -\zeta^6 E_\alpha + J \zeta^6 E_{-\alpha}, \\
\mathcal{D}'^{(5)} &= \frac{1}{2} \partial_x J \zeta^5 E_{-\alpha} + \frac{1}{2} J \zeta^5 h, \\
\mathcal{D}'^{(4)} &= -\frac{1}{2} J \zeta^4 E_\alpha + \left(\frac{1}{4} \partial_x^2 J + \frac{1}{2} J^2 \right) \zeta^4 E_{-\alpha} + \frac{1}{4} \partial_x J \zeta^4 h, \\
\mathcal{D}'^{(3)} &= \left(\frac{1}{8} \partial_x^3 J + \frac{3}{4} J \partial_x J \right) \zeta^3 E_{-\alpha} + \left(\frac{1}{8} \partial_x^2 J + \frac{3}{8} J^2 \right) \zeta^3 h \\
\mathcal{D}'^{(2)} &= \left(-\frac{1}{8} \partial_x^2 J - \frac{3}{8} J^2 \right) \zeta^2 E_\alpha + \left(\frac{1}{16} \partial_x^3 J + \frac{3}{8} J \partial_x J \right) \zeta^2 h \\
&\quad + \left(\frac{1}{16} \partial_x^4 J + \frac{3}{8} (\partial_x J)^2 + \frac{1}{2} J \partial_x^2 J + \frac{3}{8} J^3 \right) \zeta^2 E_{-\alpha} \\
\mathcal{D}'^{(1)} &= \left(\frac{1}{32} \partial_x^5 J + \frac{5}{8} (\partial_x J)(\partial_x^2 J) + \frac{5}{16} J(\partial_x^3 J) + \frac{15}{16} J^2(\partial_x J) \right) \zeta E_{-\alpha} \\
&\quad + \left(\frac{1}{32} (\partial_x^4 J) + \frac{5}{32} (\partial_x J)^2 + \frac{5}{16} J(\partial_x^2 J) + \frac{5}{16} J^3 \right) \zeta h \\
&\quad + \mathcal{D}'^{(0)} = \left(-\frac{1}{32} (\partial_x^4 J) - \frac{5}{32} (\partial_x J)^2 - \frac{5}{16} J(\partial_x^2 J) - \frac{5}{16} J^3 \right) E_\alpha \\
&\quad + \left(\frac{1}{64} (\partial_x^6 J) + \frac{5}{16} (\partial_x^2 J)^2 + \frac{15}{32} (\partial_x J)(\partial_x^3 J) \right) E_{-\alpha} \\
&\quad + \left(\frac{3}{16} J(\partial_x^4 J) + \frac{35}{32} J(\partial_x J)^2 + \frac{25}{32} J^2(\partial_x^2 J) + \frac{5}{16} J^4 \right) E_{-\alpha} \\
&\quad + \left(\frac{1}{64} (\partial_x^5 J) + \frac{5}{16} (\partial_x J)(\partial_x^2 J) + \frac{5}{32} J(\partial_x^3 J) + \frac{15}{32} J^2(\partial_x J) \right) h.
\end{aligned} \tag{A.4}$$

A.2. Equivalence between mKdV and KdV variables

We now verify the equivalence between mKdV and KdV variables from (4.16) we find

$$\begin{aligned}
\partial_x Q &= -\frac{(1+\epsilon)}{2\sigma} \partial_x (\Lambda - p) e^{\Lambda-p} (e^q + e^{-q} + \eta) \\
&\quad - \frac{(1+\epsilon)}{2\sigma} \partial_x q e^{\Lambda-p} (e^q - e^{-q}) + \frac{2(1-\epsilon)}{\sigma} \partial_x (p - \Lambda) e^{p-\Lambda}
\end{aligned} \tag{A.5}$$

and using (4.18), (4.5) and (4.6) we obtain

$$\partial_x Q = 2\beta_- - \frac{QP}{2} + Q\Omega. \tag{A.6}$$

Consider now $P = J_1 + J_2 = \partial_x (w_1 + w_2)$. In terms of Miura transformation

$$\partial_x P = \epsilon \partial_x (v_1 + v_2) - (v_1^2 + v_2^2) = \epsilon \partial_x^2 p - \frac{1}{2} (\partial_x p)^2 - \frac{1}{2} (\partial_x q)^2. \tag{A.7}$$

Acting with ∂_x in (4.18) and using (A.7) and (4.16) we find

$$\begin{aligned} \frac{Q}{2}(\partial_x P + 2\partial_x \Omega) &= \frac{\partial_x Q}{2}(P - 2\Omega) + \frac{1}{2}Q[(\partial_x p)^2 - (\partial_x q)^2] \\ &\quad - Q\partial_x p\partial_x \Lambda + \frac{(1+\epsilon)}{2\sigma}\partial_x p\partial_x q e^{\Lambda-p}(e^q - e^{-q}) + \frac{2\epsilon}{\sigma^2}\partial_x q(e^q + e^{-q}), \end{aligned} \quad (\text{A.8})$$

where $p = v_1 + v_2$, $q = v_1 - v_2$. Substituting the equation (4.16) in the equation (4.19) we find

$$\begin{aligned} \frac{Q}{2}[(\partial_x p)^2 - (\partial_x q)^2] &= \frac{2\partial_x p}{\sigma^2}(e^q - e^{-q}) - \frac{2\epsilon}{\sigma^2}\partial_x q(e^q + e^{-q}) \\ &\quad - \frac{Q}{2}\left(-\Omega^2 + \Omega P - \frac{P^2}{4} + \frac{Q^2}{4}\right) + \beta_+ Q - \beta_-(P - 2\Omega). \end{aligned} \quad (\text{A.9})$$

Substituting this result in (A.8) and eliminating the m KdV variables using (4.5) and (4.6) we obtain

$$\partial_x P + 2\partial_x \Omega = -\frac{1}{4}(P^2 + Q^2) + \Omega P - \Omega^2 + 2\beta_+. \quad (\text{A.10})$$

Equations (A.6) and (A.10) correspond precisely to the Type-II Bäcklund for the KdV hierarchy.

A.3. Consistency with equations of motion

In this appendix we verify that the compatibility of Bäcklund transformations lead us to the equation of motion.

We start with the spatial part which is common to all N . For the KdV equation it is given by

$$\partial_x P = \frac{\beta^2}{2} - \frac{1}{2}Q^2. \quad (\text{A.11})$$

In what follows it will be useful calculate its spatial derivatives:

$$\partial_x^2 P = -Q\partial_x Q; \quad (\text{A.12})$$

$$\partial_x^3 P = -(\partial_x Q)^2 - Q(\partial_x^2 Q); \quad (\text{A.13})$$

$$\partial_x^4 P = -3(\partial_x Q)(\partial_x^2 Q) - Q(\partial_x^3 Q); \quad (\text{A.14})$$

$$\partial_x^5 P = -3(\partial_x^2 Q)^2 - 4(\partial_x Q)(\partial_x^3 Q) - Q(\partial_x^4 Q); \quad (\text{A.15})$$

$$\partial_x^6 P = -10(\partial_x^2 Q)(\partial_x^3 Q) - 5(\partial_x Q)(\partial_x^4 Q) - Q(\partial_x^5 Q); \quad (\text{A.16})$$

$$\partial_x^7 P = -10(\partial_x^3 Q)^2 - 15(\partial_x^2 Q)(\partial_x^4 Q) - 6(\partial_x Q)(\partial_x^5 Q) - Q\partial_x^6 Q; \quad (\text{A.17})$$

$$\partial_x^8 P = -35(\partial_x^3 Q)(\partial_x^4 Q) - 21(\partial_x^2 Q)(\partial_x^5 Q) - 7(\partial_x Q)(\partial_x^6 Q) - Q(\partial_x^7 Q). \quad (\text{A.18})$$

A.3.1. $N = 3$ (KdV). The temporal part of the KdV BT is given by

$$4\partial_{t_3}P = -Q(\partial_x^2Q) + \frac{1}{2}[(\partial_xQ)^2 + (\partial_xP)^2]. \quad (\text{A.19})$$

In order to verify the consistency of this transformation we act with the spatial derivative to obtain

$$4\partial_x\partial_{t_3}P = -Q\partial_x^3Q + 3(\partial_xP)(\partial_x^2P), \quad (\text{A.20})$$

eliminating the term $-Q\partial_x^3Q$ from equation (A.14) we find

$$4\partial_x\partial_{t_3}P = \partial_x^4P + 3(\partial_xP)(\partial_x^2P) + 3(\partial_xQ)(\partial_x^2Q). \quad (\text{A.21})$$

Substituting

$$\partial_xP = J_1 + J_2, \quad \partial_xQ = J_1 - J_2. \quad (\text{A.22})$$

Equation (A.21) becomes precisely the sum of two KdV equations.

A.3.2. $N = 5$. The temporal part of the BT for $N = 5$ equation is given by

$$\begin{aligned} 16\partial_{t_5}P = & -Q(\partial_x^4Q) + (\partial_xQ)(\partial_x^3Q) + 5(\partial_xP)(\partial_x^3P) + \frac{5}{2}(\partial_x^2P)^2 - \frac{1}{2}(\partial_x^2Q)^2 \\ & + \frac{5}{2}(\partial_xP)[(\partial_xP)^2 + 3(\partial_xQ)^2]. \end{aligned} \quad (\text{A.23})$$

Acting ∂_x in the equation (A.23) we obtain

$$\begin{aligned} 16\partial_x\partial_{t_5}P = & -Q(\partial_x^5Q) + 10(\partial_x^2P)(\partial_x^3P) + 5(\partial_xP)(\partial_x^4P) + \frac{15}{2}(\partial_xP)^2(\partial_x^2P) \\ & + \frac{15}{2}(\partial_x^2P)(\partial_xQ)^2 + 15(\partial_xP)(\partial_xQ)(\partial_x^2Q). \end{aligned} \quad (\text{A.24})$$

Then we isolate the term $-Q(\partial_x^5Q)$ from equation (A.16) to find

$$\begin{aligned} 16\partial_x\partial_{t_5}P = & \partial_x^6P + 10(\partial_x^2Q)(\partial_x^3Q) + 5(\partial_xQ)(\partial_x^4Q) + 10(\partial_x^2P)(\partial_x^3P) + 5(\partial_xP)(\partial_x^4P) \\ & + \frac{15}{2}(\partial_x^2P)[(\partial_xP)^2 + (\partial_xQ)^2] + 15(\partial_xP)(\partial_xQ)(\partial_x^2Q). \end{aligned} \quad (\text{A.25})$$

Substituting (A.22) we obtain the sum of two equations for $N = 5$, i.e., equation (2.17).

A.3.3. $N = 7$. The temporal BT for the $N = 7$ equation is

$$\begin{aligned} 64\partial_{t_7}P = & -Q(\partial_x^6Q) + (\partial_xQ)(\partial_x^5Q) + 7(\partial_xP)(\partial_x^5P) - (\partial_x^2Q)(\partial_x^4Q) + 14(\partial_x^2P)(\partial_x^4P) \\ & + \frac{1}{2}(\partial_x^3Q)^2 + \frac{21}{2}(\partial_x^3P)^2 + \frac{35}{2}(\partial_x^3P)(\partial_xP)^2 + \frac{35}{2}(\partial_x^3P)(\partial_xQ)^2 \\ & + 35(\partial_x^3Q)(\partial_xP)(\partial_xQ) + \frac{35}{2}(\partial_xP)[(\partial_x^2P)^2 + (\partial_x^2Q)^2] + 35(\partial_x^2P)(\partial_x^2Q)^2(\partial_xQ) \\ & + \frac{35}{8}[(\partial_xP)^2 + (\partial_xQ)^2] + \frac{105}{4}(\partial_xQ)^2(\partial_xP)^2. \end{aligned} \quad (\text{A.26})$$

Likewise we did for other values of N , acting ∂_x in the above equation. Then we isolate $-Q(\partial_x^7Q)$ from equation (A.18) to find

$$\begin{aligned}
64\partial_x\partial_{t_7}P = & \partial_x^8P + (\partial_xP)(\partial_x^6P) + 7(\partial_xQ)(\partial_x^6Q) + 21(\partial_x^2P)(\partial_x^5P) + 21(\partial_x^2Q)(\partial_x^5Q) \\
& + 35(\partial_x^3P)(\partial_x^4P) + 35(\partial_x^3Q)(\partial_x^4Q) + \frac{35}{2}(\partial_x^4P)[(\partial_xP)^2 + (\partial_xQ)^2] \\
& + 35(\partial_xP)(\partial_xQ)(\partial_x^4Q) + 70(\partial_xP)[(\partial_x^2P)(\partial_x^3P) + (\partial_x^2Q)(\partial_x^3Q)] \\
& + 70(\partial_xQ)[(\partial_x^2Q)(\partial_x^3P) + (\partial_x^2P)(\partial_x^3Q)] + \frac{35}{2}(\partial_x^2P)^3 \\
& + \frac{105}{2}(\partial_x^2P)(\partial_x^2Q)^2 + \frac{35}{2}(\partial_x^2P)(\partial_xP)^3 + \frac{35}{2}(\partial_x^2Q)(\partial_xQ)^3 \\
& + \frac{105}{2}(\partial_x^2Q)(\partial_xQ)(\partial_xP)^2 + \frac{105}{2}(\partial_x^2P)(\partial_xP)(\partial_xQ)^2.
\end{aligned} \tag{A.27}$$

Substituting (A.22) we obtain the equation of motion for $N = 7$ (2.18).

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