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Topological Structural Stability of Partial Differential Equations on Projected Spaces

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Abstract In this paper we study topological structural stability for a family of nonlinear semigroups $T_h(\cdot)$ on Banach space X_h depending on the parameter h. Our results shows the robustness of the internal dynamics and characterization of global attractors for projected Banach spaces, generalizing previous results for small perturbations of partial differential equations. We apply the results to an abstract semilinear equation with Dumbbell type domains and to an abstract evolution problem discretized by the finite element method.

Keywords Structural stability · Attractors · Gradient semigroups · Dumbbell domains

1 Introduction

One of the main concepts in the modern theory of infinite-dimensional dynamical systems is the global attractor. Indeed, dissipative dynamical systems and the study of attracting compact invariant sets have shown very helpful to obtain essential information for a huge

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range of models of PDEs (see [9,20,26,33,37,41,45]). One of the main properties of the global attractor is that it is robust under perturbation of the terms in the equations. Robustness can be understood at the level of sets (upper semicontinuity [29,31], and lower semicontinuity, [19,30,43]), but also on the internal dynamics in the attractors (topological [1,18,26] and structural stability [32,38]). In recent years this kind of results have been generalized for non-autonomous perturbations [11,12,18].

We will say that a dynamically gradient system (in the sense of Definition 11) is *topologically* structurally stable if the system remains the same after a small perturbation of terms in the equations, measured as the robustness of Morse decomposition or the existence of Lyapunov functions associated to the global attractors. Note that, in the literature (see, for instance, [32,38]), structuctural stability is associated to the robustness of Morse sets and connections among them, so that we get an homeomorphism on the structure of attractors under perturbation as observed, for example, in Morse–Smale systems [14,15].

In the autonomous framework, i.e. for nonlinear semigroups acting on a fixed Banach space X, the topological structural stability has been proved, for instance, in [17] or [1,18], where the authors are able to prove the robustness of the characterization of attractors for some evolution PDEs under singular [6] or regular [1] perturbations of the terms in the equations.

On the other hand, the study of the (upper and lower) continuity of the attractors for PDEs in dumbbell domains has been widely studied in [3–5]. Consider the evolution equation of parabolic type of the form

$$\begin{cases} u_t^\epsilon(x,t) - \Delta u^\epsilon(x,t) + u^\epsilon(x,t) = f(u^\epsilon(x,t)), \quad x \in \Omega_\epsilon,\ t>0, \\ \dfrac{\partial u^\epsilon(x,t)}{\partial n} = 0, \quad x \in \partial\Omega_\epsilon, \end{cases} \tag{1.1}$$

where $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, $\epsilon \in (0,1]$ is a parameter, $\frac{\partial}{\partial n}$ is the outside normal derivative and $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable function which is bounded and has bounded derivatives up to the second order. The domain Ω_ϵ is a dumbbell type domain consisting of two disconnected domains, that we denote by Ω, joined by a thin channel, R_ϵ, which degenerates to a line segment as the parameters ϵ approaches zero, see Fig. 1. Under standard dissipative assumption on the nonlinearity f of the type,

$$\limsup_{|s|\to+\infty} \frac{f(s)}{s} < 0,$$

Eq. (1.1) has an attractor $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$, for $\epsilon \in (0,1]$.

Passing to the limit as $\epsilon \to 0$, the limit "domain" will consist of the domain Ω and a line in between. We denote by P_0 and P_1 the points where the line segment touches Ω, see Fig. 2.

Fig. 1 Dumbbell domain

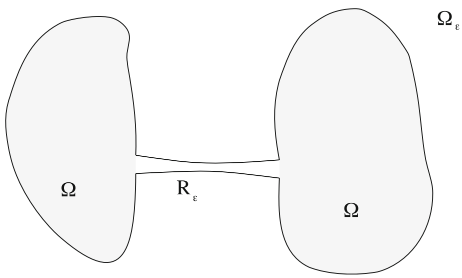
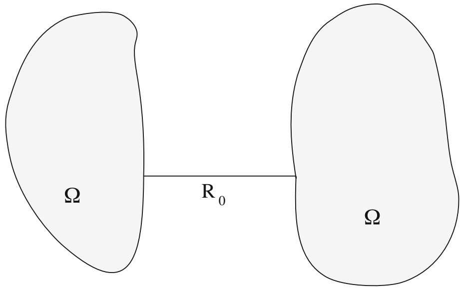


Fig. 2 Limit domain



The limiting equation is

$$\begin{cases} w_t(x,t) - \Delta w(x,t) + w(x,t) = f(w(x,t)), \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial w(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \\ v_t(s,t) - Lv(s,t) + v(s,t) = f(v(s,t)), \quad s \in R_0, \\ v(0) = w(P_0),\ v(1) = w(P_1), \end{cases} \tag{1.2}$$

where w is a function in Ω and v lives in the line segment R_0. Moreover, L is a differential operator which depends on the geometry of the channel R_ϵ, more exactly, on the way the channel R_ϵ collapses to the segment line R_0. More specifically, $Lv = \frac{1}{g}(gv_x)_x$ where g will be defined in Sect. 4. Again, this system has an attractor $\mathcal{A}_0$ in $H^1(\Omega) \times H^1(R_0) =: H^1(\Omega_0)$.

Note that both Eqs. (1.1) and (1.2) are posed in different space domains, which produces a drastic change in the nature of equations. Indeed, since (1.1) is in Ω_ϵ and (1.2) is defined in $\Omega \cup R_0$, we need to deal with solutions in these two different sets. This leads to the comparison of semigroups $\{T_\epsilon(t) : t \geq 0\}_{\epsilon \in (0,1]}$,$\{T_0(t) : t \geq 0\}$ and associated attractors $\mathcal{A}_\epsilon$, $\mathcal{A}_0$ on different Banach spaces $H^1(\Omega_\epsilon)$, $H^1(\Omega_0)$. Moreover, since the study of structural stability of dynamical systems requires a deep understanding of the geometrical description of the attractors and its behaviour under perturbation, we need to generalize the existing theoretical results in the literature on topological structural stability (see [1,18]) for fixed Banach spaces to the case of Banach spaces depending on parameters.

We also present another example given by discretization of an abstract Cauchy problem using the finite element method. Consider the abstract evolution problem in the Hilbert space X as

$$\begin{cases} \dot{u} + Au = F(u),\ t > 0, \\ u(0) = u^0 \in X^{1/2}, \end{cases} \tag{AP}$$

where the operator $A : D(A) \subset X \to X$ is given by $Au = -Lu$ for $u \in D(A)$, L is the operator in (4.9) and $F : X^{1/2} \to X$ is the Nemitskii's operator associated to f. Under certain conditions on the function f, the problem (AP) has an attractor $\mathcal{A}$ in $X^{1/2}$.

In order to discretize problem (AP), we introduce the space $X_h^{1/2}$ (see Assumption 1 and (4.19)) which has finite dimension. So that the problem (AP) can be approximated by

$$\begin{cases} \dot{u}_h + A_h u_h = F_h(u_h) \\ u_h(0) = u_h^0 \in X_h^{1/2}, \end{cases} \tag{AP$_h$}$$

where $A_h : X_h^{1/2} \to X_h^{1/2}$ is given by (4.24), $F_h := P_h F : X_h^{1/2} \to X_h^{1/2}$, for all $h \in (0, 1]$ and $P_h : X \to X_h^{1/2}$ is the projection operator (see (4.22)). Problem (AP$_h$) has an attractor

$\mathcal{A}_h$ in $X_h^{1/2}$ for all $h \in (0,1]$. Again, we deal with the comparison of semigroups $\{T_h(t) : t \geq 0\}_{h\in(0,1]}$, $\{T(t) : t \geq 0\}$ and associated attractors $\mathcal{A}_h$, $\mathcal{A}$ on different Banach spaces $X_h^{1/2}$, $X^{1/2}$. We will show that under certain conditions the topological structural stability of semigroups also holds in this case.

Thus, in this paper we show sufficient conditions for a dynamical system to be topological structurally stable on projected Banach spaces, which is then well suited to apply for our parabolic equations in dumbbell domains and an abstract evolution problem discretized via the finite element method. The difficulties to deal with changing Banach spaces depending on a parameter lead to introduce generalizations of all concepts and results already known in the previous literature. Section 2 introduce several concepts and results on the existence and characterization of attractors for dynamically gradient systems, as the important definition of $\mathscr{P}$-convergence on a family of paramatrized Banach spaces. In Sect. 3 we prove the main result of this paper (see Theorem 8) on the topological structural stability on projected Banach spaces, which is then used in Sect. 4 to prove the robustness of the characterization and internal dynamics for the attractors of (1.1) and (1.2) and also for the attractors of (AP) and (AP_h).

2 Basic Concepts and Results

We introduce the necessary basic notions and results on attractors and $\mathscr{P}$-convergence on Banach spaces.

2.1 Theory of Global Attractors

Firstly we recall the definition of a global attractor for a nonlinear semigroup $\{T(t) : t \geq 0\}$ (see [9,20,26,37,41,45]).

Let X be a metric space with metric $d : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+ = [0,\infty)$, and denote by $\mathscr{C}(X)$ the set of continuous maps from X into X. Given a subset A of X and $\epsilon > 0$, the ϵ-neighborhood of A is the set $\mathcal{O}_\epsilon(A) := \{x \in X : d(x,a) < \epsilon \text{ for some } a \in A\}$.

Now, we introduce the notion of semigroup in the metric space X.

Definition 1 A family $\{T(t) : t \geq 0\} \subset \mathscr{C}(X)$ is a semigroup in X if

(i) $T(0) = I_X$, I_X is the identity map in X.
(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$ and
(iii) $(t,x) \mapsto T(t)x \in X$ is continuous from $\mathbb{R}^+ \times X$ into X.

For simplicity, we will refer to "$T(\cdot)$" rather than "the semigroup $\{T(t) : t \geq 0\}$". A solution of $T(\cdot)$ corresponding to the initial condition $x(0) = x_0$ is the mapping $t \mapsto T(t)x_0$ from $\mathbb{R}^+$ into X.

We begin with the necessary definitions to define the global attractor for the semigroup.

Definition 2 A set $A \subset X$ is invariant under $T(\cdot)$ if $T(t)A = A$ for all $t \geq 0$.

Remark 1 Let $(A_\lambda)_{\lambda\in L}$ be a family of invariants subsets of X under $T(\cdot)$, then the union $\bigcup_{\lambda\in L} A_\lambda$ is invariant under $T(\cdot)$. In fact, for any $t \geq 0$, $T(t)\left(\bigcup_{\lambda\in L} A_\lambda\right) = \bigcup_{\lambda\in L} T(t)A_\lambda$ and, by the assumption, $T(t)A_\lambda = A_\lambda$, for all $\lambda \in L$ and $t \geq 0$.

The notion of invariant set is intimately related to that of global solution.

Definition 3 A global solution for $T(\cdot)$ is a continuous function $\xi : \mathbb{R} \to X$ with the property that $T(t)\xi(s) = \xi(t+s)$ for all $s \in \mathbb{R}$ and for all $t \in \mathbb{R}^+$. We say that $\xi : \mathbb{R} \to X$ is a global solution through $x \in X$ if it is a global solution with $\xi(0) = x$. The orbit of a global solution is

$$\gamma(\xi) = \bigcup_{t\in\mathbb{R}} \{\xi(t)\}.$$

The concepts of invariant set and global solution are connected by the following result.

Proposition 1 *A subset A of X is invariant under $T(\cdot)$ if and only if it consists of a union of orbits of global solutions.*

Proof See Lemma 1.4 in [18]. □

Next, we will introduce the notions of attraction and absorption. For that we recall the definition of Hausdorff semidistance.

Definition 4 Given A and B nonempty subset of X, we define the Hausdorff semidistance from A to B as

$$dist_X(A, B) := \sup_{a\in A} d_X(a, B) = \sup_{a\in A} \inf_{b\in B} d(a, b).$$

Remark 2 The Hausdorff semidistance fulfills the triangle inequality.

Note that, $dist_X(A, B) = 0$ implies only that $\overline{A} \subseteq \overline{B}$, where $\overline{D}$ denotes the closure of D in X; we only have $dist_X(A, B) = 0$ implying $A \subset B$ provided that B is closed.

Definition 5 Given two subsets A, B of X we say that A attracts B under $T(\cdot)$ if $dist_X(T(t)B, A) \overset{t\to\infty}{\longrightarrow} 0$ and we say that A absorbs B under $T(\cdot)$ if there is a $t_B > 0$ such that $T(t)B \subset A$ for all $t \geq t_B$.

Definition 6 $T(\cdot)$ is asymptotically compact if, for any sequence $\{t_k\}_{k\in\mathbb{N}}$ in $[0, \infty)$ with $t_k \overset{k\to\infty}{\longrightarrow} \infty$ and bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ in X, the sequence $\{T(t_k)x_k\}_{k\in\mathbb{N}}$ has a convergent subsequence in X.

With this we are in a position to define global attractors.

Definition 7 A subset $\mathcal{A}$ of X is a global attractor for $T(\cdot)$ if it is compact, invariant under $T(\cdot)$ and for every bounded subset B of X we have that $\mathcal{A}$ attracts B under $T(\cdot)$.

This definition in fact yields the minimal compact set that attracts each bounded subset of X and the maximal closed and bounded invariant set (see [18,26]). The global attractor for the semigroup is unique (see [20,41,45]).

Next, as a consequence of Proposition 1, the global attractor can be characterized as the union of the orbits of all globally defined bounded solutions.

Theorem 1 *If $T(\cdot)$ has a global attractor $\mathcal{A}$, then*

$$\mathcal{A} = \{y \in X : \textit{ there is a bounded global solution } \xi : \mathbb{R} \to X \textit{ with } \xi(0) = y\}.$$

2.2 Dynamically Gradient Semigroups

In this section we recall the notions of a dynamically gradient semigroup for a global attractor (see [17,18]). We first define the concept of isolated invariant sets.

Definition 8 We say that $\mathscr{S} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ is a family of isolated invariant sets (for $T(\cdot)$) if there exists a $\delta > 0$ such that

$$\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \emptyset, \quad 1 \leq i < j \leq n,$$

and Ξ_i is the maximal invariant subset (with respect to $T(\cdot)$) of $\mathcal{O}_\delta(\Xi_i)$.

Definition 9 Let $T(\cdot)$ be a semigroup and let $\mathscr{S} = \{\Xi_1, \Xi_2, \ldots, \Xi_n\}$ be a family of isolated invariant sets. A homoclinic structure in $\mathscr{S}$ is a non-empty subset $\{\Xi_{\ell_1}, \Xi_{\ell_2}, \ldots, \Xi_{\ell_k}\}$ of $\mathscr{S}$ (where $k \leq n$), together with a set of global solutions $\{\xi_j : \mathbb{R} \to X : 1 \leq j \leq k\}$ such that

$$\lim_{t\to-\infty} dist_X(\xi_j(t), \Xi_{\ell_j}) = 0 \quad \text{and} \quad \lim_{t\to\infty} dist_X(\xi_j(t), \Xi_{\ell_{(j+1)}}) = 0,$$

where $\Xi_{\ell_{k+1}} := \Xi_{\ell_1}$ and, if $k = 1$, the orbit $\gamma(\xi_1)$ is not contained in Ξ_{ℓ_1}.

Definition 10 The unstable set of an invariant (under $T(\cdot)$) set Ξ is defined by

$$W^u(\Xi) := \{x \in X : \text{ there is a global solution } \xi : \mathbb{R} \to X \\ \text{such that } \xi(0) = x \text{ and } \lim_{t\to-\infty} dist_X(\xi(t), \Xi) = 0\}.$$

Given $\delta > 0$, the local unstable set of Ξ associated to δ is the set

$$W^{u,\delta}(\Xi) := \{x \in W^u(\Xi) : dist_X(\xi(t), \Xi) < \delta, \forall\, t \leq 0\}. \tag{2.1}$$

We are now ready to define dynamically gradient semigroups (see [18, Sect. 5.1]).

Definition 11 A semigroup $T(\cdot)$ with a global attractor $\mathcal{A}$ is dynamically gradient with respect to the family $\mathscr{S} = \{\Xi_1, \Xi_2, \cdots, \Xi_n\}$ of isolated invariant bounded sets, or dynamically $\mathscr{S}$-gradient, if it satisfies the following two properties:

(G1) Given a global solution $\xi : \mathbb{R} \to X$ in $\mathcal{A}$, there exist $i, j \in \{1, 2, \ldots, n\}$ such that

$$\lim_{t\to-\infty} dist_X(\xi(t), \Xi_i) = 0 \text{ and } \lim_{t\to\infty} dist_X(\xi(t), \Xi_j) = 0.$$

(G2) The collection $\mathscr{S}$ does not contains homoclinic structures.

Next, we introduce a class of semigroup which will be important to explain the purpose of the results in this paper.

Definition 12 We say that a semigroup $T(\cdot)$ with a global attractor $\mathcal{A}$ and a family of isolated invariant bounded sets $\mathscr{S} = \{\Xi_1, \Xi_2, \ldots, \Xi_n\}$ is a gradient semigroup with respect to $\mathscr{S}$, or an $\mathscr{S}$-gradient semigroup, if there is a continuous function $V : X \to \mathbb{R}$ such that

(i) The real function $[0, \infty) \ni t \longmapsto V(T(t)x) \in \mathbb{R}$ is non-increasing for each $x \in X$;
(ii) V is constant in each Ξ_i for each $i = 1, \ldots, n$; and
(iii) $V(T(t)x) = V(x)$ for all $t \geq 0$ if and only if $x \in \bigcup_{i=1}^{n} \Xi_i$.

A function V with these properties is called a generalized Lyapunov function for $T(\cdot)$ with respect to $\mathscr{S}$, or an $\mathscr{S}$-Lyapunov function for $T(\cdot)$.

The following result characterizes a gradient semigroup in terms of its dynamical properties: backwards and forwards convergence to isolated invariant sets and the absence of homoclinic structures.

Theorem 2 *Let $T(\cdot)$ be a semigroup with a global attractor $\mathcal{A}$, and let $\mathscr{S}$ be a finite collection of isolated invariant bounded sets. Then $T(\cdot)$ is $\mathscr{S}$-gradient if and only if it is dynamically $\mathscr{S}$-gradient.*

Proof See Theorem 1.1 in [1] or Theorem 5.5 in [18]. □

As an immediate consequence of this theorem we obtain the following characterization of the attractor.

Corollary 3 *Let $T(\cdot)$ be a gradient semigroup with respect to $\mathscr{S} = \{\Xi_1, \Xi_2, \ldots, \Xi_n\}$ a family of bounded isolated invariant sets. If $T(\cdot)$ has a global attractor $\mathcal{A}$, then*

$$\mathcal{A} = \bigcup_{j=1}^{n} W^u(\Xi_j). \tag{2.2}$$

Proof We first note that, given $y \in \mathcal{A}$ and using Theorem 1, it follows that there is a bounded global solution $\xi : \mathbb{R} \to X$ with $\xi(0) = y$. Also, from Theorem 2, there exists a $\Xi \in \mathscr{S}$ so that $\lim_{t\to-\infty} dist_X(\xi(t), \Xi) = 0$, then $\xi(t) \in W^u(\Xi)$. Hence, $\mathcal{A} \subset \bigcup_{j=1}^{n} W^u(\Xi_j)$.

Conversely, given $x \in W^u(\Xi_\ell)$, for some $\Xi_\ell \in \mathscr{S}$ where $\ell \in \{1, \ldots, n\}$, we have there is a global solution $\xi : \mathbb{R} \to X$ for $T(\cdot)$ with $\xi(0) = x$ and $\lim_{t\to-\infty} dist_X(\xi(t), \Xi_\ell) = 0$. Since Ξ_ℓ is a bounded set, then $\{\xi(t) : t \leq -\tau\}$ is bounded set for some $\tau \in \mathbb{R}^+$. From Theorem 2, we know also that the set $\{\xi(t) : t \geq \tau_0\}$ is bounded for some $\tau_0 \in \mathbb{R}^+$. Thus, we see from Theorem 1 that $x \in \mathcal{A}$. □

2.3 Perturbation of Global Attractors

A perturbation of the semigroup $T(\cdot)$ on a space X produces a variation in its global attractor $\mathcal{A}$ contained in X, which is a very natural fact in real phenomena. Perturbation of the semigroup is reflected in dependence on a parameter $h \in (0, 1]$, that is, $T_h(\cdot)$, and similarly for the global attractor $\mathcal{A}_h$. The study of the behavior of this effect is given in two parts:

(a) The continuity of global attractors: This fact is given by the combination of

($a1$) Upper semicontinuity of global attractors under perturbation (see [27,29,31,43]), i.e., when it holds that

$$dist_X(\mathcal{A}_h, \mathcal{A}) \to 0 \text{ as } h \to 0 \quad \text{and}$$

($a2$) Lower semicontinuity of global attractors under perturbation (see [19,30,41,43]), written as

$$dist_X(\mathcal{A}, \mathcal{A}_h) \to 0 \text{ as } h \to 0.$$

(b) Topological structural stability (see [17,18]): We call topological structural stability if dynamically gradient semigroups (see [18, Definition 5.4] or (G1) and (G2) at Definition 11) are kept under perturbation. This means that the internal dynamics in global attractor is robust under perturbation.

The next section is devoted to introduce the concept of discrete convergence, used for comparing problems on different Banach spaces.

2.4 $\mathscr{P}$-convergence

$\mathscr{P}$-convergence, also called *discrete convergence*, was proposed by Stummel (see [44,46–48]), and it is specially well-suited for the analysis of comparison of solutions for discretization of PDEs under, for instance, finite element methods. In the present section we recall some fundamental notions and results concerning to the $\mathscr{P}$-convergence of elements and operators. In the papers [3–5,19,46–49] a general scheme was studied that allows to analyze convergence properties of numerical discretizations along with some applications to the continuity of attractors for some evolution PDEs. We now collect some results that are necessary for the development of the paper.

Let $\{X_h\}_{h\in[0,1]}$ be a family of Banach spaces and $\mathscr{P} = \{P_h\}_{h\in(0,1]}$ a family with $P_h : X_0 \to X_h$ such that $P_h \in \mathscr{L}(X_0, X_h) = \{T_h : X_0 \to X_h : T_h$ is a bounded linear operator$\}$, for $h \in (0, 1]$, with the following property:

$$\|P_h u_0\|_{X_h} \overset{h\to 0}{\longrightarrow} \|u_0\|_{X_0}, \quad \forall\, u_0 \in X_0. \tag{2.3}$$

Usually, in applications, the X_h are finite dimensional; but in the abstract theory this assumption is not necessary.

Definition 13 A family $\{u_h\}_{h\in(0,1]}$ with $u_h \in X_h$ $\mathscr{P}$-converges to $u_0 \in X_0$ if

$$\|u_h - P_h u_0\|_{X_h} \to 0 \quad \text{as} \quad h \to 0.$$

We write this as $u_h \overset{\mathscr{P}}{\longrightarrow} u_0$.

Similarly, the $\mathscr{P}$-convergence of sequences is defined as: A sequence $\{u_{h_n}\}_{n\in\mathbb{N}}$ with $u_{h_n} \in X_{h_n}$, such that $h_n \overset{n\to\infty}{\longrightarrow} 0$, $\mathscr{P}$-converges to $u_0 \in X_0$ if

$$\|u_{h_n} - P_{h_n} u_0\|_{X_{h_n}} \to 0 \quad as \quad n \to \infty.$$

Lemma 4 *Let $P_h \in \mathscr{L}(X_0, X_h)$ be satisfying (2.3) for all $h \in (0, 1]$, then there exist a constant $1 \le M < \infty$ and $h_0 \in (0, 1]$ such that*

$$\sup_{h\in[0,h_0]} \|P_h\|_{\mathscr{L}(X_0,X_h)} \le M.$$

Proof Let $F_n = \{u_0 \in X_0 : \exists \bar{h} \in (0, 1]$ s.t. $\|P_h u_0\|_{X_h} \le n, \forall\, h \in [0, \bar{h}]\} \subset X_0$. Then, $\cup_{n=1}^{\infty} F_n \subset X_0$. We observe that F_n is closed set because $P_h \in \mathscr{L}(X_0, X_h)$. From (2.3), we know that, given $\varepsilon > 0$ and $u_0 \in X_0$, there exists an $h_1 \in (0, 1]$ such that

$$\|P_h u_0\|_{X_h} \le |\,\|P_h u_0\|_{X_h} - \|u_0\|_{X_0}| + \|u_0\|_{X_0} < \varepsilon + \|u_0\|_{X_0}, \forall\, h \in [0, h_1]. \tag{2.4}$$

So that, $\|P_h u_0\|_{X_h} \le \|u_0\|_{X_0} \le n_1$, for some $n_1 \in \mathbb{N}$ and for all $h \in [0, h_1]$. Hence, $X_0 = \bigcup_{n=1}^{\infty} F_n$. Using the Baire Theorem, there is an $n_0 \in \mathbb{N}$ such that int $F_{n_0} \ne \emptyset$. Given $u_0 \in X_0$ and $r > 0$ such that $B_{X_0}(u_0, r) \subset F_{n_0}$. From this, there exists an h_2 such that $\|P_h u_0\|_{X_h} \le n_0$, for all $h \in [0, h_2]$. Thus, taking $u = u_0 + rz$ for $z \in B_{X_0}(0, 1)$, follows that $u \in B_{X_0}(u_0, r)$. Hence, there is an $h_3 \in (0, 1]$ such that $\|P_h u\|_{X_h} \le n_0$, for all $h \in [0, h_3]$. Then,

$$r\|P_h z\|_{X_h} \le \|P_h u\|_{X_h} + \|P_h u_0\|_{X_h} \le 2n_0.$$

Therefore, there exists an $h_0 = \min\{h_2, h_3\} \in (0, 1]$ such that

$$\|P_h\|_{\mathscr{L}(X_0,X_h)} \leq \frac{2n_0}{r} =: M,$$

for all $h \in [0, h_0]$. □

Let us present some properties of $\mathscr{P}$-convergence that follow immediately.

Proposition 2 *The $\mathscr{P}$-convergence has the following properties:*

i) If $u_h \xrightarrow{\mathscr{P}} u_0$ and $u_h \xrightarrow{\mathscr{P}} v_0$, then $u_0 = v_0$.
ii) If $u_h \xrightarrow{\mathscr{P}} u_0$ and $v_h \xrightarrow{\mathscr{P}} v_0$, then $\alpha u_h + \beta v_h \xrightarrow{\mathscr{P}} \alpha u_0 + \beta v_0$, where $\alpha, \beta \in \mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.
iii) If $u_h \xrightarrow{\mathscr{P}} u_0$, then $\|u_h\|_{X_h} \xrightarrow{h\to 0} \|u_0\|_{X_0}$.
iv) $u_h \xrightarrow{\mathscr{P}} 0$ if and only if $\|u_h\|_{X_h} \xrightarrow{h\to 0} 0$.
v) Given $u_0 \in X_0$, then $P_h u_0 \xrightarrow{\mathscr{P}} u_0$.
vi) If $u_0 \in X_0$, $\{u^{(h)}\} \subset X_0$ such that $\|u^{(h)} - u_0\|_{X_0} \xrightarrow{h\to 0} 0$, then $P_h u^{(h)} \xrightarrow{\mathscr{P}} u_0$.

Assume that $\{Y_h\}_{h\in[0,1]}$ is a family of Banach spaces and $\mathscr{Q} = \{Q_h\}_{h\in(0,1]}$ a sequence of bounded linear operators $Q_h : Y_0 \to Y_h$ with the following property:

$$\|Q_h v_0\|_{Y_h} \xrightarrow{h\to 0} \|v_0\|_{Y_0}, \ \forall \ v_0 \in Y_0. \tag{2.5}$$

Definition 14 A family $\{A_h\}_{h\in(0,1]}$ of operators $A_h : X_h \to Y_h$ $\mathscr{P}\mathscr{Q}$-converges to an operator $A_0 : X_0 \to Y_0$ as $h \to 0$ if

$$A_h u_h \xrightarrow{\mathscr{Q}} A_0 u_0 \quad \text{whenever} \quad u_h \xrightarrow{\mathscr{P}} u_0.$$

We write $A_h \xrightarrow{\mathscr{P}\mathscr{Q}} A_0$ as $h \to 0$.

The following result is very useful to show $\mathscr{P}\mathscr{Q}$-convergence of linear operators.

Proposition 3 *Let $A_h \in \mathscr{L}(X_h, Y_h)$, for all $h \in (0, 1]$, $A_0 \in \mathscr{L}(X_0, Y_0)$, the bounded linear operators $P_h : X_0 \to X_h$ and $Q_h : Y_0 \to Y_h$ satisfying (2.3) and (2.5), respectively. The following statements are equivalent:*

(i) $A_h \xrightarrow{\mathscr{P}\mathscr{Q}} A_0$ as $h \to 0$.
(ii) For any $u_0 \in X_0$, $\|A_h P_h u_0 - Q_h A_0 u_0\|_{Y_h} \xrightarrow{h\to 0} 0$ and $\limsup_{h\to 0} \|A_h\|_{\mathscr{L}(X_h,Y_h)} < \infty$.
(iii) For any $v \in E$, $\|A_h P_h v - Q_h A_0 v\|_{Y_h} \xrightarrow{h\to 0} 0$ where E is dense in X_0 and $\limsup_{h\to 0} \|A_h\|_{\mathscr{L}(X_h,Y_h)} < \infty$.

Proof See Theorem 2.2.8 in [46] or Lema 4.1 in [49]. □

Denote by $\mathscr{U}(X_h, Y_h)$ the set of uniformly Lipschitz continuous maps from X_h into Y_h, for $h \in [0, 1]$. In the following result observe that the operators are simply continuous.

Corollary 5 *Let $A_h \in \mathscr{U}(X_h, Y_h)$, for $h \in (0, 1]$, the bounded linear operators $P_h : X_0 \to X_h$ and $Q_h : Y_0 \to Y_h$ satisfying (2.3) and (2.5), respectively. $A_h \xrightarrow{\mathscr{P}\mathscr{Q}} A_0$ as $h \to 0$ if and only if $\|A_h P_h u_0 - Q_h A_0 u_0\|_{Y_h} \xrightarrow{h\to 0} 0$, for all $u_0 \in X_0$.*

Proof The first implication is straightforward from Definition 14, since $P_h u_0 \xrightarrow{\mathscr{P}} u_0$ for all $u_0 \in X_0$. Thus, $A_h P_h u_0 \xrightarrow{\mathscr{Q}} A_0 u_0$, for all $u_0 \in X_0$. Conversely, let $u_h \xrightarrow{\mathscr{P}} u_0$ as $h \to 0$. Since $A_h \in \mathscr{U}(X_h, Y_h)$, then there is a positive constant α, which is independent of h, such that

$$\|A_h u_h - Q_h A_0 u_0\|_{Y_h} \le \alpha \|u_h - P_h u_0\|_{X_h} + \|A_h P_h u_0 - Q_h A_0 u_0\|_{Y_h} \xrightarrow{h \to 0} 0.$$

Therefore, $A_h \xrightarrow{\mathscr{P}\mathscr{Q}} A_0$. □

2.5 Continuity of Attractors Under $\mathscr{P}$-convergence

Now, we define the continuity of global attractors in the sense of $\mathscr{P}$-convergence.

Definition 15 Let $\mathcal{A}_h \subset X_h$, where X_h is a Banach space, for all $h \in [0, 1]$.

(i) We say that the family of sets $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-upper semicontinuous at $h = 0$ if

$$dist_{X_h}(\mathcal{A}_h, P_h \mathcal{A}_0) := \sup_{u_h \in \mathcal{A}_h} \inf_{u \in \mathcal{A}_0} \|u_h - P_h u\|_{X_h} \xrightarrow{h \to 0} 0. \tag{2.6}$$

(ii) We say that the family of sets $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-lower semicontinuous at $h = 0$ if

$$dist_{X_h}(P_h \mathcal{A}_0, \mathcal{A}_h) := \sup_{u \in \mathcal{A}_0} \inf_{u_h \in \mathcal{A}_h} \|u_h - P_h u\|_{X_h} \xrightarrow{h \to 0} 0. \tag{2.7}$$

(ii) We say that the family of sets $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-continuous at $h = 0$ if it is $\mathscr{P}$-upper semicontinuous and $\mathscr{P}$-lower semicontinuous at $h = 0$.

Remark 6 The $\mathscr{P}$-upper and $\mathscr{P}$-lower semicontinuity of sets have the following characterizations (see [18,19]):

(1) If for each sequence $h_k \to 0$ and $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \mathcal{A}_{h_k}$ there exists a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ such that $\{x_{k_l}\}_{l\in\mathbb{N}}$ is $\mathscr{P}$-convergent to some limit u_0 belonging to $\mathcal{A}_0$, then $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-upper semicontinuous at $h = 0$.

(2-a) If $\mathcal{A}_0$ is compact and for any $u_0 \in \mathcal{A}_0$ and $h_k \to 0$ there is a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ and a sequence $\{u_l\}_{l\in\mathbb{N}}$ with $u_l \in \mathcal{A}_{h_{k_l}}$, which $\mathscr{P}$-converges to u_0, then $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-lower semicontinuous at $h = 0$.

(2-b) If $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-lower semicontinuous at $h = 0$, given $u_0 \in \mathcal{A}_0$ and $h_k \to 0$, there is a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ and a sequence $\{u_l\}_{l\in\mathbb{N}}$ with $u_l \in \mathcal{A}_{h_{k_l}}$, which $\mathscr{P}$-converges to u_0.

Proof Just for completeness, let us proof item (2-b).

Indeed, since $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-lower semicontinuous at $h = 0$ we have, in particular, that

$$\lim_{k\to\infty} dist_{X_{h_k}}(P_{h_k}\mathcal{A}_0, \mathcal{A}_{h_k}) = 0.$$

Then, given $u_0 \in \mathcal{A}_0$ we get

$$\lim_{k\to\infty} d_{X_{h_k}}(P_{h_k} u_0, \mathcal{A}_{h_k}) = 0.$$

From that, we may take a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ such that

$$d_{X_{h_{k_l}}}(P_{h_{k_l}} u_0, \mathcal{A}_{h_{k_l}}) < \frac{1}{l},$$

and by definition of $d_X(\cdot,\cdot)$ we can take a sequence $\{u_l\}_{l\in\mathbb{N}}$ with $u_l \in \mathcal{A}_{h_{k_l}}$ which satisfies

$$d_{X_{h_{k_l}}}(P_{h_{k_l}}u_0, u_l) < \frac{1}{l}$$

and the proof is complete. □

3 Topological Structural Stability of Global Attractors

In this section we develop the theory of topological structural stability for different Banach spaces parameterized by $h \in (0,1]$, say X_h, which dimension may be finite (to apply for discretizations of PDEs) or infinite. With this we generalize the theory of topological structural stability (see [1,18]) given on a fixed Banach space.

We begin with a definition that is a generalization of the concept of collectively asymptotically compact (see [1, Definition 4.2] or [18, Definition 3.16]) for variable Banach spaces.

Again for simplicity, we will refer to $\{T_h(\cdot)\}_{h\in[0,1]}$ rather than the family of nonlinear semigroups $\{T_h(t) : t \geq 0\}_{h\in[0,1]}$.

Definition 16 Let $\{T_h(\cdot)\}_{h\in[0,1]}$ be a family of semigroups in the Banach space X_h, for all $h \in [0,1]$. We say that $\{T_h(\cdot)\}_{h\in(0,1]}$ is $\mathscr{P}$-collectively asymptotically compact at $h = 0$ if, for any sequence $\{h_k\}_{k\in\mathbb{N}}$ in $(0,1]$ with $h_k \overset{k\to\infty}{\longrightarrow} 0$, uniformly bounded sequence $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in X_{h_k}$, sequence $\{t_k\}_{k\in\mathbb{N}}$ in $(0,\infty)$ with $t_k \overset{k\to\infty}{\longrightarrow} \infty$ the sequence $\{T_{h_k}(t_k)x_k\}_{k\in\mathbb{N}}$ has a $\mathscr{P}$-convergent subsequence to some element in X_0.

Remark 7 In case $X_h = X_0$ and $P_h = I_{X_0}$, for all $h \in (0,1]$, $\{T_h(\cdot)\}_{h\in[0,1]}$ is collectively asymptotically compact at $h = 0$ (see Definition 3.16 in [18]).

Definition 17 Let $\{T_h(\cdot)\}_{h\in[0,1]}$ be a family of nonlinear semigroups in X_h, for $h \in [0,1]$. We say that $T_h(\cdot) \overset{\mathscr{P}\mathscr{P}}{\longrightarrow} T_0(\cdot)$ uniformly in compact subsets of $\mathbb{R}^+ \times X_0$ if for each $\tau > 0$, we have

$$\lim_{h\to 0} \sup_{t\in[0,\tau]} \|T_h(t)u_h - P_h T_0(t)u_0\|_{X_h} = 0$$

whenever

$$\lim_{h\to 0} \|u_h - P_h u_0\|_{X_h} = 0.$$

We are now ready to state the following theorem on the behavior of global attractors under perturbation, which also implies perturbation of space. This result also shows the continuity of global attractors in the sense of $\mathscr{P}$-convergence.

Theorem 8 (Topological structural stability) *Let $\{T_h(\cdot)\}_{h\in[0,1]}$ be a family of nonlinear semigroups $\mathscr{P}$-collectively asymptotically compact at $h = 0$ on a Banach space X_h, for $h \in [0,1]$ and that $T_h(\cdot) \overset{\mathscr{P}\mathscr{P}}{\longrightarrow} T_0(\cdot)$ uniformly on compact subsets of $\mathbb{R}^+ \times X_0$. Suppose that*

(a) for each $h \in [0,1]$, the semigroup $T_h(\cdot)$ has a global attractor $\mathcal{A}_h$, with $\sup_{h\in[0,1]} \sup_{x_h\in\mathcal{A}_h} \|x_h\|_{X_h} < \infty$;

(b) there is a $p \in \mathbb{N}$ such that for any $h \in [0,1]$, the semigroup $T_h(\cdot)$ has a disjoint family of isolated invariant bounded sets $\mathscr{S}_h := \{\Xi_{1,h}, \Xi_{2,h}, \ldots, \Xi_{p,h}\}$ that behave $\mathscr{P}$-continuously at $h = 0$, that is, for each $i = 1, 2, \ldots, p$, we have

$$dist_{X_h}(\Xi_{i,h}, P_h\Xi_{i,0}) + dist_{X_h}(P_h\Xi_{i,0}, \Xi_{i,h}) \to 0 \quad as \quad h \to 0; \tag{3.1}$$

(c) *$T_0(\cdot)$ is a gradient semigroup with respect to $\mathscr{S}_0 = \{\Xi_{1,0}, \Xi_{2,0}, \ldots, \Xi_{p,0}\}$;*
(d) *the family of local unstable manifold of $\Xi_{i,h}$ behaves $\mathscr{P}$-continuously at $h = 0$, that is, there exists a $\rho > 0$ such that*

$$dist_{X_h}\left(W_h^{u,\rho}(\Xi_{i,h}), P_h W_0^{u,\rho}(\Xi_{i,0})\right) + dist_{X_h}(P_h W_0^{u,\rho}\left(\Xi_{i,0}), W_h^{u,\rho}(\Xi_{i,h})\right) \to 0$$

as $h \to 0$; and
(e) *There exists a $\delta_0 > 0$ such that, for all $h \in [0, 1]$ and $j = 1, \cdots, p$, $\Xi_{j,h}$ is the maximal invariant set for the semigroup $T_h(\cdot)$ inside $\mathcal{O}_{\delta_0}(P_h \Xi_{j,0})$. Besides, δ_0 is such that the δ_0-neighborhoods of the $P_h \Xi_{j,0}$'s are disjoints.*

Then,

1. *the family $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-upper semicontinuous at $h = 0$ whenever (a) holds;*
2. *the family $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\mathscr{P}$-lower semicontinuous at $h = 0$ whenever (a), (b), (c) and (d) hold; and*
3. *there exist an $h_0 \in (0, 1]$ such that, for al $h \in [0, h_0]$, $\{T_h(\cdot)\}_{h\in(0,1]}$ is a gradient semigroup with respect to $\mathscr{S}_h = \{\Xi_{1,h}, \Xi_{2,h}, \ldots, \Xi_{p,h}\}$ whenever (a), (b), (c) and (e) hold. Consequently,*

$$\mathcal{A}_h = \bigcup_{i=1}^{p} W^u(\Xi_{i,h}), \quad \forall\, h \in [0, h_0].$$

Remark 9 It is important to note that hypothesis (e) does not follow from hypothesis (b) along with the $\mathscr{P}\mathscr{P}$-convergence of the nonlinear semigroups, since hypothesis (e) assumes that the δ_0 must be uniform for every $h \in [0, 1]$ and $j = 1, \ldots, p$.

To be more precise, although for every $h \in [0, 1]$ and $j = 1, \cdots, p$, $\Xi_{j,h}$ is an isolated invariant set, which means that for every $h \in [0, 1]$ there exists $\delta_h > 0$ such that $\Xi_{j,h}$ is the maximal invariant set for $T_h(\cdot)$ in $\mathcal{O}_{\delta_h}(\Xi_{j,h})$, and there exists $h_0 > 0$ such that $\Xi_{j,h} \subset \mathcal{O}_{\delta_0}(P_h \Xi_{j,0})$ for all $h \in [0, h_0]$ (due to hypothesis (b)), without (e) we do not get that $\Xi_{j,h}$ is the maximal invariant set for $T_h(\cdot)$ inside $\mathcal{O}_{\delta_0}(P_h \Xi_{j,0})$. Since the neighborhood $\mathcal{O}_{\delta_h}(\Xi_{j,h})$, for h small, might be lower than $\mathcal{O}_{\delta_0}(P_h \Xi_{j,0})$, so we could not say nothing about an arbitrary invariant set E_h for $T_h(\cdot)$ which is inside the difference $\mathcal{O}_{\delta_0}(P_h \Xi_{j,0}) \setminus \mathcal{O}_{\delta_h}(\Xi_{j,h})$. By (e), we can conclude that $E_h \subset \Xi_{j,h}$.

The following lemma is crucial for the development of the proof for our main result. It shows the existence of a $\mathscr{P}$-convergent subsequence of global solutions for semigroups in variable Banach spaces to a global solution for a semigroup on a fixed Banach space.

Lemma 10 *Let $\{T_h(\cdot)\}_{h\in(0,1]}$ be a family of semigroups $\mathscr{P}$-collectively asymptotically compact at $h = 0$ on the Banach spaces X_h, for $h \in [0, 1]$, such that $T_h(\cdot) \xrightarrow{\mathscr{P}\mathscr{P}} T_0(\cdot)$ uniformly on compact subsets of $\mathbb{R}^+ \times X_0$.*

Given $\{h_k\}_{k\in\mathbb{N}}$ in $(0, 1]$ with $h_k \xrightarrow{k\to\infty} 0$ and a sequence $\{J_k\}_{k\in\mathbb{N}}$ with $J_k := [-t_k, \infty)$ for some increasing sequence of positive numbers $\{t_k\}_{k\in\mathbb{N}}$ with $t_k \xrightarrow{k\to\infty} \infty$. Suppose that $\{x_k\}_{k\in\mathbb{N}}$ is a uniformly bounded sequence with $x_k \in X_{h_k}$.

Defining for each natural number k, $\xi_k : J_k \to X_{h_k}$ by $\xi_k(t) = T_{h_k}(t + t_k)x_k$, $\forall\, t \in J_k$, there is a subsequence of $(\xi_k)_{k\in\mathbb{N}}$ that $\mathscr{P}$-converges to ξ_0 uniformly on compact sets of $\mathbb{R}$, for some global solution $\xi_0 : \mathbb{R} \to X_0$ of $T_0(\cdot)$.

Proof Since the family $\{T_h(\cdot)\}_{h\in(0,1]}$ is $\mathscr{P}$-collectively asymptotically compact, let $\mathbb{N}_0$ be an infinite subset of $\mathbb{N}$ such that the sequence $\{T_{h_k}(t_k)x_k\}_{k\in\mathbb{N}_0}$ $\mathscr{P}$-converges to $z_0 \in X_0$, that is, $\lim\limits_{\substack{k\to\infty\\ k\in\mathbb{N}_0}} \|T_{h_k}(t_k)x_k - P_{h_k}z_0\|_{X_{h_k}} = 0$. Define $\xi_0^{(0)} : \mathbb{R}^+ \to X_0$ by $\xi_0^{(0)}(t) = T_0(t)z_0$.

Using the above argument, consider $\mathbb{N}_1$ an infinite subset of $\mathbb{N}_0$ such that $t_k > 1$ for all $k \in \mathbb{N}_1$ and the sequence $\{T_{h_k}(t_k-1)x_k\}_{k\in\mathbb{N}_1}$ $\mathscr{P}$-converges to $z_1 \in X_0$. Define $\xi_0^{(1)}$: $[-1,0] \to X_0$ by $\xi_0^{(1)}(t) = T_0(t+1)z_1$.

We note that if a sequence $\mathscr{P}$-converges then all its subsequences $\mathscr{P}$-converge to the same limit. Now,

$$\begin{aligned}\|\xi_0^{(1)}(0) - \xi_0^{(0)}(0)\|_{X_0} &= \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_1}} \|P_{h_k}T_0(1)z_1 - P_{h_k}z_0\|_{X_{h_k}}\\ &\le \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_1}} \|T_{h_k}(1)T_{h_k}(t_k-1)x_k - P_{h_k}T_0(1)z_1\|_{X_{h_k}}\\ &\quad + \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_1}} \|T_{h_k}(t_k)x_k - P_{h_k}z_0\|_{X_{h_k}} = 0,\end{aligned}$$

thus $\xi_0^{(1)}(0) = \xi_0^{(0)}(0)$.

Analogously, let $\mathbb{N}_2$ be an infinite subset of $\mathbb{N}_1$ such that $t_k > 2$ for all $k \in \mathbb{N}_2$ and the subsequence $\{T_{h_k}(t_k-2)x_k\}_{k\in\mathbb{N}_2}$ $\mathscr{P}$-converges to $z_2 \in X_0$.

Define $\xi_0^{(2)} : [-2,-1] \to X_0$ by $\xi_0^{(2)}(t) = T_0(t+2)z_2$.

As before, we obtain

$$\begin{aligned}&\|\xi_0^{(2)}(-1) - \xi_0^{(1)}(-1)\|_{X_0}\\ &\quad\le \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_2}} \|T_{h_k}(1)T_{h_k}(t_k-2)x_k - P_{h_k}T_0(1)z_2\|_{X_{h_k}}\\ &\quad\quad + \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_2}} \|T_{h_k}(t_k-1)x_k - P_{h_k}z_1\|_{X_{h_k}} = 0,\end{aligned}$$

then $\xi_0^{(2)}(-1) = \xi_0^{(1)}(-1)$.

Repeating the same argument, we obtain a decreasing sequence of infinite subsets of $\mathbb{N}$, that is, $\mathbb{N} \supset \mathbb{N}_0 \supset \mathbb{N}_1 \supset \cdots \supset \mathbb{N}_n \supset \cdots$, so that for each $n \in \{0,1,2,\ldots\}$; there exist $z_n \in X_0$ and $t_k > n$ for all $k \in \mathbb{N}_n$ such that the sequence $\{T_{h_k}(t_k-n)x_k\}_{k\in\mathbb{N}_n}$ $\mathscr{P}$-converges to z_n.

Define $\xi_0^{(n)} : [-n, 1-n] \to X_0$ by $\xi_0^{(n)}(t) = T_0(t+n)z_n$, and

$$\begin{aligned}\|\xi_0^{(n)}(1-n) - \xi_0^{(n-1)}(1-n)\|_{X_0} &= \lim_{h\to 0} \|P_hT_0(1)z_n - P_hz_{n-1}\|_{X_h}\\ &\le \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_n}} \|T_{h_k}(1)T_{h_k}(t_k-n)x_k - P_{h_k}T_0(1)z_n\|_{X_{h_k}}\\ &\quad + \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_{n-1}}} \|T_{h_k}(t_k-n+1)x_k - P_{h_k}z_{n-1}\|_{X_{h_k}} = 0\end{aligned}$$

Thus,

$$\xi_0^{(n)}(1-n) = \xi_0^{(n-1)}(1-n), \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

Finally put $\xi_0 : \mathbb{R} \to X_0$ by

$$\xi_0(t) := \begin{cases}\xi_0^{(0)}(t),\ t \ge 0\\ \xi_0^{(n)}(t),\ t \in [-n, 1-n],\ n \in \mathbb{N}.\end{cases} \tag{3.3}$$

We note that $\xi_0 : \mathbb{R} \to X_0$ is well defined by (3.2).

Now, we claim that $\xi_0 : \mathbb{R} \to X_0$ is a global solution for $T_0(\cdot)$.

In fact, given $t, s \geq 0$ along with the definition of ξ_0, we have

$$T_0(t)\xi_0(s) = T_0(t)T_0(s)z_0 = T_0(t+s)z_0 = \xi_0(t+s),$$

since $t + s \geq 0$.

We see that $T_0(n)z_n = z_0$ due to

$$\begin{aligned}\|T_0(n)z_n - z_0\|_{X_0} &= \lim_{h\to 0} \|P_h T_0(n)z_n - P_h z_0\|_{X_h}\\ &\leq \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_n}} \|T_{h_k}(n)T_{h_k}(t_k - n)x_k - P_{h_k}T_0(n)z_n\|_{X_{h_k}}\\ &\quad + \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_{n-1}}} \|T_{h_k}(t_k)x_k - P_{h_k}z_0\|_{X_{h_k}} = 0.\end{aligned}$$

If $s < 0$, we choose $n \in \mathbb{N}$ such that $s \in [-n, 1-n]$, therefore $\xi_0(s) = T_0(s+n)z_n$.

For $t + s \geq 0$, we have

$$T_0(t)\xi_0(s) = T_0(t)T_0(s+n)z_n = T_0(t+s)T_0(n)z_n = T_0(t+s)z_0 = \xi_0(t+s).$$

For $s + t < 0$, there is a $m \in \mathbb{N}$ such that $m \leq n$ and $s + t \in [-m, 1-m]$. Thus

$$\begin{aligned}T_0(t)\xi_0(s) &= T_0(t)T_0(s+n)z_n = T_0([t+s+m] + [n-m])z_n\\ &= T_0(t+s+m)T_0(n-m)z_n\\ &= T_0(t+s+m)z_m = \xi_0^{(m)}(t+s) = \xi_0(t+s),\end{aligned}$$

for $T_0(n-m)z_n = z_m$. Then, $\xi_0 : \mathbb{R} \to X_0$ is global solution for $T_0(\cdot)$.

On the other hand, define the set $\mathbb{N}^*$ so that its n-th element is the n-th element of $\mathbb{N}_n$, in increasing order of the natural numbers; note that $\mathbb{N}^*$ is an infinite set.

Moreover, considering the restriction $\{\xi_k\}_{k\in\mathbb{N}^*}$, it follows that $\{\xi_k\}_{k\in\mathbb{N}^*}$ is a subsequence of $\{\xi_k\}_{k\in\mathbb{N}}$ that $\mathscr{P}$-converges to ξ_0 uniformly on compact sets of $\mathbb{R}$.

In fact, let $a, b \in \mathbb{R}$ be such that $0 \leq a < b$. It follows from $T_h(\cdot) \xrightarrow{\mathscr{P}\mathscr{P}} T_0(\cdot)$ uniformly on compact subsets of $\mathbb{R}^+ \times X_0$ that

$$\begin{aligned}&\lim_{\substack{k\to\infty\\ k\in\mathbb{N}_0}} \sup_{t\in[a,b]} \|\xi_k(t) - P_{h_k}\xi_0(t)\|_{X_{h_k}}\\ &\quad= \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_0}} \sup_{t\in[a,b]} \|T_{h_k}(t+t_k)x_k - P_{h_k}T_0(t)z_0\|_{X_{h_k}}\\ &\quad= \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_0}} \sup_{t\in[a,b]} \|T_{h_k}(t)T_{h_k}(t_k)x_k - P_{h_k}T_0(t)z_0\|_{X_{h_k}} = 0.\end{aligned}$$

For any fixed $n \in \mathbb{N}$, $k \in \mathbb{N}_n$ and $t \in [-n, 1-n]$, we obtain

$$\begin{aligned}&\lim_{\substack{k\to\infty\\ k\in\mathbb{N}_n}} \sup_{t\in[-n,1-n]} \|\xi_k(t) - P_{h_k}\xi_0(t)\|_{X_{h_k}}\\ &\quad= \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_n}} \sup_{t\in[-n,1-n]} \|T_{h_k}(t+t_k)x_k - P_{h_k}T_0(t+n)z_n\|_{X_{h_k}}\\ &\quad= \lim_{\substack{k\to\infty\\ k\in\mathbb{N}_n}} \sup_{t\in[-n,1-n]} \|T_{h_k}(t+n)T_{h_k}(t_k - n)x_k - P_{h_k}T_0(t+n)z_n\|_{X_{h_k}} = 0.\end{aligned}$$

Finally, the general case follows from the fact that every compact set $K \subset \mathbb{R}$ is contained in a finite union of intervals of the kind considered above, and so the proof is complete. □

3.1 Proof of Theorem 8

(1) We will use the part (1) of Remark 6. For that, we take sequences $h_k \to 0$ and $\{x_k\}_{k\in\mathbb{N}}$ with $x_k \in \mathcal{A}_{h_k}$. Then, for each k there is a bounded global solution $\xi_k : \mathbb{R} \to X_{h_k}$ for $T_{h_k}(\cdot)$ with $\xi_k(0) = x_k$.

By Lemma 10, we get a subsequence $(\xi_{k_l})_{l\in\mathbb{N}}$ of $(\xi_k)_{k\in\mathbb{N}}$ and a bounded (this by the hypothesis (a)) global solution $\xi_0 : \mathbb{R} \to X_0$ for $T_0(\cdot)$ such that $(\xi_{k_l})_{l\in\mathbb{N}}$ $\mathscr{P}$-converges to ξ_0 uniformly on compact sets of $\mathbb{R}$.

In particular, $\xi_{k_l}(0) = x_{k_l}$ $\mathscr{P}$-converges to $\xi_0(0) =: x_0$ and being x_0 a point of $\mathcal{A}_0$, we get the $\mathscr{P}$-upper semicontinuity at $h = 0$ of the family of attractors $\{\mathcal{A}_h\}_{h\in[0,1]}$.

(2) By part (2-a) of Remark 6, since $\mathcal{A}_0$ is compact, to show this result we just need to see that, given $u_0 \in \mathcal{A}_0$ and $h_k \to 0$ there is a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ and a sequence $\{u_l\}_{l\in\mathbb{N}}$ with $u_l \in \mathcal{A}_{h_{k_l}}$, which $\mathscr{P}$-converges to u_0.

Indeed, since $u_0 \in \mathcal{A}_0$ there exists a $\ell \in \{1, 2, \ldots, p\}$ such that $u_0 \in W_0^u(\Xi_{\ell,0})$. That is why there is a global solution $\xi_0 : \mathbb{R} \longrightarrow X_0$ for $T_0(\cdot)$ with $\xi_0(0) = u_0$ and $\lim_{t\to-\infty} d_{X_0}(\xi_0(t), \Xi_{\ell,0}) = 0$.

Now, let $\rho > 0$ such that, by hypothesis (d), the family of local unstable set of $\Xi_{i,h}$ behaves $\mathscr{P}$-continuously at $h = 0$ and take $t_0 < 0$ such that $d_{X_0}(\xi_0(t), \Xi_{\ell,0}) < \rho$ for every $t \leq t_0$, then is easy to see that $\xi_0(t_0) \in W_0^{u,\rho}(\Xi_{\ell,0})$.

Being $\left(W_h^{u,\rho}(\Xi_{\ell,h})\right)_{h\in[0,1]}$ $\mathscr{P}$-lower semicontinuous at $h = 0$, by part (2-b) of Remark 6, there exists subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ and a sequence $\{y_l\}_{l\in\mathbb{N}}$ with $y_l \in W_{h_{k_l}}^{u,\rho}(\Xi_{\ell,h_{k_l}})$, which $\mathscr{P}$-converges to $\xi_0(t_0)$.

Finally, by $T_h(\cdot) \overset{\mathscr{P}\mathscr{P}}{\longrightarrow} T_0(\cdot)$ uniformly on compact subsets of $\mathbb{R}^+ \times X_0$ follows that $\mathcal{A}_{h_{k_l}} \ni T_{h_{k_l}}(-t_0)y_l$ $\mathscr{P}$-converges to $T_0(-t_0)\xi_0(t_0) = \xi_0(0) = u_0$, as we wish.

(3) First, by Theorem 2 it suffices to show that $T_h(\cdot)$ is dynamically $\mathscr{S}_h$-gradient.

We begin by proving that $(G1)$ is stable under perturbation, that is, there is an $h_1 \in (0, 1]$ such that $T_h(\cdot)$ satisfies $(G1)$ with respect to $\mathscr{S}_h$ for all $h \in [0, h_1]$.

In fact, we remark that by assumption (e) we may take $\delta \in (0, \delta_0)$ and, for this, if $h \in (0, 1]$, $\xi_h : \mathbb{R} \longrightarrow X_h$ is a global solution for $T_h(\cdot)$ that lies in $\mathcal{A}_h$ and there exists $t_0 \in \mathbb{R}$ satisfying

$$d_{X_h}(\xi_h(t), P_h\Xi_{j,0}) \leq \delta \quad \text{for all} \quad t \in [t_0, \infty), \text{ for some } \Xi_{j,0} \in \mathscr{S}_0. \tag{3.4}$$

Then, if $\omega_h(\xi_h)$ indicates the omega limite set of the solution ξ_h with respect to the semigroup $T_h(\cdot)$, we get that $\omega_h(\xi_h) \subset \overline{\mathcal{O}_\delta(P_h\Xi_{j,0})} \subset \mathcal{O}_{\delta_0}(P_h\Xi_{j,0})$ and, by the hypothesis (e), we must have $\omega_h(\xi_h) \subset \Xi_{j,h}$ what clearly means that

$$\lim_{t\to\infty} dist_{X_h}(\xi_h(t), \Xi_{j,h}) = 0. \tag{3.5}$$

Let us now show that there exists $h_1 \in (0, 1]$ such that the condition give in (3.4) is valid for every $h \in [0, h_1]$.

Indeed, arguing by contradiction we suppose there are sequence $(h_k)_{k\in\mathbb{N}}$ in $(0, 1]$ with $h_k \overset{k\to\infty}{\longrightarrow} 0$ and, for each natural k, a global solution $\xi_k : \mathbb{R} \to X_{h_k}$ for $T_{h_k}(\cdot)$ that lies in $\mathcal{A}_{h_k}$, such that

$$\sup_{t\geq t_0} d_{X_{h_k}}\left(\xi_k(t), \bigcup_{j=1}^{p} P_{h_k}\Xi_{j,0}\right) > \delta, \text{ for all } k \in \mathbb{N} \text{ and } t_0 \in \mathbb{R}. \tag{3.6}$$

We can use Lemma 10 to suppose the existence of a subsequence (which we relabel) such that $\xi_k(t)$ $\mathscr{P}$-converges to $\xi^{(0)}(t)$ uniformly for t on compacts of $\mathbb{R}$, where $\xi^{(0)}:\mathbb{R}\to X_0$ is a global solution for $T_0(\cdot)$ that lies in $\mathcal{A}_0$, this for the hyphotesis (a).

Since $T_0(\cdot)$ is dynamically $\mathscr{S}_0$-gradient, it follows that there is an isolated invariant $\Xi_{i_0,0}\in\mathscr{S}_0$ such that $\lim_{t\to\infty} d_{X_0}(\xi^{(0)}(t),\Xi_{i_0,0})=0$. Thus, by Lemma 4, we obtain, for every real t,

$$\begin{aligned}d_{X_{h_k}}(\xi_k(t),P_{h_k}\Xi_{i_0,0}) &\le \|\xi_k(t)-P_{h_k}\xi^{(0)}(t)\|_{X_{h_k}}+d_{X_{h_k}}(P_{h_k}\xi^{(0)}(t),P_{h_k}\Xi_{i_0,0})\\ &\le \|\xi_k(t)-P_{h_k}\xi^{(0)}(t)\|_{X_{h_k}}+Md_{X_0}(\xi^{(0)}(t),\Xi_{i_0,0})\overset{k\to\infty}{\longrightarrow}0.\end{aligned}$$

Then, given $r\in\mathbb{N}$ (with $\frac{1}{r}<\delta$) there exist $k_r\in\mathbb{N}$ and $t_r\in\mathbb{R}$ such that

$$d_{X_{h_k}}(\xi_k(t_r),P_{h_k}\Xi_{i_0,0})<\frac{1}{r}\quad\text{whenever}\quad k\ge k_r. \tag{3.7}$$

Also, from (3.6) it follows the existence of $t_r'>t_r$ such that

$$d_{X_{h_{k_r}}}(\xi_{k_r}(t),P_{h_{k_r}}\Xi_{i_0,0})<\delta\quad\text{for all}\quad t\in[t_r,t_r')$$

and

$$d_{X_{h_{k_r}}}\left(\xi_{k_r}(t_r'),P_{h_{k_r}}\Xi_{i_0,0}\right)=\delta.$$

We can see that $t_r'-t_r\to\infty$ as $r\to\infty$. In fact, for otherwise, we may assume that $t_r'-t_r\overset{r\to\infty}{\longrightarrow}\bar{\tau}$, for some $\bar{\tau}\ge 0$. Then, by (3.7) and $\xi_{k_r}(t_r')=T_{h_{k_r}}(t_r'-t_r)\xi_{k_r}(t_r)$, we get the existence of a point in $\Xi_{i_0,0}$ distanced $\delta>0$ to $\Xi_{i_0,0}$, but this is a contradiction.

Use Lemma 10 again to take a subsequence such that $\xi_r^{(1)}(t)$ $\mathscr{P}$-converges to $\xi^{(1)}(t)$ uniformly for t on compact sets of $\mathbb{R}$, where $\xi_r^{(1)}(t):=\xi_{k_r}(t+t_r')$ for all $t\in[-(t_r'-t_r),\infty)$ and $\xi^{(1)}:\mathbb{R}\to X_0$ a global solution for $T_0(\cdot)$ that lies in $\mathcal{A}_0$. We see that

$$\begin{aligned}d_{X_0}(\xi^{(1)}(t),\Xi_{i_0,0})&=\lim_{h\to 0}d_{X_h}\left(P_h\xi^{(1)}(t),P_h\Xi_{i_0,0}\right)\\ &\le\lim_{r\to\infty}\|P_{h_{k_r}}\xi^{(1)}(t)-\xi_r^{(1)}(t)\|_{X_{h_{k_r}}}\\ &\quad+\lim_{r\to\infty}d_{X_{h_{k_r}}}\left(\xi_r^{(1)}(t),P_{h_{k_r}}\Xi_{i_0,0}\right)\le\delta,\end{aligned} \tag{3.8}$$

for all $t\le 0$, and the property $(G1)$ of $T_0(\cdot)$ implies that

$$\lim_{t\to-\infty}d_{X_0}(\xi^{(1)}(t),\Xi_{i_0,0})=0.$$

Since $T_0(\cdot)$ satisfies $(G1)$ and $(G2)$,

we must have that $\lim_{t\to\infty}d_{X_0}(\xi^{(1)}(t),\Xi_{i_1,0})=0$, for some isolated invariant set $\Xi_{i_1,0}\in\mathscr{S}_0$ with $i_1\ne i_0$, because if it was $i_1=i_0$, since $d_{X_0}(\xi^{(1)}(0),\Xi_{i_0,0})=\delta$, $\xi^{(1)}$ would be an homoclinic solution, what cannot be.

Now, from the fact that $\xi_r^{(1)}(t)$ $\mathscr{P}$-converges to $\xi^{(1)}(t)$ uniformly for t on compact sets of $\mathbb{R}$, it follows that (analogously as in (3.1)) for each $m\in\mathbb{N}$ (with $\frac{1}{m}<\delta$) there are $r_m\in\mathbb{N}$ and $t_m\in\mathbb{R}$ such that

$$d_{X_{h_{k_r}}}\left(\xi_r^{(1)}(t_m),P_{h_{k_r}}\Xi_{i_1,0}\right)<\frac{1}{m}\quad\text{whenever}\quad r\ge r_m.$$

Again, by (3.6) we have there is a $t_m'>t_m$ so that

$$d_{X_{h_{k_{r_m}}}}\left(\xi_{r_m}^{(1)}(t),P_{h_{k_{r_m}}}\Xi_{i_1,0}\right)<\delta\ \text{ for all }\ t\in[t_m,t_m')$$

and

$$d_{X_{h_{k_{r_m}}}}\left(\xi^{(1)}_{r_m}(t'_m),\, P_{h_{k_{r_m}}}\Xi_{i_1,0}\right)=\delta.$$

Similar to what we did in the previous case, we obtain that $t'_m - t_m \to \infty$ as $m \to \infty$ and there exists a global solution $\xi^{(2)} : \mathbb{R} \to X_0$ for $T_0(\cdot)$ such that $\xi^{(2)}_m(t)$ $\mathscr{P}$-converges to $\xi^{(2)}(t)$ uniformly for t on compact sets of $\mathbb{R}$, where $\xi^{(2)}_m(t) := \xi_{r_m}(t+t'_m)$ for all $t \in [-(t'_m - t_m), \infty)$. Following the argument in (3.8), we must have $d_{X_0}(\xi^{(2)}(t), \Xi_{i_1,0}) \le \delta$ for all $t \le 0$, and along with the property $(G1)$ of $T_0(\cdot)$ implies that $\lim_{t\to-\infty} d_{X_0}(\xi^{(2)}(t), \Xi_{i_1,0}) = 0$.

Again, since $T_0(\cdot)$ satisfies $(G1)$ follows that $\lim_{t\to\infty} d_{X_0}(\xi^{(2)}(t), \Xi_{i_2,0}) = 0$, for some isolated invariant set $\Xi_{i_2,0} \in \mathscr{S}_0$. Since $T_0(\cdot)$ also satisfies $(G2)$, it must be that $i_2 \notin \{i_0, i_1\}$, for otherwise there would be a homoclinic structure for $T_0(\cdot)$ associated to $\mathscr{S}_0$.

Then we can repeat the previous argument, which should stop after a finite number of steps, as the set $\mathscr{S}_0$ is finite and, as we saw in the last step will be forced to find a homoclinic structure in the attractor $\mathcal{A}_0$ which is a contradiction. Then our initial assumption (3.4) is true.

One can prove that there is an $h_2 \in (0, h_1]$ along with the counterpart of (3.5) as $t \to -\infty$ by a similar argument.

Finally, we will prove that there is an $h_0 \in (0, h_2]$ such that $T_h(\cdot)$ satisfies $(G2)$ for all $h \in [0, h_0]$. To show this, we again argue by contradiction. Suppose there exists a sequence $h_k \to 0$ for which there exist $(\mathscr{S}'_k)_{k\in\mathbb{N}}$ of sets with $\mathscr{S}'_k := \{\Xi_{\ell_1,h_k}, \Xi_{\ell_2,h_k}, \dots, \Xi_{\ell_m,h_k}\} \subset \mathscr{S}_{h_k}$, and a sequence of global solutions $\{\xi_{k,j} : \mathbb{R} \to X_{h_k} : 1 \le j \le m\}_{k\in\mathbb{N}}$ such that $\xi_{k,j}$ is a global solution for $T_{h_k}(\cdot)$ that lie in $\mathcal{A}_{h_k}$, satisfying

$$\lim_{t\to-\infty} d_{X_{h_k}}(\xi_{k,j}(t), \Xi_{\ell_j,h_k}) = 0 \quad \text{and} \quad \lim_{t\to\infty} d_{X_{h_k}}(\xi_{k,j}(t), \Xi_{\ell_{(j+1)},h_k}) = 0,$$

where $\Xi_{\ell_{(m+1)},h_k} := \Xi_{\ell_1,h_k}$.

With this, we get for all $t \in \mathbb{R}$,

$$d_{X_{h_k}}(\xi_{k,j}(t), P_{h_k}\Xi_{\ell_j,0}) \le d_{X_{h_k}}(\xi_{k,j}(t), \Xi_{\ell_j,h_k}) + dist_{X_{h_k}}(\Xi_{\ell_j,h_k}, P_{h_k}\Xi_{\ell_j,0}) \overset{k\to\infty}{\longrightarrow} 0.$$

Changing to a subsequence if necessary, we can assume that for each $k \in \mathbb{N}$ and every $j = 1, 2, \dots, m-1$ there is a real $t_k^{(j+1)}$ such that

$$d_{X_{h_k}}\left(\xi_{k,j}(t_k^{(j+1)}),\, P_{h_k}\Xi_{\ell_j,0}\right) < \frac{1}{k}$$

and note that for all k and j, we see that there is also $t_k^{(j+1)'} > t_k^{(j+1)}$ such that

$$d_{X_{h_k}}(\xi_{k,j}(t), P_{h_k}\Xi_{\ell_j,0}) < \delta, \ \text{for all } t \in [t_k^{(j+1)}, t_k^{(j+1)'})$$

and

$$d_{X_{h_k}}\left(\xi_{k,j}\left(t_k^{(j+1)'}\right),\, P_{h_k}\Xi_{\ell_j,0}\right) = \delta.$$

Because, otherwise, we should have $d_{X_{h_k}}(\xi_{k,j}(t), P_{h_k}\Xi_{\ell_j,0}) < \delta < \delta_0$ for all $t \in \mathbb{R}$ therefore, by maximality of Ξ_{j,h_k} in $\mathcal{O}_{\delta_0}(P_{h_k}\Xi_{l_j,0})$ given by the hypothesis (e), we get $\xi_{k,j}(t) \in \Xi_{\ell_j,h_k}$ for all real t, what is in contradiction to the fact that $(\mathscr{S}'_k)_{k\in\mathbb{N}}$ with $\{\xi_{k,j} : \mathbb{R} \to X_{h_k} : 1 \le j \le m\}_{k\in\mathbb{N}}$ define an homoclinic structure.

In this way, we have created all necessary conditions for using the same argument we have used to conclude the stability of $(G1)$ and thus the theorem is proved. □

Remark 3 The *Fundamental Theorem of Dynamical Systems* [39] states that every dynamical system on a compact metric space (in our case, the one defined on a global attractor) has a geometrical structure described by a (finite or countable) number of sets $\{\Xi_i\}_{i\in I}$ with an intrinsic recurrent dynamics and gradient-like dynamics outside them. In other words, the global attractor can be always described by a (finite or countable) number of invariants and connections between them. Thus, our theory would include the case of periodic orbits, homoclinic structures joining a set of equilibria, or even invariants inside the attractor with a chaotic behaviour. In applications, the problem with all of this kind of invariant structures is how to prove their robustness under perturbation. Hyperbolicity of equilibria has been show to be robust under autonomous and non-autonomous perturbation (see [16,33]) and normally hyperbolic periodic orbits are stable under autonomous perturbation (see, for example, [28]). But even in this last case the proof of the persistence of associated stable and unstable manifolds (needed for any result on, for instance, lower semicontinuity of attractors), is something, although expected, to be done in full development. In this work we will apply our main results to the case where the attractor is made of a finite set of hyperbolic equilibria, for which a perturbation leads to some projected spaces (see Sect. 4). Application of our results to more general attractors, as the ones described in [39] or [21] in finite dimensions, or as in [40,42] for infinite dimensional dynamical systems, would be very welcome, but out of the aims of the present paper.

4 Applications

In this section we study the application of our abstract results to parabolic equations in dumbbell domains and an abstract PDE with discretization by the finite element method.

4.1 Parabolic Equations in Dumbbell Domains

We consider the evolution equation of parabolic (see [3–5]) type of the form

$$\begin{cases} u_t^\epsilon(x,t) - \Delta u^\epsilon(x,t) + u^\epsilon(x,t) = f(u^\epsilon(x,t)), \quad x \in \Omega_\epsilon,\ t>0, \\ \frac{\partial u^\epsilon(x,t)}{\partial n} = 0, \quad x \in \partial\Omega_\epsilon, \end{cases} \tag{4.1}$$

where $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geq 2$, is a bounded smooth domain, $\epsilon \in (0,1]$ is a parameter, $\frac{\partial}{\partial n}$ is the outside normal derivative and $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable function which is bounded and has bounded derivatives up to the second order.

The domain Ω_ϵ is a dumbbell type domain (see Fig. 1) consisting of two disconnected domains, that we denote by Ω, joined by a thin channel, R_ϵ. Under standard dissipative assumption on the nonlinearity f of the type,

$$\limsup_{|s|\to+\infty} \frac{f(s)}{s} < 0,$$

Eq. (4.1) has an attractor $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$, for $\epsilon \in (0,1]$.

On the other hand, the limit domain consist of the domain Ω and a line in between. We denote by P_0 and P_1 the points where the line segment touches Ω, see Fig. 2.

The limiting equation is

$$\begin{cases} w_t(x,t) - \Delta w(x,t) + w(x,t) = f(w(x,t)), \quad x \in \Omega,\ t > 0, \\ \frac{\partial w(x,t)}{\partial n} = 0, \quad x \in \partial\Omega, \\ v_t(s,t) - Lv(s,t) + v(s,t) = f(v(s,t)), \quad s \in R_0, \\ v(0) = w(P_0),\ v(1) = w(P_1), \end{cases} \tag{4.2}$$

where w is a function that lives in Ω and v lives in the line segment R_0. Moreover, L is a differential operator depending on the way the channel R_ϵ collapses to the segment line R_0, i.e., $Lv = \frac{1}{g}(gv_x)_x$ where g will be defined below. Again, this system has an attractor $\mathcal{A}_0$ in $H^1(\Omega) \times H^1(R_0) =: H^1(\Omega_0)$.

Definition 18 A dumbbell domain Ω_ϵ consists of a fixed domain Ω attached to a thin handle R_ϵ that approaches a line segment as the parameter ϵ approaches zero; that is, $\Omega_\epsilon = \Omega \cap R_\epsilon$. More precisely, let $\mathbb{R}^N$, with $N \geq 2$, Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$, $N \geq 2$, be a fixed open bounded and smooth domain such that there is an $l > 0$ satisfying

$$\Omega \cap \left\{(s,x') : s^2 + |x'|^2 < l^2\right\} = \left\{(s,x') : s^2 + |x'|^2 < l^2,\ s < 0\right\},$$
$$\Omega \cap \left\{(s,x') : (s-1)^2 + |x'|^2 < l^2\right\} = \left\{(s,x') : (s-1)^2 + |x'|^2 < l^2,\ s > 1\right\},$$
$$\Omega \cap \left\{(s,x') : 0 < s < 1, |x'| < l\right\} = \emptyset,$$

with $\left\{(0,x') : |x'| < l\right\} \cup \left\{(1,x') : |x'| < l\right\} \subset \partial\Omega$. We are using the standard notation $\mathbb{R}^N \ni x = (s,x')$, with $s \in \mathbb{R}$, $x' = (x_2,\ldots,x_N) \in \mathbb{R}^{N-1}$.

The channel that we consider will be defined as $R_\epsilon = \{(s,\epsilon x') : (s,x') \in R_1\}$ and R_1 is a smooth domain given by $R_1 = \{(s,x') : 0 \leq s \leq 1,\ x' \in \Gamma_1^s\}$ and for all $s \in [0,1]$, Γ_1^s is diffeomorphic to the unit ball in $\mathbb{R}^{N-1}$. That is, we assume that for each $s \in [0,1]$, there exists a C^1 dipheomorphism $L_s : B(0,1) \to \Gamma_1^s$. Moreover, if we define

$$\begin{aligned} L : (0,1) \times B(0,1) &\to R_1 \\ (s,z) &\to (s, L_s(z)) \end{aligned} \tag{4.3}$$

then L is a C^1 dipheomorphism.

The function $[0,1] \ni s \mapsto g(s) := |\Gamma_1^s| \in \mathbb{R}$, where $|\Gamma_1^s|$ denotes the $(N-1)$-dimensional Lebesgue measure of the set Γ_1^s. From the smoothness of R_1, we may assume that g is a smooth function defined in $[0,1]$. In particular, there exist $d_0, d_1 > 0$ such that $d_0 \leq g(s) \leq d_1$ for all $s \in [0,1]$. Moreover, fixed $\epsilon > 0$, $g(s) = \epsilon^{1-N}|\Gamma_\epsilon^s|$, for all $s \in [0,1]$ and the channel R_ϵ collapses to the line segment $R_0 = \{(s,0) : 0 \leq s \leq 1\}$.

Remark 11 A very important class of channels will be those whose transversal sections Γ_1^s are disks centered at the origin of radius $r(s)$, that is

$$R_1 = \{(s,x'), |x'| < r(s), 0 \leq s \leq 1\}.$$

For this channel, $g(s) = \omega_{N-1} r(s)^{N-1}$ where ω_{N-1} is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.

The dumbbell domain will be the domain $\Omega_\epsilon = \Omega \cup R_\epsilon$ for $\epsilon \in (0,1]$. Observe that we did not specify any connectedness property for Ω. Therefore we can have the situation described in Fig. 1 or, for instance, as in Fig. 3. Now, we define appropriate spaces U_ϵ^p, for $1 < p < \infty$ and $\epsilon \in [0,1]$ as follows, $U_\epsilon^p := L^p(\Omega_\epsilon)$, for $\epsilon \in (0,1]$ with norm

$$\|\cdot\|_{U_\epsilon^p} := \|\cdot\|_{L^p(\Omega)} + \epsilon^{\frac{1-N}{p}} \|\cdot\|_{L^p(R_\epsilon)}$$

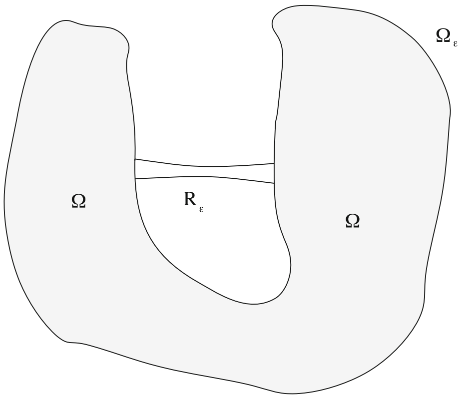


Fig. 3 Dumbbell domain with a connected Ω

and $U_0^p = L^p(\Omega) \oplus L^p(R_0)$ with norm

$$\|(w, v)\|_{U_0^p} := \|w\|_{L^p(\Omega)} + \|v\|_{L_g^p(0,1)},$$

where $L_g^p(0, 1)$ is the space $L^p(0, 1)$ with the norm

$$\|v\|_{L_g^p(0,1)} := \left(\int_0^1 g(s)|v(s)|^p \mathrm{d}s \right)^{\frac{1}{p}}.$$

We will also consider the spaces $H_\epsilon^1 = H^1(\Omega) \oplus H^1(R_\epsilon)$ with the norm

$$\| \cdot \|_{H_\epsilon^1} = \| \cdot \|_{H^1(\Omega)} + \epsilon^{\frac{1-N}{2}} \| \cdot \|_{H^1(R_\epsilon)}$$

and $H_0^1 = H^1(\Omega) \oplus H^1(R_0)$.

Since both Eqs. (4.1) and (4.2) are posed in different space domains, the first one in Ω_ϵ and the second one in $\Omega \cup R_0$, we need to devise a tool to compare functions defined in these two different sets. A tool given in [2,3] is to define the extension operator $E_\epsilon : U_0^p \to U_\epsilon^p$ as follows

$$E_\epsilon(w, v)(x) := \begin{cases} w(x), \ x \in \Omega, \\ v(s), \ x = (s, y) \in R_\epsilon. \end{cases}$$

With this, problem (4.1) can be written as a semilinear abstract equation of form

$$\begin{cases} u_t^\epsilon + A_\epsilon u^\epsilon = F_\epsilon(u^\epsilon), \quad t > 0 \\ u^\epsilon(0) = u_0^\epsilon \in U_\epsilon^p \end{cases} \tag{4.4}$$

for family of spaces U_ϵ^p, where $A_\epsilon : D(A_\epsilon) \subset U_\epsilon^p \to U_\epsilon^p$, for $\epsilon \in (0, 1]$. Also, problem (4.2) can also be written as

$$\begin{cases} u_t + A_0 u = F_0(u), \quad t > 0 \\ u(0) = u_0 \in U_0^p \end{cases} \tag{4.5}$$

in space U_0^p, where $A_0 : D(A_0) \subset U_0^p \to U_0^p$. The nonlinearity $F_\epsilon : U_\epsilon \to U_\epsilon$ is the Nemitskĭi operator generated by f, that is $F_\epsilon(u^\epsilon)(x) = f(u^\epsilon(x))$.

Proposition 4 ([3]) *For* $\epsilon \in (0, 1]$, $E_\epsilon : U_0^p \to U_\epsilon^p$ *is a bounded linear operator and*

$$\|E_\epsilon(w, v)\|_{U_\epsilon^p} = \|(w, v)\|_{U_0^p} \quad \textit{for all} \quad (w, v) \in U_0^p.$$

We also define the projection operator $M_\epsilon : U^p_\epsilon \to U^p_0$ given by $M_\epsilon(\psi_\epsilon) = (w_\epsilon, v_\epsilon)$ with $w_\epsilon(x) = \psi_\epsilon(x)$, $x \in \Omega$ and $v_\epsilon(s) = T^s_\epsilon \psi_\epsilon$, $s \in (0, 1)$, where

$$T^s_\epsilon \psi_\epsilon(x) = \frac{1}{|\Gamma^s_\epsilon|} \int_{\Gamma^s_\epsilon} \psi_\epsilon(s, y)\mathrm{d}y, \quad \Gamma^s_\epsilon = \{y : (s, y) \in R_\epsilon\}.$$

Proposition 5 ([3]) *For* $\epsilon \in (0, 1]$, $M_\epsilon \in \mathscr{L}(U^p_\epsilon, U^p_0)$ *and* $\|M_\epsilon(w, v)\|_{\mathscr{L}(U^p_\epsilon, U^p_0)} \leq 1$. *Moreover,* $M_\epsilon \circ E_\epsilon = I_{U^p_0}$.

We can see that the family of linear operators $E_\epsilon : U^p_0 \to U^p_\epsilon$ satisfy the property

$$\|E_\epsilon u\|_{U^p_\epsilon} \xrightarrow{\epsilon \to 0} \|u\|_{U^p_0}, \quad \text{for all } u \in U^p_0, \tag{4.6}$$

by using of Proposition 4.

Definition 19 We say that a sequence $\{u_\epsilon\}_{\epsilon \in (0,1]}$, $u_\epsilon \in U^p_\epsilon$, E-converges to $u_0 \in U^p_0$ if $\|u_\epsilon - E_\epsilon u\|_{U^p_\epsilon} \xrightarrow{\epsilon \to 0} 0$. We write this as $u_\epsilon \xrightarrow{E} u_0$.

With all of this, the continuity of attractors for (4.1) and (4.2) has been studied in [3–5]. On the other hand, by the theory in [2], the authors obtained (see [2, Theorem 8.4]) a general result on the rate of convergence of local unstable manifolds and attractors. These results apply to our dumbbell domain model, as shown in the next theorem.

Theorem 12 *Let* $T_\epsilon(\cdot)$ *the solution operator associated to (4.1) and (4.2) and* $\mathcal{A}_\epsilon$ *be its global attractor,* $\epsilon \in [0, 1]$. *Then, there are* $\epsilon_0 > 0$, $L > 0$, $\beta > 0$, $\gamma \in (0, 1)$ *and* $C > 0$ *such that*

(a)

$$\|T_\epsilon(t)u_\epsilon - E_\epsilon T_0(t)M_\epsilon v_\epsilon\|_{L^p(\Omega_\epsilon)} \leq Ce^{\beta t} t^{-\gamma} \left(\|u_\epsilon - v_\epsilon\|_{U^p_\epsilon} + \epsilon^{\frac{\theta N}{p}}\right),$$
$$\|T_\epsilon(t)u_\epsilon - E_\epsilon T_0(t)M_\epsilon v_\epsilon\|_{U^p_\epsilon} \leq Ce^{\beta t} t^{-\gamma} \left(\|u_\epsilon - v_\epsilon\|_{U^p_\epsilon} + \epsilon^{\frac{\theta}{p}}\right),$$

for each $p > N$, $\theta \in (1/2, 2p/(N + 2p))$, *for all* $t > 0$.

(b) If all equilibrium points $\mathcal{E}_0 = \{u^{1,0}_*, \ldots, u^{\mathfrak{n},0}_*\}$ *of (4.2) are hyperbolic (hence there are only a finitely many of them), the semigroup* $\{T_\epsilon(t) : t \geq 0\}$ *has a set of exactly* $\mathfrak{n}$ *equilibria,* $\mathcal{E}_\epsilon = \{u^{1,\epsilon}_*, \ldots, u^{\mathfrak{n},\epsilon}_*\}$, *all of them hyperbolic, for* $p > N$, $\|u^{i,\epsilon}_* - E_\epsilon u^{i,0}_*\|_{L^p(\Omega_\epsilon)} \leq C\epsilon^{\frac{N}{p}}$ *and* $\|u^{i,\epsilon}_* - E_\epsilon u^{i,0}_*\|_{U^p_\epsilon} \leq C\epsilon^{\frac{1}{p}}$, $1 \leq i \leq \mathfrak{n}$.

(c) There is a $\rho > 0$ *such that, if* $W^u_\rho(u^{i,\epsilon}_*) = W^u(u^{i,\epsilon}_*) \cap B^{L^p(\Omega_\epsilon)}_\rho(u^{i,\epsilon}_*)$ *(or* $W^u_\rho(u^{i,\epsilon}_*) = W^u(u^{i,\epsilon}_*) \cap B^{U^p_\epsilon}_\rho(u^{i,\epsilon}_*)$*), there is a* $C_\theta > 0$ *such that*

$$dist^{L^p(\Omega_\epsilon)}\left(W^u_\rho\left(u^{i,\epsilon}_*\right), E_\epsilon W^u_\rho\left(u^{i,0}_*\right)\right) + dist^{L^p(\Omega_\epsilon)}\left(E_\epsilon W^u_\rho\left(u^{i,0}_*\right), W^u_\rho\left(u^{i,\epsilon}_*\right)\right) \leq C_\theta \epsilon^{\frac{\theta N}{p}},$$
$$\left(or\, dist^{U^p_\epsilon}\left(W^u_\rho\left(u^{i,\epsilon}_*\right), E_\epsilon W^u_\rho\left(u^{i,0}_*\right)\right) + dist^{U^p_\epsilon}\left(E_\epsilon W^u_\rho\left(u^{i,0}_*\right), W^u_\rho\left(u^{i,\epsilon}_*\right)\right) \leq C_\theta \epsilon^{\frac{\theta}{p}}\right),$$

where $dist^X(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_X$ *is the Hausdorff semi-distance between the subsets* A *and* B *of the Banach space* X.

We now present the main result of this section.

Theorem 13 *Under the same conditions of Theorem 12. Suppose that* $T_0(\cdot)$ *is a gradient nonlinear semigroup in* U_0^p *with respect to* $\mathcal{E}_0 = \{u_*^{1,0}, \ldots, u_*^{\mathfrak{n},0}\}$*, then there is an* $\epsilon_0 > 0$ *such that the family* $\{T_\epsilon(\cdot)\}_{\epsilon\in[0,1]}$ *of nonlinear semigroups in* U_ϵ^p *is gradient with respect to* $\mathcal{E}_\epsilon = \{u_*^{1,\epsilon}, \ldots, u_*^{\mathfrak{n},\epsilon}\}$*, for all* $\epsilon \in [0, \epsilon_0]$*. Consequently,*

$$\mathcal{A}_\epsilon = \bigcup_{i=1}^{\mathfrak{n}} W^{\mathrm{u}}(u_*^{i,\epsilon}), \quad \forall\, \epsilon \in [0, \epsilon_0].$$

Proof We need to prove the hypotheses of Theorem 8.

EE-convergence of $T_\epsilon(\cdot)$ to $T_0(\cdot)$ on compact subsets of $\mathbb{R}^+ \times U_0^p$, follows directly from the item (a) of Theorem 12 by writing $v_\epsilon = E_\epsilon u_0$, where u_ϵ E-converges to u_0.

Now, let us prove the E-collectively asymptotically compactness at $\epsilon = 0$. We know that the nonlinear semigroup $T_0(t) : U_0^p \to U_0^p$ is compact for $t > 0$ (see [4, p. 198]), so that for every bounded sequence in U_0^p, say $\{u^k\}_{k\in\mathbb{N}}$ in B_0 with $B_0 \subset U_0^p$ bounded, it follows that there are a subsequence of $\{T_0(t)u^k\}_{k\in\mathbb{N}}$ (denoted by the same) and some $u^0 \in U_0^p$ such that $\|T_0(t)u^k - u^0\|_{U_0^p} \to 0$ as $k \to \infty$. From item (a) of Theorem 12 and [5, Remark 4.5], that is, $\beta < 0$, we obtain

$$\begin{aligned}\|T_\epsilon(t)v_\epsilon - E_\epsilon u^0\|_{U_\epsilon^p} &\leq Ce^{\beta t}t^{-\gamma}\epsilon^{\frac{\theta}{p}} + \|E_\epsilon T_0(t)M_\epsilon v_\epsilon - E_\epsilon u^0\|_{U_\epsilon^p}\\ &\leq Ce^{\beta t}t^{-\gamma}\epsilon^{\frac{\theta}{p}} + \|T_0(t)M_\epsilon v_\epsilon - u^0\|_{U_0^p}.\end{aligned}$$

Thus, given sequences $\epsilon_k \to 0$, $t_k \to \infty$ as $k \to \infty$ and $\{v_{\epsilon_k}\}_{k\in\mathbb{N}}$ in B_{ϵ_k} a bounded subset of $U_{\epsilon_k}^p$, the compact asymptotic E-collectivity at $\epsilon = 0$ follows by taking $u^k = M_{\epsilon_k}v_{\epsilon_k}$ along with Proposition 5.

The existence of global attractor $\mathcal{A}_\epsilon \subset U_\epsilon^p$ for $T_\epsilon(\cdot)$, $\epsilon \in [0, 1]$, was shown in [4,5]. In general, the attractors $\mathcal{A}_\epsilon$ lie in more regular spaces. In particular, they lie in U_ϵ^∞, $\epsilon \in [0, 1]$. With this, the condition (a) of Theorem 8 holds.

Clearly, item (b) of Theorem 8 is valid due to item (b) of Theorem 12 (for more details see [3]).

From item (c) of Theorem 12 follows condition (d) of Theorem 8.

To see hypothesis (e) of Theorem 8 observe that, by Theorem 12, all of the equilibrium points $\mathcal{E}_\epsilon = \{u_*^{1,\epsilon}, \ldots, u_*^{\mathfrak{n},\epsilon}\}$, for the semigroup $T_\epsilon(\cdot)$ in U_ϵ^p are hyperbolic, and by estimate $\|u_*^{i,\epsilon} - E_\epsilon u_*^{i,0}\|_{U_\epsilon^p} \leq C\epsilon^{\frac{1}{p}}$, $1 \leq i \leq \mathfrak{n}$, we can see that $u_*^{i,\epsilon}$ is the maximal invariant set in $\mathcal{O}_{\delta_0}(E_\epsilon u_*^{i,0})$ for $\delta_0 > 0$ small enough. Because in this neighborhood $u_*^{i,\epsilon}$ is the only global solution for $T_\epsilon(\cdot)$, if D_ϵ is an invariant set for $T_\epsilon(\cdot)$ inside $\mathcal{O}_{\delta_0}(E_\epsilon u_*^{i,0})$, it is an union of global solutions $\xi_\epsilon : \mathbb{R} \longrightarrow U_\epsilon^p$ for $T_\epsilon(\cdot)$ (by using Proposition 1), therefore each one of these solutions ξ_ϵ must be equal to $u_*^{i,\epsilon}$, consequently $D_\epsilon = \{u_*^{i,\epsilon}\}$ and (e) is satisfied. Thus, conditions of Theorem 8 are satisfied, then the result follows. □

4.2 Discretization by the Finite Element Method

In this section we consider the application of Theorem 8 to certain approximation schemes via finite element for abstract semilinear parabolic problems (see [10,19,24,25,35]). Consider the boundary value problem (BVP)

$$\begin{cases} u_t = Lu + f(u), \quad t > 0,\ x \in \Omega \\ u = 0, \quad t > 0,\ x \in \partial\Omega, \\ u(x, 0) = u^0(x). \end{cases} \tag{4.7}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded smooth domain, $u^0 \in H_0^1(\Omega)$, L is a second order elliptic operator

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) - \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} - (c(x)+\lambda)u, \tag{4.8}$$

with smooth coefficients a_{ij}, b_j, c and a dissipative nonlinearity f. The parameter λ will be specified below. We assume that L is a uniformly strongly elliptic operator, that is, there is a constant $\vartheta > 0$, such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \vartheta\left(\sum_{k=1}^{n}\xi_k^2\right),\ \forall\, x \in \Omega,\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{4.9}$$

As usual, problem (4.7) can be written as an abstract evolution equation in the Hilbert space $X = L^2(\Omega)$ given by

$$\begin{cases} \dot{u} + Au = F(u),\ t > 0, \\ u(0) = u^0 \in X^{1/2}, \end{cases} \tag{AP}$$

where the operator $A : D(A) \subset X \to X$ is given by $Au = -Lu$ for all $u \in D(A) = H^2(\Omega)\cap H_0^1(\Omega)$ and Nemitskii's operator $F : X^{1/2} \to X$ associated to f, that is, $F(u(t))x = f(u(t,x))$. We know that A is a sectorial operator in X, then $-A$ generates an analytic and compact C_0-semigroup $\{e^{-At} : t \geq 0\}$. Assume that λ is chosen such that the spectrum of A is located to the right of the imaginary axis with $\mathrm{Re}\,\sigma(\mathrm{A}) > 0$. Then, we can define the fractional powers A^α of A and the corresponding fractional power spaces $X^\alpha := D(A^\alpha)$ endowed with the graph norm, $\alpha \in [0, \infty)$. It is well know that $X^1 = D(A)$, $X^0 = L^2(\Omega)$ and $X^{1/2} = H_0^1(\Omega)$.

Define u^* a equilibrium point of (AP) if $u^* \in D(A)$ such that u^* is a solution of

$$Au^* = F(u^*). \tag{4.10}$$

Denote by $\mathcal{E}$ the set of solutions of (4.10).

We assume that $f : \mathbb{R} \to \mathbb{R}$ is a $C^2(\mathbb{R})$ function and

$$|f'(s)| \leq C(1 + |s|^{\gamma-1}), \tag{C}$$

for all $s \in \mathbb{R}$ where $1 \leq \gamma < \frac{n+2}{n-2}$ if $n \geq 3$ and $\gamma \geq 1$ if $n = 2$.

Under this growth condition the problem (AP) is locally well posed in $X^{1/2}$ and F is locally Lipschitz continuous and Frechet differentiable (see [8,23,36]). Following the ideas in [23,36], one can show that

$$\|F(u) - F(v)\|_X \leq C(R)\|u - v\|_{X^{1/2}}, \tag{4.11}$$

and

$$\|F'(u) - F'(v)\|_{\mathscr{L}(X^{1/2},X)} \leq C(R)\|u - v\|_{X^{1/2}} \tag{4.12}$$

for any $u, v \in B_R := \{z \in X^{1/2} : \|z\|_{X^{1/2}} \leq R\}$.

If in addition we assume that

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} \leq 0 \tag{D}$$

then, all solutions of (AP) are globally defined (see [20]). In this case we have, for each $u^0 \in X^{1/2}$, a globally defined solution $t \mapsto u(t,u^0) \in X^{1/2}$, $t \geq 0$, which defines the compact nonlinear semigroup $\{T(t) : t \geq 0\}$ where $T(t)u^0 = u(t,u^0)$, $t \geq 0$.

Under the above hypotheses (see [7,20]) the nonlinear semigroup $\{T(t) : t \geq 0\}$ associated to (AP) has a global attractor $\mathcal{A}$ in $X^{1/2}$ and the attractor $\mathcal{A}$ satisfies

$$\sup_{u\in\mathcal{A}} \|u\|_{L^\infty(\Omega)} < \infty. \tag{4.13}$$

This bound enables us to cut off the nonlinearity f in such a way that it becomes bounded with bounded derivatives up to second order. Now, as the operator A is sectorial, it can be associated with a sesquilinear form $\sigma(\cdot,\cdot) : X^{1/2} \times X^{1/2} \to \mathbb{C}$ such that

$$\sigma(u,v) = \langle Au, v\rangle_X, \quad u \in D(A),\ v \in X^{1/2}, \tag{4.14}$$

$$|\sigma(u,v)| \leq c_1\|u\|_{X^{1/2}}\|v\|_{X^{1/2}}, \quad u, v \in X^{1/2} \tag{4.15}$$

$$\operatorname{Re}\sigma(u,u) \geq c_2\|u\|^2_{X^{1/2}}, \quad u \in X^{1/2}, \tag{4.16}$$

where the constants $c_1, c_2 = \vartheta/2$ are positive. Also, there are constants $\theta_1 \in (0,\pi/2)$ and $M_1 > 0$ such that

$$\mathcal{S}_{0,\theta_1} = \{z \in \mathbb{C} : \theta_1 \leq |\arg(z)| \leq \pi, z \neq 0\} \subset \rho(A) \quad \text{and} \tag{4.17}$$

$$\|(z-A)^{-1}\|_{\mathscr{L}(X)} \leq \frac{M_1}{|z|}, \quad \forall\, z \in \mathcal{S}_{0,\theta_1}. \tag{4.18}$$

With the following hypotheses (see [24, Assumption 3.1]) we can now make the discretization to problem (AP):

Assumption 1 Let $\Omega \subset \mathbb{R}^n$ be a polyhedral domain with $n \geq 2$ which has $\{\mathcal{T}^h\}_{h\in(0,1]}$ a quasi-uniform family of subdivisions with positive constant ρ. Let $(K, \mathcal{P}, \mathcal{N})$ be a reference element of class C^0, satisfying K is star-shaped with respect to some ball, $\mathcal{P}_1 \subseteq \mathcal{P} \subseteq W^{2,\infty}(K)$ and $\mathcal{N} \subseteq \big(C(\overline{K})\big)'$.

From this assumption, we obtain that the space

$$X_h^{1/2} := \{\mathcal{I}^h v : v \in C(\overline{\Omega}), v\big|_{\partial\Omega} = 0\} \subset X^{1/2} \cap C(\Omega), \tag{4.19}$$

has finite dimension, where $\mathcal{I}^h$ is the interpolation operator. Moreover, there exists positive constants C and $\hat{C}$ (see [13, Theorems 4.4.20 and 4.5.11]) such that

$$\|v - \mathcal{I}^h v\|_X + h\|v - \mathcal{I}^h v\|_{X^{1/2}} \leq Ch^2\|v\|_{X^1},\ \forall\, v \in X^1 \ \text{ and} \tag{4.20}$$

$$\|v_h\|_{X^{1/2}} \leq \hat{C}h^{-1}\|v_h\|_{L^2(\Omega)},\ \forall\, v_h \in X_h^{1/2}. \tag{4.21}$$

We define the orthogonal projection $P_h : X \to X_h^{1/2}$ by

$$\langle P_h u, \phi_h\rangle_X = \langle u, \phi_h\rangle_X, \quad \forall\, \phi_h \in X_h^{1/2}. \tag{4.22}$$

The operator P_h is bounded (uniformly in h) with respect to the X-norm and the $X^{1/2}$-norm (see [24, Lemmata 2 and 5]).

Lemma 14 *If* (4.15) *and* (4.16) *hold, then there is a positive constant* C *such that*

$$\|P_h v - v\|_{X^s} \leq Ch^{2-2s}\|v\|_{X^1(\Omega)}, \forall\, v \in X \text{ and } s = 0, 1/2. \tag{4.23}$$

Proof See [24, Lemma 3]. □

Definition 20 We say that a sequence $\{u_h\}_{h\in(0,1]}$, $u_h \in X_h^{1/2}$, $\mathscr{P}$-converges to $u \in X$ if $\|u_h - P_h u\|_X \overset{h\to 0}{\longrightarrow} 0$. We write this as $u_h \overset{\mathscr{P}}{\longrightarrow} u$.

In this framework, the finite element approximation $A_h : X_h^{1/2} \to X_h^{1/2}$ of the operator A is given by

$$\langle A_h\phi_h, \psi_h\rangle_X = \sigma(\phi_h, \psi_h), \quad \phi_h, \psi_h \in X_h^{1/2}. \tag{4.24}$$

That is, the operator A_h is associated with the sesquilinear form $\sigma_h(\cdot,\cdot)$ which is the restriction of $\sigma(\cdot,\cdot)$ to $X_h^{1/2} \times X_h^{1/2}$. Then the problem (AP) can be discretized in the following form

$$\begin{cases} \dot{u}_h + A_h u_h = F_h(u_h) \\ u_h(0) = u_h^0 \in X_h^{1/2}, \end{cases} \tag{AP$_h$}$$

where $F_h := P_h F : X_h^{1/2} \to X_h^{1/2}$, for all $h \in (0,1]$ (for instance, see [25, §5]).

Let $A_h : D(A_h) \subseteq X_h^{1/2} \to R(A_h) \subseteq X_h^{1/2}$ be the approximation operator of the operator A in the space $X_h^{1/2}$ for $h \in (0,1]$, defined by (4.24).

The family of equilibrium points $\{u_h^*\}_{h\in(0,1]}$ of the discrete problem (AP$_h$) is given for $u_h \in X_h^{1/2}$ satisfying

$$A_h u_h^* = F_h(u_h^*), \ \forall\, h \in (0,1]. \tag{4.25}$$

As before, denote by $\mathcal{E}_h$ the set of solutions of (4.25) in $X_h^{1/2}$.

We denote $N_h := \dim(X_h^{1/2}) < \infty$. We can see that $\mathrm{Ker}(A_h) = \{0\}$ by using (4.16). Then, from Rank-Nullity Theorem (see [34, p. 17]), we have that $D(A_h) = X_h^{1/2} = R(A_h)$. Hence $A_h : X_h^{1/2} \to X_h^{1/2}$ is a linear bijection.

Theorem 15 *The linear operator A_h is sectorial in $X_h^{1/2}$, for all $h \in (0,1]$.*

Proof From the construction of space $X_h^{1/2}$, the sesquilinear form σ satisfies (4.15) and (4.16) for elements in $X_h^{1/2}$. Thus, by [50, Theorem 2.1] the result follows.

On the other hand, we need other orthogonal projection $\tilde{P}_h : X^{1/2} \to X_h^{1/2}$, which associates each function $v \in X^{1/2}$ with a function $\tilde{P}_h v \in X_h^{1/2}$, so that the difference $\tilde{P}_h v - v$ on the finite element space $X_h^{1/2}$ is perpendicular with respect to the $X^{1/2}$-inner product. For this reason, we introduce a *Ritz or elliptic projection* operator $\tilde{P}_h : X^{1/2} \to X_h^{1/2}$ with respect to $X^{1/2}$-inner product given by

$$\sigma(\tilde{P}_h u, \phi_h) = \sigma(u, \phi_h), \ \forall\, \phi_h \in X_h^{1/2}. \tag{4.26}$$

The operator $\tilde{P}_h$ is bounded (uniformly in h) with respect to the $X^{1/2}$-norm (see [24, Lemma 4]).

Theorem 16 *If Assumption 1 holds, then there exists a constant $C > 0$ such that*

$$\|u - \tilde{P}_h u\|_X \le Ch\|u\|_{X^{1/2}}, \quad u \in X^{1/2}, \tag{4.27}$$

$$\|u - \tilde{P}_h u\|_{X^{1/2}} \le Ch\|u\|_{X^1}, \quad u \in X^1, \tag{4.28}$$

$$\|u - \tilde{P}_h u\|_X \le Ch^2\|u\|_{X^1}, \quad u \in X^1. \tag{4.29}$$

Proof See [24, Theorem 4]. □

Definition 21 We say that a sequence $\{u_h\}_{h\in(0,1]}$, $u_h \in X_h^{1/2}$, $\tilde{\mathscr{P}}$-converges to $u \in X^{1/2}$ if $\|u_h - \tilde{P}_h u\|_{X^{1/2}} \overset{h\to 0}{\longrightarrow} 0$. We write this as $u_h \overset{\tilde{\mathscr{P}}}{\longrightarrow} u$.

With relation to the resolvent operators of A and A_h the following result holds.

Theorem 17 *Under Assumption 1, there exist a positive constant C and an acute angle θ_1 such that for any $f \in X$ and $z \in \mathcal{S}_{0,\theta_1}$ we have*

$$\|(z-A)^{-1}f - (z-A_h)^{-1}P_h f\|_{X^{1/2}} \le Ch\|f\|_X, \tag{4.30}$$

$$\|(z-A)^{-1}f - (z-A_h)^{-1}P_h f\|_{X} \le Ch^2\|f\|_X, \tag{4.31}$$

$$\|(z-A)^{-1}f - (z-A_h)^{-1}P_h f\|_{X} \le Ch|z|^{-1/2}\|f\|_X, \tag{4.32}$$

where $(z-A)^{-1}$ and $(z-A_h)^{-1}$ are the resolvent operators of A and A_h, respectively.

Proof See [24, Theorem 1]. □

Lemma 18 (Discrete local solution) *Under the growing hypothesis (C), there exists a constant $h_0 > 0$ such that the problem (AP_h) has a local solution in $X_h^{1/2}$, for all $h \in (0, h_0]$.*

Proof We know that $\mathrm{Re}\,\sigma(A) > 0$, by using [22, Corollary 4.7] and [3, Theorem 4.10(i)] there exists an $h_0 > 0$ such that $\mathrm{Re}\,\sigma(A_h) > 0$ for all $h \in (0, h_0]$. For other hand, for each $u_h, v_h \in B_R^h = \{w_h \in X_h^{1/2} : \|w_h\|_{X^{1/2}} \le R\}$, we have

$$\|F_h(u_h) - F_h(v_h)\|_X = \|P_h(F(u_h) - F(v_h))\|_X \le C(R)\|u_h - v_h\|_{X^{1/2}},$$

where we have used (4.11). Therefore, using [20, Theorem 2.1.1] the result follows. □

Lemma 19 *Let $u_h = u_h(t, u_h^0) \in X_h^{1/2}$ be a local solution of the problem (AP_h) with $u_h^0 \in X_h^{1/2}$ and $t \in [0, \tau_0)$ and $h \in [0, h_0]$ (τ_0 and h_0 are given by Lemma 18). Suppose that (D) holds, so that for any $\varepsilon > 0$, there exists a positive constant $m := m(\varepsilon)$ independent of h such that for every $s \in \mathbb{R}$,*

$$sf(s) \le \varepsilon s^2 + m. \tag{4.33}$$

Then,

$$\sup_{t\in[0,\tau_0)} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \le C_0 \max\left\{\max\{\|u_h^0\|_X, m|\Omega|\}, 1\right\}, \tag{4.34}$$

where $C_0 = C_0(n, m, |\Omega|, \|u_h^0\|_{L^\infty(\Omega)})$ is a positive constant independent of h.

Proof Multiplying the equation (AP_h) by $u_h^{2^k-1}$, $k = 1, 2, \ldots$, and integrating over Ω, we obtain

$$\left\langle \dot{u}_h, u_h^{2^k-1}\right\rangle_X = \left\langle -A_h u_h, u_h^{2^k-1}\right\rangle_X + \left\langle P_h F(u_h), u_h^{2^k-1}\right\rangle_X. \tag{4.35}$$

Now, using (4.22), we get

$$\frac{1}{2^k}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_h^{2^k}\,\mathrm{d}x = \left\langle Lu_h, u_h^{2^k-1}\right\rangle_X + \left\langle F(u_h), u_h^{2^k-1}\right\rangle_X. \tag{4.36}$$

In the same way as in [23, Lemma 5] or [20, Lemma 9.3.1], we show

$$\sup_{t\in[0,\tau_0)} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \le C_0 \max\left\{\sup_{t\in[0,\tau_0)} \|u_h(t, u_h^0)\|_X, 1\right\}, \tag{4.37}$$

where $C_0 = C_0(n, m, |\Omega|, \|u_h^0\|_{L^\infty(\Omega)})$ is a positive constant. Similarly as we did above, we multiply the equation (AP$_h$) by u_h and integrating over Ω, we get

$$\sup_{t\in[0,\tau_0)} \|u_h(t, u_h^0)\|_X \leq \max\{\|u_h^0\|_X, m|\Omega|\}, \tag{4.38}$$

by using the estimates (4.9) and (4.33). Therefore, the inequality (4.34) follows using (4.37) and (4.38). □

Theorem 20 (Discrete global solution) *Assume that* (C) *and* (D) *holds, then the solution* $u_h = u_h(t, u_h^0) \in X_h^{1/2}$ *of problem* (AP$_h$) *with* $u_h^0 \in X_h^{1/2}$ *is globally defined for* $h \in [0, h_0]$. *Furthermore, given* $R > 0$, *there are positive constants* K_∞ *and* K_1 *such that*

$$\limsup_{t\to\infty} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \leq K_\infty \tag{4.39}$$

and

$$\limsup_{t\to\infty} \|u_h(t, u_h^0)\|_{H_0^1(\Omega)} \leq K_1, \tag{4.40}$$

for all $h \in [0, h_0]$.

Proof Using Lemma 19, we obtain that $u_h \in L^\infty(\Omega)$, for $t \in [0, \tau_0)$ (where $\tau > 0$ comes from Lemma 18). Hence, there is a positive constant $C = C(\|u_h\|_{L^\infty(\Omega)}) > 0$ such that

$$\|F(u_h)\|_X \leq |\Omega|^{1/2} \sup_{\{y\in\mathbb{R}:|y|\leq\|u_h\|_{L^\infty(\Omega)}\}} |f(y)| =: C(\|u_h\|_{L^\infty(\Omega)}). \tag{4.41}$$

From Lemma 18, we know that u_h is a local solution of (AP$_h$) in $X_h^{1/2}$ which satisfy

$$T_h(t)u_h^0 = e^{-tA_h}u_h^0 + \int_0^t e^{-(t-s)A_h} F_h(T_h(s)u_h^0)\mathrm{d}s, \quad \forall\, t \geq (0, \tau_0]. \tag{4.42}$$

Since A_h is sectorial and $\operatorname{Re}\sigma(A_h) > \beta > 0$ for $h \in [0, h_0]$ (by using [3, Theorem 4.10(i)] and [22, Corollary 4.7]).

With all of this, we obtain

$$\|e^{-A_h t}v\|_X \leq Ce^{-\beta t}\|v\|_X, \quad \forall\, v \in X,\ h \in [0, h_0]. \tag{4.43}$$

Now, using (4.16), (4.43) and [33, Theorem 1.3.4], we get

$$\begin{aligned}\|e^{-A_h t}v\|^2_{X^{1/2}} &\leq \frac{1}{c_2}\operatorname{Re}\sigma(e^{-A_h t}v, e^{-A_h t}v) \leq \frac{1}{c_2}\left|\left\langle A_h e^{-A_h t}v, e^{-A_h t}v\right\rangle_X\right| \\ &\leq \frac{1}{c_2}\|A_h e^{-A_h t}v\|_X\|e^{-A_h t}v\|_X \leq Ct^{-1}e^{-2\beta t}\|v\|_X^2,\end{aligned}$$

for all $v \in X$. From this, it follows that there exists a constant $\bar{C} > 0$ independent of h such that

$$\|e^{-A_h t}v\|_{X^{1/2}} \leq \bar{C}t^{-1/2}e^{-\beta t}\|v\|_X, \quad \forall\, v \in X,\ h \in [0, h_0]. \tag{4.44}$$

From (4.44) and (4.22), we have

$$\begin{aligned}\|u_h(t, u_h^0)\|_{X^{1/2}} &\leq \|e^{-A_h t}u_h^0\|_{X^{1/2}} + \int_0^t \|e^{-A_h(t-s)}F_h(u_h(s, u_h^0))\|_{X^{1/2}}\mathrm{d}s \\ &\leq Ce^{-\beta t}\|u_h^0\|_{X^{1/2}} + C(\|u_h\|_{L^\infty(\Omega)})\int_0^t \bar{C}(t-s)^{-1/2}e^{-\beta(t-s)}\mathrm{d}s \\ &= C\|u_h^0\|_{X^{1/2}} + \bar{C}C(\|u_h\|_{L^\infty(\Omega)})\beta^{-1/2}\sqrt{\pi} =: c(\|u_h^0\|_{X^{1/2}}).\end{aligned} \tag{4.45}$$

Using the property (4.22), we obtain

$$\|F_h(u_h)\|_X = \|P_h F(u_h)\|_X \leq \|F(u_h)\|_X \leq g(\|u_h\|_{X^{1/2}}). \tag{4.46}$$

Thus, by [20, Theorem 3.1.1] we get that the solution $u_h(t, u_h^0)$ is globally defined. Thanks to inequality (4.37), we have

$$\sup_{t\geq 0} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \leq C_0 \max\{1, \sup_{t\geq 0} \|u_h(t, u_h^0)\|_X\}. \tag{4.47}$$

On the other hand, from inequality (33) in [23, Lemma 5] and [20, Lemma 1.2.4], we get

$$\limsup_{t\to\infty} \|u_h(t, u_h^0)\|_X \leq (m_{\varepsilon_0}|\Omega|)^{1/2}, \quad \text{for some } \varepsilon_0 > 0.$$

Hence, we obtain

$$\limsup_{t\to\infty} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \leq C_0 \max\{1, (m_{\varepsilon_0}|\Omega|)^{1/2}\} = K_\infty,$$

which proves (4.39).

Lastly, using (4.42) for $t \geq t_0 > 0$ where t_0 is large enough such that $\|u_h(s, u_h^0)\|_{L^\infty(\Omega)} \leq K_\infty + \eta$ for $s \geq t_0$ and $\eta > 0$ along with (4.44), we obtain

$$\begin{aligned}
\|u_h(t, u_h^0)\|_{X^{1/2}} &\leq \|e^{-tA_h} u_h(t_0, u_h^0)\|_{X^{1/2}} + \int_{t_0}^t \|e^{-(t-s)A_h} P_h F(u_h(s, u_h^0))\|_{X^{1/2}} \mathrm{d}s \\
&\leq \bar{C}(t-t_0)^{-1/2} e^{-\beta(t-t_0)} \|u_h(t_0, u_h^0)\|_X + \int_{t_0}^t \|e^{-(t-s)A_h} F(u_h(s, u_h^0))\|_{X^{1/2}} \mathrm{d}s \\
&\leq \bar{C}(t-t_0)^{-1/2} e^{-\beta(t-t_0)} \|u_h(t_0, u_h^0)\|_{L^\infty(\Omega)} + \bar{C}C(K_\infty) \int_{t_0}^t (t-s)^{-1/2} e^{-\beta(t-s)} \mathrm{d}s.
\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty} \|u(t, u^0)\|_{X^{1/2}} \leq \bar{C}C(K_\infty) \int_0^\infty r^{-1/2} e^{-\beta r} dr = \bar{C}C(K_\infty)\sqrt{\pi}\beta^{-1/2} =: K_1.$$

□

As a consequence of Theorem 20, we can define the family of nonlinear semigroups $\{T_h(t) : t \geq 0\}_{h\in(0,h_0]}$, with $T_h(t) \in C(X_h^{1/2})$, given by $T_h(t)u_h^0 = u_h(t, u_h^0)$ for all $t \geq 0$. From [20, Remark 3.1.1] it follows that $\{T_h(t) : t \geq 0\}$ is a C_0-semigroup in $X_h^{1/2}$ for all $h \in (0, h_0]$. Due to [20, Theorem 3.1.1], $\{T_h(t) : t > 0\}$ is a compact operator, for all $h \in (0, h_0]$.

From Theorem 20 and following the ideas in the proof of [20, Theorem 4.1.1], we have

Corollary 21 *Under the hypothesis of Theorem 20, we have that there exists an $h_0 > 0$ such that the family of nonlinear semigroup $\{T_h(t) : t \geq 0\}_{h\in(0,h_0]}$ is point dissipative in $X_h^{1/2}$.*

The following result shows the existence of attractor for (AP_h) for h sufficiently small.

Theorem 22 *Under the hypothesis of Theorem 20, there exists a positive constant h_0 such that the nonlinear semigroup $T_h(\cdot)$ associated to (AP_h) has an global attractor $\mathcal{A}_h$ in $X_h^{1/2}$, for all $h \in (0, h_0]$.*

Proof From Lemma 18 we have that [20, Assumption 2.1.1] is satisfied. We also know that, for all $t > 0$, $T_h(\cdot)$ is a compact operator in $X_h^{1/2}$, for all $h \in (0, h_0]$. The family $\{A_h^{-1}\}_{h\in(0,1]}$ consists of compact linear operators since all of them have finite rank. Thus, A_h has a compact resolvent in $X_h^{1/2}$. Finally, using Corollary 21 and [20, Corollary 1.1.6] we have the result. □

As a consequence of Lemma 19, we have

Theorem 23 *Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$-function satisfying (C) and (D), then there exist positive constants K_0 and h_0 such that*

$$\sup_{h\in(0,h_0]} \sup_{v_h\in\mathcal{A}_h} \|v_h\|_{L^\infty(\Omega)} \le K_0. \tag{4.48}$$

Proof Suppose that for any $r > 0$, there is $\ell \ge 1$ such that for h small enough we have that $B_r^h(0) \subset V_{\ell R_0}^h = \{v_h \in X_h^{1/2} : |v_h(x)| \le \ell R_0, \forall\ x \in \Omega\}$ with $\ell = C_0$ and $R_0 := \max\{1, \max\{r, (m_{\varepsilon_0}|\Omega|)^{1/2}\}\}$. Now, from inequalities (4.47), (33) in [23, Lemma 5], [20, Lemma 1.2.4] and Theorem 20, for each $h \in (0, h_0]$, we get

$$\sup_{t\ge0} \|u_h(t, u_h^0)\|_{L^\infty(\Omega)} \le C_0 \max\left\{1, \max\{\|u_h^0\|_X, (m_{\varepsilon_0}|\Omega|)^{1/2}\}\right\} \le \ell R_0,$$

for all $u_h^0 \in B_r^h(0) \subset V_{\ell R_0}^h$. Hence, $u_h(t, u_h^0) \in V_{\ell R_0}^h$. Furthermore, using (4.45) we have that $\|u_h(t, u_h^0)\|_{X^{1/2}} \le c(u_h^0)$, for all $u_h^0 \in V_{\ell R_0}^h$. Also, for all $u_h^0 \in V_{\ell R_0}$, the assumptions on f imply (4.46). The requirements $(A2)$ for $u_h^0 \in V_{\ell R_0}^h$ in [20, Section 3.1.1] hold. Thus, from [20, Corollary 3.1.2] and [20, Theorem 4.2.1] and the uniqueness of the attractor on $V_{\ell R_0}^h$, it follows that $\mathcal{A}_h \subset V_{\ell R_0}^h$. Therefore, the result follows assuming $K_0 := \ell R_0$. □

The existence of attractors and stationary points for approximation schemes has been well studied in [19] and [10]. We now present the main result of this section.

Theorem 24 *Suppose that $T(\cdot)$ is a gradient nonlinear semigroup in $X^{1/2}$ with respect to $\mathcal{E} = \{u_1^*, \dots, u_{\mathfrak{p}}^*\}$ (all hyperbolic points), then there is an $h_0 > 0$ such that the family $\{T_h(\cdot)\}_{h\in[0,1]}$ of nonlinear semigroups in $X_h^{1/2}$ is gradient with respect to $\mathcal{E}_h = \{u_{h,1}^*, \dots, u_{h,\mathfrak{p}}^*\}$, for all $h \in [0, h_0]$. Consequently,*

$$\mathcal{A}_h = \bigcup_{i=1}^{\mathfrak{p}} W^{\mathrm{u}}(u_{i,h}^*), \quad \forall\ h \in [0, h_0].$$

Moreover, if the family of local unstable manifold of $u_{h,i}^* \in \mathcal{E}_h$ behaves $\tilde{\mathscr{P}}$-continuously at $h = 0$, that is, there exists a $\rho > 0$ such that

$$dist_{X^{1/2}}\left(W_h^{u,\rho}(u_{h,i}^*), \tilde{P}_h W^{u,\rho}(u_i^*)\right) + dist_{X^{1/2}}\left(\tilde{P}_h W^{u,\rho}(u_i^*), W_h^{u,\rho}(u_{h,i}^*)\right) \xrightarrow{h\to0} 0,$$

then the family $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\tilde{\mathscr{P}}$-continuous at $h = 0$.

Proof Following the ideas of Theorem 13, we will prove hypotheses of Theorem 8.

First, let us show that $T_h(\cdot)$ is a semigroup gradient, for h enough small.

Theorem 17 shows the $\mathscr{P}$-convergence with uniform convergence of resolvents. On the other hand, due that $(z - A)^{-1}$ is compact for some z, then the inequality (4.31) yields (where $\mu(\cdot)$ is measure of noncompactness)

$$\mu((z - A_h)^{-1}u_h) \le \mu((z - A)^{-1}u_h) + \overline{\lim_{h\to0}} \|(z - A)^{-1}u_h - (z - A_h)^{-1}u_h\|_X = 0$$

and therefore the resolvents converge compactly as $h \to 0$. Then $\Delta_{cc} \neq \emptyset$. Now, suppose that $u_h \xrightarrow{\tilde{\mathscr{P}}} u$, then

$$\begin{aligned}\|F_h(u_h) - P_h F(u)\|_X &\leq \|F(u_h) - F(u)\|_X \\ &\leq \|F(u_h) - F(\tilde{P}_h u)\|_X + \|F(\tilde{P}_h u) - F(u)\|_X \\ &\leq C(R)(\|u_h - \tilde{P}_h u\|_{X^{1/2}} + \|\tilde{P}_h u - u\|_{X^{1/2}}) \to 0 \text{ as } h \to 0,\end{aligned}$$

where we have (4.11) and (4.28). Hence, $F_h(u_h) \xrightarrow{\mathscr{P}} F(u)$.

With the above results, we can see that the conditions $[A1]$ in [19] holds. Consequently, $T_h(\cdot)$ is $\tilde{\mathscr{P}}$-collectively asymptotically compactness at $h = 0$ and $\tilde{\mathscr{P}}\tilde{\mathscr{P}}$-convergence of $T_h(\cdot)$ to $T(\cdot)$ on compact subsets of $\mathbb{R}^+ \times X^{1/2}$ by using of Theorems 4.3 and 4.7 in [19].

The hypothesis (a) of Theorem 8 follows directly of Theorems 22 and 23.

By other hand, using [22, Theorem 2.9] where $\tilde{P}_h \equiv R_h$, we have the hypothesis (b) of Theorem 8.

To show the hypothesis (e) of Theorem 8 observe that, by [22, Theorem 2.9], all the equilibrium points $\mathcal{E}_h = \{u^*_{h,1}, \ldots, u^*_{h,\mathfrak{p}}\}$, for the semigroup $T_h(\cdot)$ in $X_h^{1/2}$ are hyperbolic and there is $\eta > 0$ such that $\|u^*_{h,i} - \tilde{P}_h u^*_i\|_{X^{1/2}} \leq \eta$, $1 \leq i \leq \mathfrak{p}$. Arguing as was done in Theorem 13, we have that $u^*_{h,i}$ is the maximal invariant set in $\mathcal{O}_{\eta_0}(\tilde{P}_h u^*_i)$ for $\eta_0 \in (0, \eta)$.

Therefore, the conditions of the Theorem 8, item (3), are satisfied, then the result follows.

Finally, from Theorem 8, items (1) and (2), we obtain that the family $\{\mathcal{A}_h\}_{h\in(0,1]}$ is $\tilde{\mathscr{P}}$-continuous at $h = 0$. □

Remark 25 We can see that, if $b_j = 0$ for all $j = 1, \ldots, n$, in the operator L given by (4.8), then the semigroup $T(\cdot)$ is gradient (see [26, p. 78] and [33, p. 124]). Therefore, the results of Theorem 24 are valid.

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