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# $C^1$ -Genericity of symplectic diffeomorphisms and lower bounds for topological entropy

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## ABSTRACT

There is a  $C^1$ -residual (Baire second class) subset  $\mathcal{R}$  of symplectic diffeomorphisms on  $2d$ -dimensional manifold,  $d \geq 1$ , such that for every non-Anosov  $f$  in  $\mathcal{R}$ , its topological entropy is lower bounded by the supremum of the Lyapunov exponents of their hyperbolic periodic points in the *unbreakable central sub-bundle* (i.e. central direction with no dominated splitting) of  $f$ . The previous result deals with the fact that for  $f$  in a  $C^1$ -residual set  $\tilde{\mathcal{R}}$  of symplectic diffeomorphisms (containing  $\mathcal{R}$ ) satisfies a trichotomy: or  $f$  is Anosov or  $f$  is robustly transitive partially hyperbolic with *unbreakable centre* of dimension  $2m$ ,  $0 < m < d$ , or  $f$  has totally elliptic periodic points dense on  $M$ . In the second case, we also show the existence of a sequence of  $m$ -elliptic periodic points converging to  $M$ . Indeed,  $\tilde{\mathcal{R}}$  contains an  $C^1$  open and dense subset of symplectic diffeomorphisms.

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## 1. Introduction

The concept of topological entropy of a dynamical system provides information about its complexity and it is invariant by conjugacy. Topological entropy is a positive real number that, roughly, measures the rate of exponential growth of the number of distinguishable orbits with finite but arbitrary precision as time advances. Precisely, let  $(X, d)$  be a compact metric space and  $f: X \rightarrow X$  be a continuous map. For each natural number  $n$ , we define the metric

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i \leq n\}.$$

Notice that, given any  $\varepsilon > 0$  and  $n \geq 1$ , two points of  $X$  are  $\varepsilon$ -close with respect to this metric if their first  $n$  iterates are  $\varepsilon$ -close. A subset  $E$  of  $X$  is said to be  $(n, \varepsilon)$ -separated if each pair of distinct points of  $E$  has distance greater than  $\varepsilon$  in the metric  $d_n$ . Denote by  $N(n, \varepsilon)$  the maximum cardinality of an  $(n, \varepsilon)$ -separated set. The *topological entropy* of the map  $f$  is defined by

$$h_{\text{top}}(f) = \lim_{\varepsilon \rightarrow 0} \left( \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, \varepsilon) \right).$$

Recall that the above limit always exists in the extended real line (but could be infinite).

Lyapunov exponents are another useful tool to measure the complexity of a dynamical system. They are important constants to measure the asymptotic behaviour of dynamics in the tangent space level. Positive Lyapunov exponents indicate orbital divergence and long-term unpredictability of a dynamical system because the omnipresent uncertainty in determining its initial state grows exponentially fast in time. In other words, Lyapunov exponents tell us the rate of divergence of nearby trajectories. More precisely, given a diffeomorphism  $f$  over a manifold  $M$ , we say that a real number  $\lambda(x)$  is a *Lyapunov exponent* of  $x \in M$  if there exists a non-zero vector  $v \in T_x M$  such that

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df^n(x) v\| = \lambda(x).$$

Using the lack of hyperbolicity inside the central direction, in [Theorem B](#), we relate these two different ways to measure the complexity of a system. Roughly, we provide lower bounds of the topological entropy for a class of symplectic systems using Lyapunov exponents of the hyperbolic periodic points in the central direction, i.e. taking  $v$  above in the central sub-bundle. Hence, we obtain an estimate to topological complexity of symplectic systems via differential properties of its hyperbolic periodic points for a class of symplectic diffeomorphisms. By class we mean a residual subset of  $\text{Diff}_\omega^1(M)$  in the complement of symplectic Anosov diffeomorphisms. The estimate we obtain for the topological entropy of a diffeomorphism  $f$  depends on the level of partial hyperbolicity of  $f$ , namely, the centre dimension. Hence, it is worthwhile to know conditions that allow to identify the level of partial hyperbolicity of a symplectic diffeomorphisms. In this sense, we obtain [Theorem A](#), which allows to specify the level of partial hyperbolicity of a symplectic diffeomorphism through a study of the periodic set of the symplectic diffeomorphism.

Let us make precise the statements.

We say that a diffeomorphism  $f: M \rightarrow M$  is *partially hyperbolic* if there exists a continuous  $Df$ -invariant splitting  $TM = E^s \oplus E^c \oplus E^u$  with non-trivial extremal sub-bundles  $E^s$  and  $E^u$ , such that for every  $x \in M$  and every  $n$  large enough,

- the splitting is *dominated*:

$$\|Df^n|E^i(x)\| \|Df^{-n}|E^j(f^n(x))\| \leq \frac{1}{2}, \quad \text{for any } (i, j) = (s, c), (s, u), (c, u); \text{ and}$$

- the extremal subbundles are hyperbolic:

$$\|Df^n|E^s(x)\| \leq \frac{1}{2} \quad \text{and} \quad \|Df^{-n}|E^u(x)\| \leq \frac{1}{2}.$$

We say that a partially hyperbolic diffeomorphism  $f \in \text{Diff}_\omega^1(M)$  has *unbreakable centre bundle* if the centre bundle  $E^c$  has no dominated sub-splitting for  $f$ . If the centre bundle  $E^c$  is trivial, then  $f$  is hyperbolic, that is,  $f$  is an *Anosov diffeomorphism*.

Here,  $(M^{2d}, \omega)$  denotes a compact, connected, and boundaryless symplectic manifold with dimension  $2d$  and  $\text{Diff}_\omega^1(M^{2d})$  denotes the set of  $C^1$ -diffeomorphisms on  $(M^{2d}, \omega)$  that preserve the symplectic form  $\omega$ . Recall that a partially hyperbolic symplectic

diffeomorphism has even dimensional unbreakable centre bundle. So, we can split the set of partially hyperbolic diffeomorphisms in subsets according to the dimension of their unbreakable centre bundle. We denote by  $\mathcal{PH}_\omega^1(m) \subset \text{Diff}_\omega^1(M^{2d})$ ,  $0 < m < d$ , the set of partially hyperbolic diffeomorphisms with  $\dim(E^c) = 2m$ . For convenience, we denote by  $\mathcal{PH}_\omega^1(0) = \mathcal{A}$  the subset of Anosov diffeomorphisms and by  $\mathcal{PH}_\omega^1(d)$  the complement of the closure of the union of all  $\mathcal{PH}_\omega^1(m)$ ,  $0 \leq m < d$ . Notice that  $\mathcal{PH}_\omega^1(i)$  and  $\mathcal{PH}_\omega^1(j)$  are disjoint subsets for every distinct  $0 \leq i, j \leq d$ . Moreover, they form a partition of  $\text{Diff}_\omega^1(M^{2d})$ .

In order to state the first result, let us recall the definition of elliptic periodic points. Let  $\text{Per}(f)$  be the set of periodic points of  $f$  in  $\text{Diff}_\omega^1(M^{2d})$ . We say that  $p \in \text{Per}(f)$  of period  $k$  is an  $m$ -elliptic periodic point,  $0 < m \leq d$ , if  $Df^k(p)$  has exactly  $2m$  non-real and simple eigenvalues of modulus one, and all other eigenvalues have modulus different from 1. Here, a  $d$ -elliptic periodic point is called *totally elliptic periodic point*.

Recall that, as hyperbolic periodic points,  $m$ -elliptic periodic points are robust for symplectic diffeomorphisms. Also, if  $f$  is a partially hyperbolic diffeomorphism and has an  $m$ -elliptic periodic point then  $\dim E^c$  must be larger than  $2m$ . In particular, from the continuity of the partially hyperbolic splitting and the robustness of  $m$ -elliptic periodic points for symplectic diffeomorphisms it follows that every  $f \in \mathcal{PH}_\omega^1(m)$  having an  $m$ -elliptic periodic point belongs to the interior of  $\mathcal{PH}_\omega^1(m)$ .

In [3], Arnaud, Bonatti, and Crovisier show that a generic partially hyperbolic symplectic diffeomorphism  $f \in \mathcal{PH}_\omega^1(m) \subset \text{Diff}_\omega^1(M^4)$ ,  $1 \leq m \leq 2$ , must have  $m$ -elliptic periodic points dense on  $M$ . They also conjectured that the same is true in  $\text{Diff}_\omega^1(M^{2d})$  for any  $d \geq 1$ . The next result provides a positive answer to the conjecture.

**Theorem A:** *There exists a residual subset  $\tilde{\mathcal{R}} \subset \text{Diff}_\omega^1(M^{2d})$ , such that if  $f \in \tilde{\mathcal{R}}$ , one of the following properties happens:*

- (a)  $f$  is an Anosov diffeomorphism;
- (b)  $f$  is a robustly transitive partially hyperbolic diffeomorphism in  $\mathcal{PH}_\omega^1(m)$ , for some  $0 < m < d$ , and there is a sequence of  $m$ -elliptic periodic points converging to  $M$  (in the Hausdorff topology);
- (c)  $f$  is not partially hyperbolic and has a sequence of totally elliptic periodic points converging to  $M$  (in the Hausdorff topology).

In particular, generically the non-existence of totally elliptical periodic points implies some level of (uniform/partial) hyperbolicity.

Let us recall some previous related results. Newhouse [14] shows that in the complement of the set of Anosov symplectic diffeomorphisms (in  $\text{Diff}_\omega^1(M^2)$ ), there is a residual subset of symplectic diffeomorphisms exhibiting 1-elliptic periodic points dense on  $M$ . Arnaud [2] prove the existence of an open and dense subset of  $\text{Diff}_\omega^1(M^4)$  such that  $f$  is Anosov, or  $f$  is partially hyperbolic, or  $f$  has a totally elliptic periodic points on  $M$ . These results were extended in two direction: in the first one, Saghin and Xia [15] generalize to  $\text{Diff}_\omega^1(M^{2d})$ , for any  $d \geq 1$ ; this result also follows from Horita and Tahzibi [12]. In the second direction, Arnaud, Bonatti, and Crovisier [3] has a four-dimensional version of our Theorem A, as we have already mentioned.

Before we state the result of estimates for the topological entropy of non-Anosov maps, we define some notations and recall some previous results. We denote by  $\tau(p, f)$  the period

of a periodic point  $p$  of  $f$ . For  $f \in \text{Diff}_\omega^1(M)$  and  $p$  a periodic point with some eigenvalue with modulus different from 1, we define

$$\lambda_{\min}(p, f) = \min\{|\lambda| : \lambda \text{ is an eigenvalue of } Df^{\tau(p, f)}(p) \text{ with } |\lambda| > 1\}.$$

and if  $f$  has hyperbolic periodic points, we define

$$s(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \lambda_{\min}(p, f) : p \text{ is a hyperbolic periodic point of } f \right\}. \quad (1)$$

Recall that generically (i.e. for a residual subset) symplectic diffeomorphisms have hyperbolic periodic points (in fact they are dense). So,  $s(f)$  is well defined for  $f$  in a residual subset of  $\text{Diff}_\omega^1(M)$ .

Newhouse [13] relates the topological entropy and  $s(\cdot)$  for certain non-Anosov symplectic diffeomorphisms on surfaces. Recently, Catalan and Tahzibi [7] generalize [13] for symplectic diffeomorphisms in any  $2d$ -dimensional manifold.

**Theorem 1.1** ([13] for  $d = 1$  and [7] for any  $d \geq 1$ ): *There is a residual subset  $\mathcal{R} \subset \text{Diff}_\omega^1(M^{2d})$  of  $C^1$  symplectic diffeomorphisms in  $M$ , such that for every non-Anosov diffeomorphism  $f \in \mathcal{R}$ , we have*

$$h_{\text{top}}(f) \geq s(f).$$

Moreover, Catalan and Tahzibi [7] obtain a stronger result for symplectic diffeomorphisms on surface: for a generic non-Anosov surface symplectic diffeomorphism  $f$ , we have  $h_{\text{top}}(f) = s(f)$ .

Roughly, in the proof of Theorem 1.1, the authors use the lack of hyperbolicity to obtain the estimate. Here we are able to use the lack of partial hyperbolicity getting better estimates to topological entropy. Let us make this precise.

Let  $A: V \rightarrow V$  be a linear operator defined on a vector space  $V$  and let  $E \subset V$  be an  $A$ -invariant subspace. We denote by  $\sigma(A|E)$  the *spectral radius* of  $A$  restrict to  $E$ . Hence, given  $f \in \mathcal{PH}_\omega^1(m)$  in  $0 < m \leq d$ , we define

$$S_m(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)) : p \text{ hyperbolic periodic point of } f \right\}.$$

Let us remark that, according to Theorem A, for a residual subset of  $\text{Diff}_\omega^1(M^{2d})$ , a non-Anosov diffeomorphism  $f$  belongs to a subset  $\mathcal{PH}_\omega^1(m)$ , for some  $0 < m \leq d$ . Indeed, it holds for an open and dense subset of  $\tilde{\mathcal{R}} \setminus \mathcal{PH}_\omega^1(0)$ . So, if  $f \in \text{Diff}_\omega^1(M^{2d})$  is a generic non-Anosov diffeomorphism, then  $S_m(f)$  is defined for some  $m$ . Clearly, if  $0 < m \leq d$ , then  $S_m(f) \geq s(f)$ . In fact, for  $f \in \mathcal{PH}_\omega^1(1)$ , the equality holds, i.e.  $S_1(f) = s(f)$ . It is not difficult to show that  $S_m(\cdot)$ ,  $0 < m \leq d$ , is a lower semi-continuous function as  $s(\cdot)$  is, see Section 4.

The next result provides lower bounds for the topological entropy of a non-Anosov symplectic diffeomorphism.

**Theorem B:** *There exists a residual subset  $\mathcal{R} \subset \text{Diff}_\omega^1(M^{2d})$ ,  $d \geq 1$ , such that if  $f \in \mathcal{R} \cap \mathcal{PH}_\omega^1(m)$ ,  $0 < m \leq d$ , then*

$$h_{\text{top}}(f) \geq S_m(f).$$

It is worth to remark that, generically, in the lack of partial hyperbolicity, i.e. for a generic  $f$  in  $\mathcal{PH}_\omega^1(d)$ , the previous result yields a lower bound to topological entropy in terms of the supremum of the largest Lyapunov exponent of all hyperbolic periodic points of  $f$ .

The paper is organized as follows: in [Section 2](#), we recall and provide some useful perturbative results in the symplectic scenario as connecting lemma, Franks lemma, and linear systems with transitions. [Section 3](#) is devoted to prove [Theorem A](#). Using periodic linear systems we show in [Section 4](#) how to perturb a symplectic diffeomorphism in order to find a nice periodic point, namely a diagonalizable periodic point, having Lyapunov exponents close to Lyapunov exponents of an arbitrary fixed periodic point, [Lemma 4.3](#). Furthermore, in [Proposition 4.2](#), we show that  $S_m(f)$  can take account just diagonalizable hyperbolic periodic points and using a technical result, [Proposition 4.5](#), we complete the proof of [Theorem B](#). We obtain, in [Section 5](#), intersections between strong stable and unstable manifolds of diagonalizable periodic points with small angles, [Lemmas 5.1](#) and [5.2](#). These lemmas are essential to prove [Proposition 4.5](#) in [Section 6](#).

We finish this section by giving a sketch of the proof of the main theorems. We point out what we should overcome from the technics used in [\[13\]](#) and [\[7\]](#) in order to prove [Theorem B](#). Also, we put how [Theorem A](#) follows from the technics developed in order to prove [Theorem B](#).

Let us recall the key points in the proof of [Theorem 1.1](#). An essential point is that in the symplectic scenario, the Palis conjecture is known, more precisely, symplectic diffeomorphisms either are approximated by symplectic Anosov diffeomorphisms or by diffeomorphisms exhibiting homoclinic tangencies, see Newhouse [\[14\]](#). Another essential point is also due to Newhouse that show how to perturb a symplectic surface diffeomorphism  $f$ , which exhibits a homoclinic tangency for a hyperbolic periodic point  $p$ , in order to create a basic hyperbolic set having topological entropy arbitrary close to the (unique) positive Lyapunov exponent of  $p$  for  $f$ . This kind of perturbation is called here by *snake perturbation*. To prove [Theorem 1.1](#) for symplectic diffeomorphisms in higher dimension, Catalan and Tahzibi developed higher dimensional snake perturbations. The hyperbolic basic set obtained after a snake perturbation has topological entropy close to the smallest positive Lyapunov exponent of a periodic point  $p$ . This is because of the natural  $Df^{(p)}$ -invariant dominated splitting in  $T_p M$  given by the eigenspaces of  $Df^{(p)}(p)$ . This is why [Theorem 1.1](#) provides lower bounds for topological entropy in terms of the smallest positive Lyapunov exponents of all hyperbolic periodic points.

Hence, to prove [Theorem B](#), following the programme of [Theorem 1.1](#), we need to perturb a symplectic diffeomorphism  $f$  in order to find a basic hyperbolic set associated to a hyperbolic periodic point  $p$  of  $f$ , having topological entropy close to non-minimal Lyapunov exponents of  $p$ . In this sense, we find an  $f$ -invariant symplectic sub-manifold  $D \subset M$  containing  $p$ , such that the stable and unstable manifolds of  $p$  inside  $D$  has a non-transversal intersection, which allows us to use the snake perturbation inside  $D$ , to construct a basic hyperbolic set having entropy close to the smallest Lyapunov exponent of  $p$  for  $f$  restrict to  $D$ .

There are many technical steps to find the sub-manifold  $D$ , cited before. One of them is to prove the existence of a periodic point  $p$  having all central eigenvalues equal to one, after a perturbation. This result is due to Horita and Tahzibi [12]. [Theorem A](#) is a consequence of this fact.

## 2. Preliminaries

In this section, we recall some techniques and provide results that we will use along the proofs of [Theorems A](#) and [B](#). They encompass perturbations of linear symplectic transformations, connection of invariant manifolds, and periodic symplectic linear systems with transitions.

### 2.1. Linear symplectic perturbations

First, let us recall some basic facts about symplectic vector spaces. Let  $(V, \omega)$  be a symplectic vector space of dimension  $2d$ . For any subspace  $W \subset V$ , we define its *symplectic orthogonal* vector space as

$$W^\omega = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in W\}.$$

The subspace  $W$  is called *symplectic* if  $W^\omega \cap W = \{0\}$ . We say  $W$  is *isotropic* if  $W \subset W^\omega$ , that is, if  $\omega|_{(W \times W)} = 0$ . When  $W = W^\omega$  we say that  $W$  is a *Lagrangian* subspace.

For a symplectic form  $\omega$  in  $V$ , there is a symplectic basis  $\mathcal{B} = \{e_1, \dots, e_{2d}\}$  of  $V$  such that, with respect to this basis,  $\omega$  is in the standard form  $\omega = \sum_{i=1}^d de_i \wedge de_{i+d}$ , i.e.  $\omega(e_i, e_{d+i}) = 1$  and  $\omega(e_i, e_j) = 0$  if  $j \neq d+i$ , for every  $1 \leq i \leq d$ . Now, if  $J$  is the canonical map on  $V$ , with respect to  $\mathcal{B}$ , such that  $J^2 = -Id$ , we say that a *linear map*  $A$  is *symplectic* if  $A^*JA = J$ . In particular,  $A$  is a symplectic map if and only if  $A^*\omega = \omega$ . Notice that, if we take an inner product on  $V$  for which  $\mathcal{B}$  is an orthonormal basis, then  $\omega(u, v) = \langle u, Jv \rangle$ .

Given two vector subspaces  $E$  and  $E'$  of a vector space  $V$  endowed with an inner product,  $\dim E = \dim E' = j$ , we say that  $E$  and  $E'$  are  $\delta$ -close if there exists orthonormal basis  $\{e_1, \dots, e_j\}$  of  $E$  and  $\{e'_1, \dots, e'_j\}$  of  $E'$  such that  $\max\{\|e_i - e'_i\| : 1 \leq i \leq j\} < \delta$ , where  $\|\cdot\|$  is induced by the inner product. For any pair of  $E$  and  $E'$  of vector subspaces of the same dimension, it is trivial to find a linear isomorphism  $A$  such that  $A(E') = E$ . Moreover, if  $E$  and  $E'$  are close, then  $A$  can be chosen close to the identity. The next lemma asserts that if  $E$  and  $E'$  are close, then  $A$  can be taken symplectic and preserving a complementary space of  $E$ .

In the remainder of this section, for sake of simplicity, we denote by  $V = (V, \omega)$  a symplectic vector space.

**Lemma 2.1:** *Suppose  $V = E \oplus F$ , where  $E$  is an isotropic subspace. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $W \subset V$  is an isotropic subspace  $\delta$ -close to  $E$ , then there exists a symplectic linear map  $B$  on  $V$   $\varepsilon$ -close to  $Id$  such that  $B(W) = E$  and  $B|_F = Id_F$ .*

**Proof:** We use here coordinates  $(x, y)$  in  $V$  with respect to the decomposition  $V = E \oplus F$ , and we fix an arbitrary norm  $\|\cdot\|$  in  $V$ .

Since  $W$  is close enough to  $E$ , there exists a linear map  $A: E \rightarrow F$  such that  $W = \{(x, A(x)) : x \in E\}$ . Moreover, given  $\varepsilon > 0$ , we can choose  $\delta > 0$  small enough such that if  $W$  is

$\delta$ -close to  $E$ , then  $\|A\| < \varepsilon$ . We define  $j: E \rightarrow V$  by  $j(x) = (x, A(x))$ . Since  $W$  is an isotropic subspace, we have  $j^*\omega = 0$  ( $j^*\omega$  is the pullback of the symplectic form  $\omega$  by  $j$ ). Analogously, if  $i: E \rightarrow V$  is the natural inclusion,  $i(x) = (x, 0)$ , we have  $i^*\omega = 0$ .

Finally, if we define  $B: V \rightarrow V$  by  $B(x, y) = (x, y - A(x))$ , then to conclude the proof we just need to show that  $B$  is indeed symplectic, since  $\|B - Id\| \leq \|A\| < \varepsilon$ ,  $B(W) = E$  and  $B|_F = Id_F$ .

Let  $\pi: V \rightarrow E$  be the projection on the first coordinate, i.e.  $\pi(x, y) = x$ . We can write  $B$  as  $B = Id + i \circ \pi - j \circ \pi$ . Therefore,

$$B^*\omega = \omega + \pi^*i^*\omega - \pi^*j^*\omega = \omega,$$

since  $i^*\omega = j^*\omega = 0$ . The proof is finished. ■

The next lemma allows to perturb inside a given symplectic subspace, keeping invariant its symplectic orthogonal space.

**Lemma 2.2:** *Let  $W \subset V$  be a symplectic subspace. For any  $\varepsilon > 0$ , there exists  $\delta > 0$ , such that if  $A: W \rightarrow W$  is a symplectic linear map  $\delta$ -close to the identity map  $Id|_W$ , then there exists a symplectic linear map  $B$  over  $V$ ,  $\varepsilon$ -close to  $Id$  such that  $B|_W = A$  and  $B|_{W^\omega} = Id|_{W^\omega}$ .*

**Proof:** If  $W = V$ , we are done, so we suppose  $\dim W = 2m < 2d = \dim V$ . Let  $\{e_1, \dots, e_{2m}\}$  be a symplectic basis of  $W$  and let  $\{e_{2m+1}, \dots, e_{2d}\}$  be a symplectic basis of  $W^\omega$ . Hence,  $\{e_1, \dots, e_{2d}\}$  is a symplectic basis of  $V$ . Let  $\langle \cdot, \cdot \rangle$  be the inner product for which this basis is orthonormal and thus  $\omega(u, v) = \langle u, Jv \rangle$ , where

$$J = \begin{bmatrix} J_W & 0 \\ 0 & J_{W^\omega} \end{bmatrix}, \text{ for } J_\iota = \begin{bmatrix} 0 & Id_\iota \\ -Id_\iota & 0 \end{bmatrix}, \quad \iota = W, W^\omega,$$

where  $Id_\iota$  is the identity matrix of order  $m \times m$ , if  $\iota = W$ , or the  $(d - m) \times (d - m)$  is the identity matrix if  $\iota = W^\omega$ .

Since  $A$  is a symplectic linear map on  $W$ , we have  $A^*J_W A = J_W$ . Defining

$$B = \begin{bmatrix} A & 0 \\ 0 & Id_{W^\omega} \end{bmatrix},$$

we have that  $B^*JB = J$ . So,  $B$  is a symplectic linear map on  $V$  such that  $B|_W = A$  and  $B|_{W^\omega} = Id_{W^\omega}$ .

Therefore, for any  $\varepsilon > 0$ , we can choose  $\delta > 0$  small enough, depending on the symplectic basis fixed at the beginning, such that if  $A$  is  $\delta$ -close to  $Id_W$ , then the linear map  $B$  is  $\varepsilon$ -close to  $Id$ . The proof is finished. ■

Now, we recall a symplectic version of the well-known Franks lemma in [9], which enable us to perform nonlinear perturbations along a finite invariant set, namely a periodic orbit, from linear perturbations (in particular, from those given in the previous lemmas).

**Lemma 2.3** (Symplectic Franks lemma): *Let  $f \in \text{Diff}_\omega^1(M)$  and let  $\mathcal{U} \subset \text{Diff}_\omega^1(M)$  be any neighbourhood of  $f$ . Then, there exists  $\delta > 0$  and  $\mathcal{U}' \subset \mathcal{U}$  a small neighbourhood of  $f$  such that given  $g \in \mathcal{U}'$ , a finite  $g$ -invariant set  $\{x_1, \dots, x_N\}$ , a neighbourhood  $U$  of  $\{x_1, \dots, x_N\}$  and symplectic linear maps  $A_i: T_{x_i}M \rightarrow T_{g(x_i)}M$  such that  $\|A_i - Dg(x_i)\| \leq \delta$  for all  $1 \leq i \leq N$*

$N$ , then there is a symplectic diffeomorphism  $\tilde{g} \in \mathcal{U}$  such that  $\tilde{g}(x) = g(x)$  if  $x \in \{x_1, \dots, x_N\} \cup (M \setminus U)$  and  $D\tilde{g}(x_i) = A_i$  for all  $1 \leq i \leq N$ .

The proof in [9] can be extended to the symplectic set using generating functions.

## 2.2. Connecting invariant manifolds.

Given a hyperbolic periodic point  $p$  of a symplectic diffeomorphism  $f$ , we define its *stable* (resp., *unstable*) *manifold*,  $W^s(p, f)$  (resp.,  $W^u(p, f)$ ), as the subset of points in  $M$  whose forward (resp., backward) orbit by  $f^{(p, f)}$  converges to  $p$ .

**Remark 2.4:** If  $f$  is a symplectic diffeomorphism over  $M$  and  $p$  is a hyperbolic periodic point of  $f$ , the stable and unstable manifolds of  $p$ ,  $W^s(p, f)$  and  $W^u(p, f)$ , are Lagrangian submanifolds of  $M$ , that is,  $T_x W^i(p, f)$  is a Lagrangian subspace for every  $x \in W^i(p, f)$ ,  $i = s$  or  $u$ . In particular,  $E^s(p, f)$  and  $E^u(p, f)$  are Lagrangian subspaces of  $T_p M$ .

**Remark 2.5:** If  $f$  is a partially hyperbolic symplectic diffeomorphism over  $M$ ,  $TM = E^{ss} \oplus E^c \oplus E^{uu}$ , then we also recall that  $E^{ss}$  and  $E^{uu}$  are isotropic sub-bundles of  $TM$ . Furthermore, any partially hyperbolic splitting for a symplectic diffeomorphism is such that  $E^c$  is a symplectic sub-bundle of  $TM$  satisfying  $(E^c(x))^\omega = E^{ss}(x) \oplus E^{uu}(x)$ , for every  $x \in M$ . In particular,  $E^{ss}(x) \oplus E^{uu}(x)$  is a symplectic subspace of  $T_x M$ .

Now, given a periodic point  $p$  of  $f \in \text{Diff}_\omega^1(M)$ , if there exists a partially hyperbolic splitting  $T_p M = E_k^{ss} \oplus E_k^c \oplus E_k^{uu}$  for  $Df^{\alpha(p, f)}$ , with  $\dim(E_k^{ss}) = \dim(E_k^{uu}) = k$  and  $\dim E_k^c = 2d - 2k$ , then by [11], there exists a  $f^{\alpha(p, f)}$  (resp.,  $f^{-\tau(p, f)}$ ) *local invariant  $k$ -dimensional strong stable* (resp., *unstable*) *manifold*  $W_{k, \text{loc}}^{ss}(p)$  (resp.,  $W_{k, \text{loc}}^{uu}(p)$ ) tangent to  $E_k^{ss}$  (resp.,  $E_k^{uu}$ ) at  $p$  varying  $C^1$ -continuously with respect to the diffeomorphism. Hence, we define the  *$k$ -dimensional strong stable* (resp., *unstable*) *manifold* of  $p$  by

$$W_k^{ss}(p) = \bigcup_{n \in \mathbb{N}} f^{-n}(W_{k, \text{loc}}^{ss}(p)) \quad (\text{resp., } W_k^{uu}(p) = \bigcup_{n \in \mathbb{N}} f^n(W_{k, \text{loc}}^{uu}(p))).$$

We say that a periodic point  $p$  of  $f \in \text{Diff}_\omega^1(M)$  is *diagonalizable* if  $Df^{\alpha(p, f)}(p)$  has only real positive eigenvalues with multiplicity one. If  $p$  is a diagonalizable periodic point, then for each  $0 < k \leq d$ , the partially hyperbolic splitting  $T_p M = E_k^{ss} \oplus E_k^c \oplus E_k^{uu}$  is defined. So, for those points  $k$ -strong invariant manifolds are defined for any  $0 < k \leq d$ . Hence, for any diffeomorphism  $f \in \text{Diff}_\omega^1(M)$  and any diagonalizable hyperbolic periodic point  $p$  of  $f$ , denoting by  $\lambda_{1, p} < \dots < \lambda_{2d, p}$ , the distinct simple eigenvalues of  $Df^{\alpha(p, f)}(p)$  and by  $E_{\lambda_{1, p}} < \dots < E_{\lambda_{2d, p}}$ , the respective eigenspaces, we set the  *$k$ -dimensional strong stable* (resp., *strong unstable*) *subspace* in  $T_p M$ ,  $0 < k \leq d$ , as follows:

$$E_k^{ss}(p) = \bigoplus_{1 \leq j \leq k} E_{\lambda_{j, p}} \quad \left( \text{resp., } E_k^{uu}(p) = \bigoplus_{2d-k+1 \leq j \leq 2d} E_{\lambda_{j, p}} \right).$$

For analogy, taking  $m = d - k$ , we denote

$$E_m^c(p) = \bigoplus_{k+1 \leq j \leq 2d-k} E_{\lambda_{j, p}},$$

which is the  $2m$ -dimensional centre subspace. It is worth to point out that according to the previous definition,  $W_d^{ss}(p) = W^s(p)$  and  $W_d^{uu}(p) = W^u(p)$ .

In this work, by  $k$ -strong homoclinic intersections,  $k < d$ , we mean non-trivial intersections between  $W_k^{ss}(p)$  and  $W_k^{uu}(p)$ . Moreover, if  $q$  is a  $k$ -strong homoclinic intersection such that  $T_q W_k^{ss}(p) \cap T_q W_k^{uu}(p) = \{0\}$ , then we say  $q$  is a  $k$ -strong quasi-transversal homoclinic intersection. To create such intersections, we use a symplectic version of Hayashi connecting lemma [10], due to Xia and Wen [16].

**Proposition 2.6** (Theorem F in [16]): *Let  $z \in M$  be a non-periodic point of  $f \in \text{Diff}_\omega^1(M)$ . For any  $C^1$ -neighbourhood  $\mathcal{U}$  of  $f$ , there are  $\rho > 1$ ,  $L \in \mathbb{N}$  and  $\delta_0 > 0$  such that, for any  $0 < \delta < \delta_0$ , and for any point  $x$  outside the tube  $\Delta = \cup_{n=1}^L f^{-n}B(z, \delta)$ , and any point  $y \in B(z, \delta/\rho)$ , if the forward  $f$ -orbit of  $x$  intersects  $B(z, \delta/\rho)$ , then there is a symplectic diffeomorphism  $g \in \mathcal{U}$  such that  $g = f \circ \text{off } \Delta$  and  $y$  is on the forward  $g$ -orbit of  $x$ .*

We emphasize that the perturbation  $g$  of  $f$  in the above proposition is a local perturbation. That is,  $g$  should be different from  $f$  only in  $\Delta$ .

**Remark 2.7:** Symmetrically, we can restate the previous proposition for a tube along the positive orbit of  $z$ , and require that the backward  $f$ -orbit of  $x$  intersects  $B(z, \delta/\rho)$ , obtaining now that  $y$  belongs to the backward  $g$ -orbit of  $x$ .

The next result is a consequence of Proposition 2.6 which permit us create  $k$ -strong homoclinic intersections.

**Lemma 2.8:** *Let  $f \in \text{Diff}_\omega^1(M)$  and let  $p, q$  be either hyperbolic or  $m$ -elliptic periodic points of  $f$  both having  $k$ -dimensional strong stable and unstable manifolds. For any neighbourhood  $\mathcal{U}$  of  $f$  and any  $y \in W_k^{ss}(p, f)$ , there exists a diffeomorphism  $g \in \mathcal{U}$  with a  $k$ -strong heteroclinic intersection  $\tilde{y}$ , which is arbitrary close to  $y$ , for the analytic continuation  $p(g)$  and  $q(g)$  of  $p$  and  $q$ , respectively, for  $g$ . That is,  $\tilde{y} \in W_k^{ss}(p(g)) \cap W_k^{uu}(q(g))$ .*

The proof of this lemma uses the fact that transitive diffeomorphisms are  $C^1$ -dense in  $\text{Diff}_\omega^1(M)$ . This is the content of [3, Theorem 1], let us state it for completeness.

We define the *homoclinic class* of a hyperbolic periodic point,  $H(p, f)$ , as the closure of transversal intersections between the stable and unstable manifolds of all points in the orbit of  $p$ :  $H(p, f) = \overline{W^s(\text{orb}(p)) \pitchfork W^u(\text{orb}(p))}$ . It is well known that homoclinic class is a transitive set and coincides with the closure of the hyperbolic periodic points homoclinically related to  $p$  (we say that a hyperbolic periodic point  $q$  is *homoclinically related* to  $p$  if  $W^s(p) \pitchfork W^u(q) \neq \emptyset$  and  $W^u(p) \pitchfork W^s(q) \neq \emptyset$ ).

**Proposition 2.9** (Theorem 1 in [3]): *There exists a residual subset  $\mathcal{R}$  of  $\text{Diff}_\omega^1(M)$  such that if  $f \in \mathcal{R}$ , then there exists a hyperbolic periodic point  $p$  off  $f$  such that  $M = H(p, f)$ . In particular,  $f$  is transitive.*

**Proof of Lemma 2.8:** By Proposition 2.9, after a perturbation, we can suppose that  $f$  is transitive. Also, by transversality, we can suppose that  $W_k^{ss}(p) \cap W_k^{uu}(q) = \emptyset$ . Assume  $z^s = y \in W_k^{ss}(p)$ , and take  $z^u \in W_k^{uu}(q)$ . Now, let  $\mathcal{U}$ ,  $\rho > 1$ ,  $L \in \mathbb{N}$ , and  $\delta_0 > 0$  satisfy simultaneously Proposition 2.6 for  $z = z^u$  and Remark 2.7 for  $z = z^s$ .

We write  $\Delta^s = \bigcup_{n=1}^L f^{-n}B(z^s, \delta)$  and  $\Delta^u = \bigcup_{n=1}^L f^n B(z^u, \delta)$ . Since  $L$  is finite and  $W_k^{ss}(p) \cap W_k^{uu}(q) = \emptyset$ , we can choose  $\delta > 0$  small enough such that  $\Delta^s \cap \Delta^u = \emptyset$  and

$$f^n(z^s) \notin \Delta^s \cup \Delta^u \text{ and } f^{-n}(z^u) \notin \Delta^s \cup \Delta^u, \text{ for every } n \in \mathbb{N}.$$

Since  $f$  is transitive, we can find  $x \in B(z^s, \delta/\rho)$  such that  $f(x) \notin \Delta^s \cup \Delta^u$  and  $f^m(x) \in B(z^u, \delta/\rho)$  for a positive integer  $m$ . Now, by choice of  $\delta$  and  $x$ , applying [Proposition 2.6](#) simultaneously for  $z^s$  and  $z^u$  (which is possible since the perturbation is a local perturbation of  $f$ ) we can find a symplectic diffeomorphism  $g$ ,  $C^1$ -close to  $f$ , such that  $f(x) = g^{-n+1}(z^u)$  and  $g^{-1}(f(x)) = z^s$ . Therefore,  $z^s$  belongs to the forward orbit of  $z^u$ , and since  $g = f$  outside  $\Delta^s \cup \Delta^u$ , we have  $z^s, z^u \in W_k^{ss}(p(g)) \cap W_k^{uu}(q(g))$ . The lemma is proved. ■

### 2.3. Periodic symplectic linear systems

We recall the notion of periodic linear systems with transitions in the symplectic scenario as done in [12]. For the general definition and more details, see [6].

Let  $f$  be a homeomorphism defined on a topological space  $\Sigma$ . Let  $\mathcal{E}$  be a locally trivial vector bundle over  $\Sigma$  such that for every  $x \in \Sigma$ ,  $\mathcal{E}(x)$  is a symplectic vector space of the same dimension and endowed with the same symplectic form  $\omega$ . We define  $\mathcal{S}(\Sigma, f, \mathcal{E})$  as the set of maps  $A : \mathcal{E} \rightarrow \mathcal{E}$  such that, for every  $x \in \Sigma$ , the induced map  $A(x, \cdot)$  is a linear symplectic isomorphism  $\mathcal{E}(x) \rightarrow \mathcal{E}(f(x))$ , that is,  $\omega(u, v) = \omega(A(u), A(v))$ . Thus,  $A(x, \cdot)$  belongs to  $\mathcal{L}_\omega(\mathcal{E}(x), \mathcal{E}(f(x)))$ , the set of linear maps from  $\mathcal{E}(x)$  to  $\mathcal{E}(f(x))$  that preserve the symplectic form  $\omega$ . We define a norm  $|\cdot|$  on  $\mathcal{L}_\omega(\mathcal{E}(x), \mathcal{E}(f(x)))$  induced by Euclidean metrics of  $\mathcal{E}(x)$  and  $\mathcal{E}(f(x))$ :

$$|A(x, \cdot)| = \sup\{|A(x, v)|, v \in \mathcal{E}(x), |v| = 1\}.$$

For  $A \in \mathcal{S}(\Sigma, f, \mathcal{E})$ , we set  $|A| = \sup\{|A(x, \cdot)| : x \in \Sigma\}$ . Then we define the *norm* of  $A \in \mathcal{S}(\Sigma, f, \mathcal{E})$  as  $\|A\| = \max\{|A|, |A^{-1}|\}$ .

A *linear symplectic system* (or *linear symplectic cocycle over  $f$* ) is a 4-tuple  $(\Sigma, f, \mathcal{E}, A)$ , where  $\Sigma$  is a topological space,  $f$  is a homeomorphism on  $\Sigma$ ,  $\mathcal{E}$  is a symplectic vector bundle defined over  $\Sigma$ , and  $A \in \mathcal{S}(\Sigma, f, \mathcal{E})$  with  $\|A\| < \infty$ . When all points in  $\Sigma$  are periodic points of  $f$ , we say that  $(\Sigma, f, \mathcal{E}, A)$  is *periodic*.

Now, we recall the concept of *linear systems with transitions*. Given a set  $\mathcal{B}$ , a *word* with letters in  $\mathcal{B}$  is a finite sequence of elements of  $\mathcal{B}$ . The product of the word  $[a] = (a_1, \dots, a_n)$  by  $[b] = (b_1, \dots, b_m)$  is the word  $(a_1, \dots, a_n, b_1, \dots, b_m)$ . We say a word is *not a power* if  $[a] \neq [b]^k$  for every word  $[b]$  and  $k > 1$ .

With this notation, for a periodic symplectic linear system  $(\Sigma, f, \mathcal{E}, A)$ , if we consider the word  $[M_A(x)] = (A(f^{n-1}(x)), \dots, A(x))$ , where  $n$  is the period of  $x \in \Sigma$ , then the matrix  $M_A(x)$  is the product of the letters of the word  $[M_A(x)]$ , that is,

$$M_A(x) = A(f^{n-1}(x))A(f^{n-2}(x)) \dots A(x).$$

A periodic linear system is *diagonalizable* at the point  $x \in \Sigma$ , if  $M_A(x)$  has only real eigenvalues of multiplicity one.

**Definition 2.10** (Definition 1.6 of [6]): Given  $\varepsilon > 0$ , a periodic symplectic linear system  $(\Sigma, f, \mathcal{E}, A)$  admits  $\varepsilon$ -transitions if for every finite family of points  $x_1, \dots, x_n = x_1 \in \Sigma$ , there is an orthonormal system of coordinates of the linear bundle  $\mathcal{E}$  so that  $(\Sigma, f, \mathcal{E}, A)$  can now be considered as a system of matrices  $(\Sigma, f, A)$ , and for any  $(i, j) \in \{1, \dots, n\}^2$ , there exist  $k(i, j) \in \mathbb{N}$  and a finite word  $[t^{i,j}] = (t_1^{i,j}, \dots, t_{k(i,j)}^{i,j})$  of symplectic matrices, satisfying the following properties:

- (1) For every  $m \in \mathbb{N}$ ,  $\iota = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$ , and  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$  consider the word

$$[W(\iota, \alpha)] = [t^{i_1, i_m}] [M_A(x_{i_m})]^{\alpha_m} [t^{i_m, i_{m-1}}] [M_A(x_{i_{m-1}})]^{\alpha_{m-1}} \dots [t^{i_2, i_1}] [M_A(x_{i_1})]^{\alpha_1},$$

where the word  $[W(\iota, \alpha)]$  is not a power. Then there is  $x(\iota, \alpha) \in \Sigma$  such that

- the length of  $[W(\iota, \alpha)]$  is the period of  $x(\iota, \alpha)$ ;
  - the word  $[M_A(x(\iota, \alpha))]$  is  $\varepsilon$ -close to  $[W(\iota, \alpha)]$  and there is an  $\varepsilon$ -symplectic perturbation  $\tilde{A}$  of  $A$  such that the word  $[M_{\tilde{A}}(x(\iota, \alpha))]$  is  $[W(\iota, \alpha)]$ .
- (2) One can choose  $x(\iota, \alpha)$  such that the distance between the orbit of  $x(\iota, \alpha)$  and any point  $x_{i_k}$  is bounded by some function of  $\alpha_k$  which tends to zero as  $\alpha_k$  goes to infinity.

Given  $\iota, \alpha$  as above, the word  $[t^{i,j}]$  is an  $\varepsilon$ -transition from  $x_j$  to  $x_i$ . We call  $\varepsilon$ -transition matrices, the matrices  $T_{i,j}$  which are the product of the letters composing  $[t^{i,j}]$ . We say a periodic linear system admits transitions if, for any  $\varepsilon > 0$ , it admits  $\varepsilon$ -transitions.

**Remark 2.11:** Let  $x_1, \dots, x_n = x_1$  be in  $\Sigma$  and let  $[t^{i,j}]$  be an  $\varepsilon$ -transition from  $x_j$  to  $x_i$ . Then, for every  $\alpha, \beta \geq 0$ , the word

$$([M_A(x_i)]^\alpha [t^{i,j}] [M_A(x_j)]^\beta)$$

is also an  $\varepsilon$ -transition from  $x_j$  to  $x_i$ . Further, if  $[t^{j,k}]$  is an  $\varepsilon$ -transition from  $x_k$  to  $x_j$ , then the word  $[t^{i,j}][t^{j,k}]$  is an  $\varepsilon$ -transition from  $x_k$  to  $x_i$ . In particular, for any  $\varepsilon > 0$  and  $x \in \Sigma$ , we can consider non-trivial  $\varepsilon$ -transitions from  $x$  to itself.

The following lemma gives an example of linear systems having symplectic transitions. It is a symplectic version of Lemma 1.9 in [6].

**Lemma 2.12** (Lemma 4.5 in [12]): Let  $f$  be a symplectic diffeomorphism and let  $p$  be a hyperbolic periodic point of  $f$ . The derivative  $Df$  induces a continuous periodic symplectic linear system with transitions on the set  $\Sigma$  formed by hyperbolic periodic points homoclinically related to  $p$ .

A nice property of periodic linear systems  $(\Sigma, f, \mathcal{E}, A)$  admitting transitions is the existence of arbitrarily small perturbation of  $A$  which is diagonalizable and defined on a dense subset of  $\Sigma$ , see [6, Lemma 4.16] (and [12, Lemma 4.7] for a symplectic version).

### 3. Proof of Theorem A

We start this section proving that after a small perturbation we obtain a diffeomorphism in  $\mathcal{PH}_\omega^1(m)$  having a *special* non-hyperbolic periodic point. In the sequel, we use this result to prove [Theorem A](#).

**Proposition 3.1:** *Let  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ . For any small neighbourhood  $\mathcal{U} \subset \mathcal{PH}_\omega^1(m)$  of  $f$  and  $\varepsilon > 0$ , there exists a diffeomorphism  $g \in \mathcal{U}$  and a periodic point  $p$  of  $g$  such that  $Dg^{\tau(p,g)}(p)|_{E_m^c} = \text{Id}$ . Moreover, the orbit of  $p$  is  $\varepsilon$ -dense in  $M$ .*

The proof of this proposition is a direct consequence of the next result and [Proposition 2.9](#). Let us mention that part of this proposition is given in [4, Theorem 3.5].

**Proposition 3.2** (Proposition 5.3 in [12]): *For any  $\varepsilon > 0$ , and  $K > 0$ , there is  $l > 0$  such that any symplectic periodic  $2d$ -dimensional linear system  $(\Sigma, f, \mathcal{E}, A)$  bounded by  $K$  (i.e.  $\|A\| < K$ ) and having symplectic transitions satisfies the following points:*

- either  $A$  admits an  $l$ -dominated splitting,
- or there are a symplectic  $\varepsilon$ -perturbation  $\tilde{A}$  of  $A$  and a point  $x \in \Sigma$  such that  $M_{\tilde{A}}(x)$  is the identity matrix.

**Remark 3.3:** We remark that the periodic point  $x$  in the second item of the previous proposition can be found with  $\varepsilon$ -dense orbit in  $\Sigma$ , for any  $\varepsilon > 0$ .

**Proof of Proposition 3.1:** Let  $TM = E^s \oplus E^c \oplus E^u$  be the partially hyperbolic splitting with unbreakable centre given by  $f$  over  $M$ . Since  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ , by [Proposition 2.9](#), we can suppose, after a perturbation, that  $M = H(p, f)$ . We denote by  $\Sigma$  the set of hyperbolic periodic points homoclinically related to  $p$ , which are dense in  $M$ , and consider thus the periodic symplectic linear system  $(\Sigma, f, E^c, Df|_{E^c})$  having symplectic transitions.

Provided that  $f \in \mathcal{PH}_\omega^1(m)$ , the vector bundle  $E^c$  admits no dominated splitting for  $Df$ . It follows from [Proposition 3.2](#) and [Remark 3.3](#) that there exists  $x \in \Sigma$  with  $\varepsilon$ -dense orbit in  $M$  and a symplectic perturbation  $\tilde{A}$  of  $Df|_{E^c}$  along the orbit of  $x$ , such that  $M_{\tilde{A}}(x) = \text{Id}$ .

Hence, let  $\delta > 0$  be a small constant and  $\mathcal{U}$  a neighbourhood of  $f$  given by Franks lemma ([Lemma 2.3](#)), we use [Lemma 2.2](#) to find symplectic linear maps  $A_i$ ,  $\delta$ -close to  $Df(f^i(x))$ ,  $0 \leq i < \tau(x)$ , such that  $A_i|_{E^c} = \tilde{A}(f^i(x))$  and  $A_i|(E^{ss} \oplus E^{uu}) = Df|(E^{ss} \oplus E^{uu})$ . Thus, there exists  $g \in \mathcal{U}$  such that  $x$  still is a periodic point of  $g$  and  $Dg(g^i(x)) = A_i$ , for any  $0 \leq i < \tau(x)$ , which implies

$$Dg^{\tau(x)}|_{E^c}(x) = M_{\tilde{A}}(x) = \text{Id}|_{E^c}(x).$$

The proof is complete. ■

Using [Proposition 3.1](#), we are able to prove [Theorem A](#).

**Proof of Theorem A:** By Dolgopyat and Wilkinson [8], for any  $1 \leq m < d$ , there exists an open and dense subset  $\widetilde{\mathcal{PH}}_\omega^1(m) \subset \text{int}(\mathcal{PH}_\omega^1(m))$  such that every  $f \in \widetilde{\mathcal{PH}}_\omega^1(m)$  is a robustly transitive partially hyperbolic symplectic diffeomorphisms.

Recall we are denoting the set of Anosov symplectic  $C^1$ -diffeomorphisms by  $\mathcal{A}$ . We write

$$\widetilde{\mathcal{PH}}_\omega^1(d) = \text{Diff}_\omega^1(M) \setminus \overline{\mathcal{A} \cup \bigcup_{1 \leq m < d} \widetilde{\mathcal{PH}}_\omega^1(m)}.$$

Notice that  $\widetilde{\mathcal{PH}}_\omega^1(d)$  coincides with  $\mathcal{PH}_\omega^1(d)$  (the complement of the closure of partially hyperbolic and Anosov diffeomorphisms). In fact, if  $f \in \text{Diff}_\omega^1(M)$  is partially hyperbolic (or Anosov), then after a perturbation we can assume that the centre bundle  $E^c$  of  $f$  has no dominated splitting, say  $\dim E^c = 2m$ ,  $0 \leq m < d$ , in a neighbourhood of  $f$ . Hence, by continuity of the partially hyperbolic splitting,  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ .

Given  $1 \leq m \leq d$  and  $n \in \mathbb{N}$ , we denote by  $\mathcal{B}_{n,m} \subset \widetilde{\mathcal{PH}}_\omega^1(m)$  the subset of diffeomorphisms  $g$  having an  $m$ -elliptic periodic point, with  $1/n$ -dense orbit in  $M$ . Since  $m$ -elliptic periodic points are robust,  $\mathcal{B}_{n,m}$  is an open set.

Let  $f \in \widetilde{\mathcal{PH}}_\omega^1(m)$  and  $n \in \mathbb{N}$ . By [Proposition 3.1](#), there exists a diffeomorphism  $g \in \text{int} \mathcal{PH}_\omega^1(m)$ ,  $C^1$ -close to  $f$ , having a periodic point  $p$  with  $1/n$ -dense orbit in  $M$ , such that  $Dg^{\tau(p,g)}|_{E^c(p)} = Id$ . Let  $\{e_1, \dots, e_{2m}\}$  be a symplectic basis in  $E^c(p)$ , where the subspace spanned by  $\{e_i, e_{m+i}\}$ ,  $1 \leq i \leq m$ , is a symplectic subspace. We rearrange the vectors and we consider the basis  $\mathfrak{B} = \{e_1, e_{m+1}, \dots, e_i, e_{m+i}, \dots, e_m, e_{2m}\}$  of  $E^c(p)$ . Thus for any small positive value  $\alpha$ , we can define a symplectic linear map in  $T_p M$  induced by the following matrix with respect to the basis  $\mathfrak{B}$ :

$$A = \begin{bmatrix} \tilde{A} & 0 & 0 & \dots & 0 \\ 0 & \tilde{A} & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & & & \tilde{A} \end{bmatrix}, \text{ where } \tilde{A} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}.$$

Note this symplectic linear map restrict to the symplectic plane generated by  $\{e_i, e_{m+i}\}$  is a small rotation, whenever  $\alpha$  is small enough, for any  $1 \leq i \leq m$ . So, we can suppose  $A$  arbitrary close to  $Id$ . Hence, by [Lemma 2.2](#) and [Remark 2.5](#), we can find a symplectic linear map  $B: T_p M \rightarrow T_p M$  arbitrary close to  $Id$ , such that  $B|_{E^c} = A$  and  $B|(E^{ss} \oplus E^{uu}) = Id|(E^{ss} \oplus E^{uu})$ . Taking  $C = B \circ Dg(g^{\tau(p)} - 1)$  which is a symplectic linear map close to  $Dg(g^{\tau(p)} - 1)$ , we can use Franks lemma to perform a local perturbation of  $g$  and find a diffeomorphism  $h$   $C^1$ -close to  $g$ , such that  $p$  still is a periodic point of  $h$  and  $Dh^{\tau(p,h)}(p) = B \circ Dg^{\tau(p,g)}(p)$ . Then  $Dh^{\tau(p,h)}|_{E^c(p)} = A$ , which implies  $p$  is an  $m$ -elliptic periodic point. Since this perturbation keeps the orbit of  $p$ , this  $m$ -elliptic periodic point still has  $(1/n)$ -dense orbit in  $M$ , which implies  $h \in \mathcal{B}_{n,m}$ . When  $m = d$ ,  $p$  is a totally elliptic periodic point.

Therefore, the sets  $\mathcal{B}_{n,m}$  are open and dense inside  $\widetilde{\mathcal{PH}}_\omega^1(m)$ , which implies

$$\mathcal{R} = \mathcal{A} \cup \left( \bigcup_{1 \leq m \leq d} \bigcap_{n \in \mathbb{N}} \mathcal{B}_{n,m} \right)$$

is a residual subset of  $\text{Diff}_\omega^1(M)$ .

To finish, we remark that diffeomorphisms in  $\mathcal{R}$  satisfies one, and only one, of the three items in [Theorem A](#). ■

#### 4. Bounds for entropy: proof of Theorem B

Using periodic symplectic linear systems with transitions, we show that the supremum in

$$S_m(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|_{E^c(p)}) : p \text{ is a periodic hyperbolic point of } f \right\}$$

is achieved taking account just diagonalizable periodic points.

**Remark 4.1:** Notice that  $S_m(\cdot)$ ,  $0 < m \leq d$ , is a lower semi-continuous map. Indeed, denote by  $\text{Per}_h^n(f)$  the set of hyperbolic periodic points of period smaller or equal to  $n$ . Provided that hyperbolic periodic points are robust, the map  $S_m^n(\cdot) := \sup \left\{ \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|_{E^c(p)}) : p \in \text{Per}_h^n(f) \right\}$  is continuous. Then  $S_m(\cdot)$  is lower semi-continuous.

We denote the set of hyperbolic periodic points of  $f$  by  $\text{Per}_h(f)$ .

**Proposition 4.2:** *There exists a residual subset  $\mathcal{R}_m \subset \text{int } \mathcal{PH}_\omega^1(m)$ ,  $0 < m \leq d$ , such that if  $f \in \mathcal{R}_m$ , then*

$$S_m(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|_{E^c(p)}) : p \in \text{Per}_h(f) \text{ is diagonalizable} \right\}.$$

A key point in the proof of this proposition is the next technical result that allow us to perturb symplectic linear system getting a diagonalizable system having the largest eigenvalue (in absolute value) as close as we want to the previous ones.

**Lemma 4.3:** *Let  $(\Sigma, f, \mathcal{E}, A)$  be a periodic symplectic linear system with transition. For any  $\varepsilon > 0$ , and  $x \in \Sigma$ , there exists  $y \in \Sigma$  and an arbitrarily small symplectic perturbation  $\tilde{A}$  of  $A$  defined on the orbit of  $y$ , such that  $M_{\tilde{A}}(y)$  is diagonalizable. Moreover, if  $\lambda_x$  (resp.  $\lambda_y$ ) denotes the eigenvalue of  $M_A(x)$  (resp.  $M_{\tilde{A}}(y)$ ) with the largest absolute value, then*

$$\left| \frac{1}{\tau(x)} \log |\lambda_x| - \frac{1}{\tau(y)} \log |\lambda_y| \right| < \varepsilon,$$

where  $\tau(x)$  (resp.  $\tau(y)$ ) denotes the period of  $x$  (resp.  $y$ ).

**Proof:** After an arbitrarily small symplectic perturbation of  $A$  along a periodic orbit  $x$ , we can assume that  $M_A(x)$  has only simple eigenvalues and that any complex eigenvalue has rational argument. If  $\mathcal{E}$  is a  $2d$ -dimensional vector bundle, we can consider the partially hyperbolic splitting  $\mathbb{R}^{2d} = F_1 \oplus \dots \oplus F_n$  given by the eigenspaces associated to the eigenvalues of  $M_A(x)$ , which implies  $\dim F_i = 1$  or  $2$ . As a consequence of the symplectic structure, we have for every distinct  $1 \leq i, j \leq n$ :

- $F_i$  is an isotropic subspace,
- $F_i \oplus F_j$  is a symplectic subspace if  $i + j = n + 1$ , and
- $(F_i \oplus F_j)^\omega = \bigoplus_{k \neq i, j} F_k$ , if  $i + j = n + 1$ .

The fact that all eigenvalues of  $M_A(x)$  has rational argument implies that there exists a positive integer  $k$  such that  $(M_A(x))^k$  has only real eigenvalues. However, if  $\dim F_i = 2$ , then  $(M_A(x))^k|_{F_i}$  has a real eigenvalue with multiplicity two. We use [Lemma 2.2](#) to find a symplectic linear map  $H_i$  arbitrary close to identity, such that  $H_i|_{F_j} = Id$ , if  $j \neq i$ ,  $n+1-i$ , and  $H_i(M_A(x))^k|_{F_i \oplus F_{n+1-i}}$  have four distinct real eigenvalues. Moreover, for any  $\varepsilon > 0$ , we can choose such  $H_i$  such that defining  $M_{1,i} = H_i(M_A(x))^k$ , if  $\xi$  is an eigenvalue of  $M_{1,i}|_{F_i \oplus F_{n+1-i}}$ , then there is an eigenvalue  $\lambda$  of  $M_A(x)|_{F_i \oplus F_{n+1-i}}$  such that

$$\left| \log |\lambda| - \frac{1}{k} \log |\xi| \right| < \frac{\varepsilon}{2}. \quad (2)$$

Hence, given  $\varepsilon > 0$ , we can use the existence of the linear maps above  $H_i$  defined on  $F_i \oplus F_{n+1-i}$ , when  $\dim E_i = 2$ , to find a symplectic linear map  $H$  arbitrarily close to  $Id$  such that  $M_1 = H(M_A(x))^k$  has only real eigenvalues with multiplicity one, and (2) holds for every  $1 \leq i \leq n$ . We denote by  $E_i$  the one-dimensional  $M_1$ -invariant eigenspaces,  $1 \leq i \leq 2d$ .

Since the linear system has transitions, there exists a non-trivial word  $[t] = (t_1, \dots, t_r)$  of symplectic matrices which is a  $(\varepsilon/2)$ -transition from  $x$  to itself, see [Remark 2.11](#). We denote by  $T$  the symplectic matrix obtained by the product of the matrices in the word  $[t]$ . After an arbitrarily small symplectic perturbation of the matrix  $t_1$ , if necessary, we can suppose that

$$T(E_{2d}) \cap (E_1 \oplus \dots \oplus E_{2d-1}) = \{0\} \text{ and } T^{-1}(E_1) \cap (E_2 \oplus \dots \oplus E_{2d}) = \{0\}.$$

Thus by the choice of the partially hyperbolic splitting on  $\mathbb{R}^{2d}$ ,  $(M_1)^j T(E_{2d})$  converges to  $E_{2d}$  when  $j$  goes to infinity. Hence, taking  $j_{2d}$  large enough, by [Lemma 2.1](#), we can find a symplectic linear map  $L_{2d}$  close to  $Id$ , such that  $L_{2d}(M_1)^{j_{2d}} T(E_{2d}) = E_{2d}$  and  $L_{2d}|_{(E_1 \oplus \dots \oplus E_{2d-1})} = Id$ .

Analogously, provided that  $(M_1)^{-j} T^{-1}(E_1)$  converges to  $E_1$  when  $j$  goes to infinity, we can choose  $j_1 > 0$  to find a symplectic linear map  $L_1$  arbitrarily close to  $Id$ , such that  $L_1(E_1) = (M_1)^{-j_1} T^{-1}(E_1)$  and  $L_1|(E_2 \oplus \dots \oplus E_{2d}) = Id$ . Therefore, defining

$$\tilde{M}_1 = L_{2d}(M_1)^{j_{2d}} T(M_1)^{j_1} L_1,$$

we have that  $\tilde{M}_1(E_1) = E_1$  and  $\tilde{M}_1(E_{2d}) = E_{2d}$ . Once again, as a consequence of the symplectic structure,  $\tilde{M}_1$  also satisfies

$$\tilde{M}_1(E_2 \oplus \dots \oplus E_{2d-1}) = E_2 \oplus \dots \oplus E_{2d-1}.$$

In fact, if this is not true, then there exist  $v \in E_2 \oplus \dots \oplus E_{2d-1}$  and  $u \in E_1 \oplus E_{2d}$  such that  $\omega(\tilde{M}_1(v), u) \neq 0$ . On the other hand, by construction,  $\tilde{M}_1^{-1}(u) \in E_1 \oplus E_{2d}$  and then  $\omega(v, \tilde{M}_1^{-1}(u)) = 0$ , which gives a contradiction since  $\tilde{M}_1$  is symplectic.

Proceeding as before, we can find positive integers  $j_2, j_{2d-1}$  sufficiently large, symplectic linear maps  $L_2$  and  $L_{2d-1}$  arbitrarily close to  $Id$ , such that  $L_2(E_2) = (M_1)^{-j_2} (\tilde{M}_1)^{-1}(E_2)$ ,  $L_2|(E_1 \oplus E_3 \oplus \dots \oplus E_{2d}) = Id$ ,  $L_{2d-1}(M_1)^{j_{2d-1}} \tilde{M}_1(E_{2d-1}) = E_{2d-1}$ , and  $L_{2d-1}|(E_1 \oplus \dots \oplus E_{2d-2} \oplus E_{2d}) = Id$ . So, defining

$$\tilde{M}_2 = L_{2d-1}(M_1)^{j_{2d-1}} \tilde{M}_1(M_1)^{j_2} L_2,$$

we have that  $\tilde{M}_2(E_i) = E_i$ , for  $i = 1, 2, 2d - 1, 2d$ , and

$$\tilde{M}_2(E_3 \oplus \cdots \oplus E_{2d-2}) = E_3 \oplus \cdots \oplus E_{2d-2}.$$

Repeating the above process finitely many times, we also can find symplectic linear maps  $L_i$  close to identity, for any  $3 \leq i \leq 2d - 2$ , such that the maps

$$\tilde{M}_k = L_{2d-k+1}(M_1)^{j_{2d-k+1}} \tilde{M}_{k-1}(M_1)^{j_k} L_k,$$

are well defined for any  $3 \leq k \leq d$ , satisfying  $\tilde{M}_k(E_i) = E_i$ , for  $i = 1, \dots, k, 2d - k + 1, \dots, 2d$  and  $\tilde{M}_k(E_{k+1} \oplus \cdots \oplus E_{2d-k}) = E_{k+1} \oplus \cdots \oplus E_{2d-k}$ , where the last equality holds only when  $k < d$ . In particular,  $\tilde{M} = \tilde{M}_d$  preserves  $E_i$  for every  $1 \leq i \leq 2d$ , i.e.  $\tilde{M}(E_i) = E_i$ .

Now, since  $E_i$  is a one-dimensional subspace,  $1 \leq i \leq 2d$ , we can choose  $l > 0$  large enough and define  $M = (M_1)^l \tilde{M}$ , such that if  $\mu_i$  and  $\xi_i$  denote the eigenvalues of  $M|_{E_i}$  and  $M_1|_{E_i}$ , respectively, and  $\tau = \tau(x)k(j+l) + r$ , where  $j = j_{d+1}, \dots, j_{2d}$  (recall  $\tau(x)$  is the period of  $x \in \Sigma$  and  $r$  is the length of  $[t]$ ), then we have

$$\left| \frac{1}{\tau} \log |\mu_i| - \frac{1}{k\tau(x)} \log |\xi_i| \right| < \frac{\varepsilon}{2}. \quad (3)$$

Hence, if we denote by  $\mu_M$  (resp.  $\lambda_x$ ) the eigenvalue of the largest absolute value of  $M$  (resp.  $M_A(x)$ ), then Equations (2) and (3) give

$$\left| \frac{1}{\tau} \log |\mu_M| - \frac{1}{\tau(x)} \log |\lambda_x| \right| < \varepsilon. \quad (4)$$

Thus, since  $[t]$  is a non-trivial  $(\varepsilon/2)$ -transition from  $x$  to itself, there exists  $y \in \Sigma$  such that  $[M_A(y)]$  is  $(\varepsilon/2)$ -close to  $[\tilde{M}] = [M_A(x)]^{k(l+j_{d+1}+\dots+j_{2d})} [t] [M_A(x)]^{j_1+\dots+j_d}$ , and moreover,  $\tau(y)$  is equal to the length of  $[\tilde{M}]$  which is  $\tau$ .

Now, since  $H$  and  $L_i$ ,  $1 \leq i \leq 2d$ , are symplectic linear maps close to  $Id$ , then the matrix  $M_A(y)$  is close to  $M$ , which implies that there exists an arbitrarily small symplectic perturbation  $\tilde{A}$  of  $A$  defined on the orbit of  $y$ , such that  $M_{\tilde{A}}(y) = M$ . Therefore,  $M_{\tilde{A}}(y)$  is diagonalizable and if  $\lambda_y$  denotes the eigenvalue with the largest absolute value of  $M_{\tilde{A}}(y)$ , we have, by (4),

$$\left| \frac{1}{\tau(y)} \log |\lambda_y| - \frac{1}{\tau(x)} \log |\lambda_x| \right| < \varepsilon.$$

This completes the proof. ■

**Remark 4.4:** In the previous lemma, if we consider the additional hypothesis that the linear bundle  $\mathcal{E}$  has a dominated splitting for  $f$ ,  $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ , then the diagonalizable periodic point  $y$  could be found such that  $M_{\tilde{A}}(y)$  keeps invariant the subbundles  $\mathcal{E}_i$  and, moreover,

$$\left| \frac{1}{\tau(x)} \log |\lambda_{x,i}| - \frac{1}{\tau(y)} \log |\lambda_{y,i}| \right| < \varepsilon,$$

where  $\lambda_{x,i}$  and  $\lambda_{y,i}$  denote the eigenvalues with the largest absolute value of  $M_A(x)|\mathcal{E}_i$  and  $M_{\tilde{A}}(y)|\mathcal{E}_i$ , respectively, for every  $1 \leq i \leq n$ .

**Proof of Proposition 4.2:** For each  $0 < m \leq d$ , we define

$$\tilde{S}_m(f) = \sup \left\{ \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)), p \in \text{Per}_h(f) \text{ is diagonalizable} \right\}.$$

Similar to  $S_m(f)$ , the maps  $\tilde{S}_m(f)$  are lower semi-continuous for each  $0 < m \leq d$ , see [Remark 4.1](#). Hence there exists a residual subset  $\mathcal{R}_m \subset \text{int } \mathcal{PH}_\omega^1(m)$  where  $\tilde{S}_m(f)$  is continuous. Taking  $f \in \mathcal{R}_m$ , for any  $\varepsilon > 0$ , there exists a small neighbourhood  $\mathcal{U} \subset \text{int } \mathcal{PH}_\omega^1(m)$  of  $f$  such that

$$\tilde{S}_m(g) < \tilde{S}_m(f) + \frac{\varepsilon}{3}, \text{ for every } g \in \mathcal{U}. \quad (5)$$

By definition of  $S_m(f)$ , there exists a hyperbolic periodic point  $p$  of  $f$  such that

$$S_m(f) < \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)) + \frac{\varepsilon}{3}. \quad (6)$$

It follows from [Lemma 2.12](#) that  $Df$  induces a periodic symplectic linear system with transition  $(\Sigma, f, TM, Df)$ , where  $\Sigma$  is the set of hyperbolic periodic points of  $f$  homoclinically related to  $p$ . We can suppose  $\Sigma$  non-trivial since  $f$  belongs to a residual subset, see [\[17\]](#).

Let  $\delta > 0$  be a small constant given by Franks lemma ([Lemma 2.3](#)) for  $f$  and the neighbourhood  $\mathcal{U}$ . Provided that  $TM = E^s \oplus E^c \oplus E^u$ , by [Lemma 4.3](#) and [Remark 4.4](#), there exist  $\tilde{p} \in \Sigma$  and a symplectic  $\delta$ -perturbation  $\tilde{A}$  of  $Df$  along the orbit of  $\tilde{p}$  such that  $\tilde{p}$  is diagonalizable and, moreover,

$$\frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)) < \frac{1}{\tau(\tilde{p})} \log \sigma(M_{\tilde{A}}(\tilde{p})|E^c) + \frac{\varepsilon}{3}.$$

Hence, using Franks lemma, we can find a symplectic diffeomorphism  $g \in \mathcal{U}$  such that  $\tilde{p}$  is a diagonalizable hyperbolic periodic point of  $g$  satisfying

$$\frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)) < \frac{1}{\tau(\tilde{p}, g)} \log \sigma(Dg^{\tau(\tilde{p}, g)}|E^c(\tilde{p})) + \frac{\varepsilon}{3}. \quad (7)$$

Using, respectively, (6) and (7), definition of  $\tilde{S}_m(g)$ , and (5), we obtain

$$\begin{aligned} S_m(f) &< \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c(p)) + \frac{\varepsilon}{3} \\ &< \frac{1}{\tau(\tilde{p}, g)} \log \sigma(Dg^{\tau(\tilde{p}, g)}|E^c(\tilde{p})) + \frac{2\varepsilon}{3} \\ &\leq \tilde{S}_m(g) + \frac{2\varepsilon}{3} < \tilde{S}_m(f) + \varepsilon. \end{aligned}$$

Therefore, since  $\varepsilon > 0$  is arbitrary, we have  $S_m(f) \leq \tilde{S}_m(f)$ , for every  $f \in \mathcal{R}_m$ , which finishes the proof, since  $\tilde{S}_m(f) \leq S_m(f)$  by definition. ■

The next proposition is the main technical result in this paper.

**Proposition 4.5:** *Let  $0 < m \leq d$  and  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ . If  $p$  is a diagonalizable hyperbolic periodic point of  $f$ , then for any neighbourhood  $\mathcal{U}$  of  $f$  and any large positive integer  $n$ , there exists a diffeomorphism  $g \in \mathcal{U}$ , such that  $p$  still is a diagonalizable hyperbolic periodic point of  $g$ . Moreover, there exists a hyperbolic basic set  $\Lambda(p, g) \subset H(p, g)$  such that*

$$h_{\text{top}}(g|_{\Lambda(p, g)}) > \frac{1}{\tau(p, g)} \log \sigma(Df^{\tau(p, g)}|_{E^c}) - \frac{1}{n}.$$

We postpone the proof of this proposition to [Section 6](#). Now, let us prove [Theorem B](#).

**Proof of Theorem B:** For any positive integer  $n > 0$ , and every  $0 < m \leq d$ , we define  $\mathcal{B}_{m, n} \subset \text{int}(\mathcal{PH}_\omega^1(m))$  the subset of diffeomorphisms  $g$  such that

$$h_{\text{top}}(g) > S_m(g) - \frac{1}{n}.$$

Since  $S_m(\cdot)$  is a lower semi-continuous map defined in  $\mathcal{PH}_\omega^1(m)$ , there is a residual subset  $\mathcal{R}_m^* \subset \text{int}(\mathcal{PH}_\omega^1(m))$  where  $S_m(\cdot)$  is continuous. We can choose  $\mathcal{R}_m^*$  as a subset of  $\mathcal{R}_m$  in [Proposition 4.2](#).

Let us fix some  $0 < m \leq d$ . Given  $f \in \mathcal{R}_m^*$  and  $n > 0$ , we consider a small neighbourhood  $\mathcal{U} \subset \text{int}(\mathcal{PH}_\omega^1(m))$  of  $f$  such that

$$S_m(f) > S_m(\xi) - \frac{1}{4n}, \text{ for every } \xi \in \mathcal{U}. \quad (8)$$

By [Proposition 4.2](#), there exists a diagonalizable hyperbolic periodic point  $p$  of  $f$  such that

$$\frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|_{E^c}) > S_m(f) - \frac{1}{4n}. \quad (9)$$

From [Proposition 4.5](#), there exists  $g \in \mathcal{U}$ , and a hyperbolic basic set  $\Lambda(p, g) \subset H(p, g)$  such that

$$h_{\text{top}}(g|_{\Lambda(p, g)}) > \frac{1}{\tau(p, g)} \log \sigma(Dg^{\tau(p, g)}|_{E^c}) - \frac{1}{4n}. \quad (10)$$

Therefore, if  $\tilde{g} \in \mathcal{U}$  is a diffeomorphism  $C^1$ -close to  $g$ , then there is a continuation of the hyperbolic basic set  $\Lambda(p, g)$  which we denote by  $\Lambda(p(\tilde{g}), \tilde{g})$ , where  $p(\tilde{g})$  is a continuation of  $p$ . Using properties of entropy [\(10\)](#), continuity of the Lyapunov exponents [\(9\)](#) and [\(8\)](#),

respectively, for  $g$  sufficiently close to  $f$ , we obtain

$$\begin{aligned} h_{\text{top}}(\tilde{g}) &\geq h_{\text{top}}(\tilde{g}|\Lambda(p(\tilde{g}), \tilde{g})) = h_{\text{top}}(g|\Lambda(p, g)) \geq \frac{1}{\tau(p, g)} \log \sigma(Dg^{\tau(p, g)}|E^c) - \frac{1}{4n} \\ &\geq \frac{1}{\tau(p, f)} \log \sigma(Df^{\tau(p, f)}|E^c) - \frac{1}{2n} \geq S_m(f) - \frac{3}{4n} \geq S_m(\tilde{g}) - \frac{1}{n}. \end{aligned}$$

Hence, every  $\tilde{g}$  sufficiently  $C^1$ -close to  $g$  belongs to  $\mathcal{B}_{m,n}$ . So,  $\mathcal{B}_{m,n}$  contains an open and dense subset of  $\text{int}(\mathcal{PH}_\omega^1(m))$  in view of  $\mathcal{R}_m^*$  is dense in  $\text{int}(\mathcal{PH}_\omega^1(m))$ . We denote this subset by  $\mathcal{B}_{m,n}^*$ .

Now, [Theorem A](#) implies that  $\text{int}(\mathcal{PH}_\omega^1(1)) \cup \dots \cup \text{int}(\mathcal{PH}_\omega^1(d)) \cup \mathcal{A}$  is an open and dense subset of  $\text{Diff}_\omega^1(M)$ . Hence,

$$\mathcal{B}_n = \bigcup_{m=1}^d \mathcal{B}_{m,n}^* \cup \mathcal{A},$$

is an open and dense subset of  $\text{Diff}_\omega^1(M)$ . Therefore,

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{B}_n$$

is a residual subset in  $\text{Diff}_\omega^1(M)$ , and satisfies the properties required. In fact, if  $f \in \mathcal{R}$  and is non-Anosov, then there exists  $0 < m \leq d$  such that  $f \in \mathcal{B}_{m,n}$  for every  $n > 0$ . Hence,

$$h_{\text{top}}(f) \geq S_m(f).$$

The proof is complete. ■

## 5. Nice properties of strong invariant manifolds

In this section, we obtain properties of the strong invariant manifolds, which are essential to prove [Proposition 4.5](#). Roughly, we get that for any two hyperbolic periodic points  $p$  and  $\tilde{p}$  having strong stable and strong unstable directions well defined with the same dimension, we can perform a symplectic perturbation in order to obtain a symplectic diffeomorphism such that the continuation of the hyperbolic periodic point  $\tilde{p}$  has a strong quasi-transversal homoclinic intersection with *angle* as close as we want to the *angle* between the strong directions of  $p$ .

Let us make precise the notion of *angle between vector subspaces*. Given a Riemannian manifold  $M$  and vectors  $v, w \in T_q M$ , we define the angle between  $v$  and  $w$  by

$$\text{ang}(v, w) = \left| \tan \left[ \arccos \left( \frac{\langle v, w \rangle}{\|v\| \|w\|} \right) \right] \right|.$$

If  $E$  is a vector subspace of  $T_q M$ , the angle between a vector  $u \in T_q M$  and  $E$  is defined by

$$\text{ang}(v, E) = \min_{w \in E, \|w\|=1} \text{ang}(v, w).$$

Finally, if  $E, F \subset T_q M$  are subspaces, we define

$$\text{ang}(E, F) = \min_{w \in E, \|w\|=1} \text{ang}(w, F).$$

First of all, we show, after a perturbation, the existence of a hyperbolic periodic point having strong stable and unstable directions with arbitrary small angle.

**Lemma 5.1:** *Let  $1 \leq m \leq d$  and  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ . For any  $\varepsilon > 0$  and any neighbourhood  $\mathcal{U}$  of  $f$ , there exists a diffeomorphism  $g \in \mathcal{U}$  with a hyperbolic periodic point  $p$  having  $d - m + 1$ -strong stable and unstable manifolds  $W_{d-m+1}^{ss}(p)$  and  $W_{d-m+1}^{uu}(p)$  such that  $\text{ang}(E_{d-m+1}^{ss}(p), E_{d-m+1}^{uu}(p)) < \varepsilon$ .*

**Proof:** The proof is similar in spirit to the proof of [Theorem A](#). Fixed  $1 \leq m \leq d$ , let  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$ . Using [Proposition 3.1](#), we can find a diffeomorphism  $g$ ,  $C^1$ -close to  $f$ , having a periodic point  $p$  such that  $Dg^{\tau(p, g)}|_{E^c(p)} = \text{Id}$ . Now, for any  $\varepsilon > 0$ , we can choose a symplectic basis  $\{e_1, \dots, e_{2m}\}$  of  $E^c(p)$ , such that  $\text{ang}(e_1, e_{m+1}) < \varepsilon$ . Hence, taking small constants  $\varepsilon_1 \geq \varepsilon_2 \geq \dots \geq \varepsilon_m > 0$ , we define a symplectic linear map over  $E^c(p)$  close to identity, induced by the symplectic matrix  $A = (a_{ij})$  of order  $2m \times 2m$ , where  $a_{ii} = 1 - \varepsilon_i$ , if  $1 \leq i \leq m$ ;  $a_{ii} = (1 - \varepsilon_i)^{-1}$ , if  $m + 1 \leq i \leq 2m$ ; and  $a_{ij} = 0$ , if  $i \neq j$ .

Hence, as done in the proof of [Theorem A](#), we can find a diffeomorphism  $h$   $C^1$ -close to  $g$ , such that  $p$  still is a periodic point of  $h$ ,  $Dh^{\tau(p)}|_{E^c(p)} = A$  and  $Dh^{\tau(p)}|(E^c(p))^\omega = Dg$ . Therefore,  $p$  is a hyperbolic periodic point of  $h$ . Moreover, by the choice of  $A$  and the symplectic basis of  $E^c(p)$ , the  $d - m + 1$ -strong stable and unstable manifolds of  $p$  are well defined, with  $e_1 \in E_{d-m+1}^{ss}(p)$  and  $e_{m+1} \in E_{d-m+1}^{uu}(p)$ . Thus,  $\text{ang}(E_{d-m+1}^{ss}(p), E_{d-m+1}^{uu}(p)) < \varepsilon$  and the lemma is proved.  $\blacksquare$

Next, we state and prove the main result in this section.

**Lemma 5.2:** *Let  $f$  be a symplectic diffeomorphism on a  $2d$ -dimensional symplectic manifold  $M$  with two hyperbolic periodic points  $p$  and  $\tilde{p}$ , both having  $k$ -strong stable and unstable manifolds, for some  $1 \leq k \leq d$ . Given  $\varepsilon > 0$ , for any neighbourhood  $\mathcal{U}$  of  $f$  and any neighbourhood  $V$  of  $\tilde{p}$ , there exists a diffeomorphism  $g \in \mathcal{U}$ , such that the analytic continuation  $p(g)$  of  $p$  for  $g$  has a  $k$ -strong quasi-transversal homoclinic intersection  $q$  in  $V$ ,  $q \in W_k^{ss}(p(g)) \cap W_k^{uu}(p(g))$ . Moreover,  $T_q W_k^{ss}(p(g))$  and  $T_q W_k^{uu}(p(g))$  are  $\varepsilon$ -close to  $T_{\tilde{p}(g)} W_k^{ss}(\tilde{p}(g))$  and  $T_{\tilde{p}(g)} W_k^{uu}(\tilde{p}(g))$ , respectively.*

**Proof:** First, we fix  $\varepsilon > 0$  small enough. We may assume  $\tilde{p}$  is a fixed point of  $f$ , replacing  $f$  by an iterate if necessary. Now, we consider in the neighbourhood  $V$  of  $\tilde{p}$  a continuous splitting  $T_V M = E \oplus F \oplus G$ , with  $\dim E = \dim G = k$ , not necessarily invariant, which extends the  $Df(\tilde{p})$ -invariant partially hyperbolic splitting on  $T_{\tilde{p}} M$ , i.e.  $E_{\tilde{p}} = E_{\tilde{p}}^{ss}$ ,  $F_{\tilde{p}} = E_{\tilde{p}}^c$ , and  $G_{\tilde{p}} = E_{\tilde{p}}^{uu}$ .

Let  $\mathcal{U}' \subset \mathcal{U}$  be a small neighbourhood of  $f$  and  $\delta > 0$  given by [Lemma 2.3](#). It follows from [Lemma 2.1](#), there exists  $\gamma > 0$  such that if  $G' \subset T_V M$  is a  $k$ -dimensional isotropic subspace, then there is a linear map  $A_{G'}$ ,  $\delta$ -close to the identity map, such that  $\text{ang}(A(G'), E \oplus F) > \gamma$ .

With respect to the previous decomposition fixed on  $V$ , we define the strong unstable cone fields  $C_\alpha$  on  $V$ :

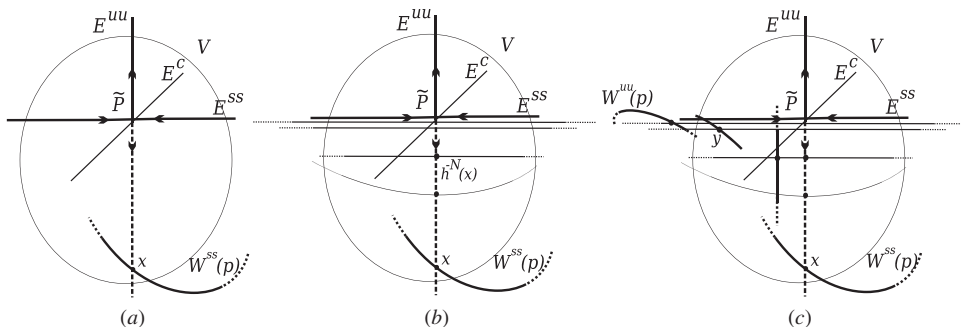
$$C_\alpha(x) = \{w \in T_x M : w = w_{cs} + w_u \text{ with } w_{cs} \in E \oplus F, w_u \in G, \text{ and } |w_{cs}| \leq \alpha |w_u|\}.$$

We fix  $\alpha > 0$  such that any vector  $v \in T_x M$  satisfying  $\text{ang}(v, E \oplus F) > \gamma$  must belong to  $C_\alpha(x)$ , for all  $x \in V$ . Now, since the decomposition at  $T_{\tilde{p}} M$  is partially hyperbolic and  $Df$ -invariant, by taking smaller neighbourhoods  $\mathcal{U}'' \subset \mathcal{U}'$  and  $V' \subset V$  of  $f$  and  $\tilde{p}$ , respectively, there exists  $l > 0$  such that, for any  $g \in \mathcal{U}''$  and any  $x \in \bigcap_{0 \leq i \leq l} g^{-i}(V')$ ,

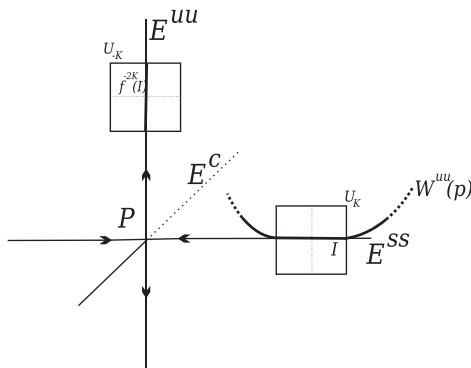
$$Dg^l(C_\alpha(x)) \subset C_\varepsilon(g^l(x)).$$

Also, for technical reasons, we can also suppose that the local strong stable manifold of  $\tilde{p}(g)$ ,  $g \in \mathcal{U}''$ , has two components outside  $V'$ .

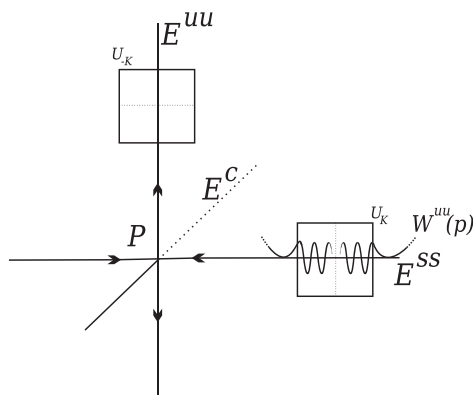
From [Lemma 2.8](#), we can perturb  $f$  to find an intersection between the  $k$ -dimensional strong unstable manifold of  $\tilde{p}$  and the  $k$ -dimensional strong stable manifold of  $p$ . See [Figure 1\(a\)](#). That is, there exists a diffeomorphism  $h \in \mathcal{U}''$ , such that there is  $x \in W_k^{ss}(p(h)) \cap W_k^{uu}(\tilde{p}(h))$ . Replacing  $x$  by a backward iterate we can suppose  $x \in V'$ , and since  $h^{-j}(x)$  converges to  $\tilde{p}(h)$ , after perturbation using [Lemmas 2.1](#) and [2.3](#), if necessary,



**Figure 1.** (a) Connecting  $W^{ss}(p)$  and  $W^{uu}(\tilde{p})$ . (b) Taking strong iterated disks of  $p$  close to  $\tilde{p}$ . (c) Connecting  $W^{uu}(p)$  and  $W^{ss}(p)$ .



**Figure 2.** Creating a  $(2d - 2m)$ -dimensional surface containing an interval of strong homoclinic intersections.



**Figure 3.** Newhouse's snakes.

we can assume  $T_{h^{-j}(x)}(W_k^{ss}(p(h)))$  converges to  $E_k^{ss}(\tilde{p}(h))$ , when  $j$  goes to infinity, using partially hyperbolic splitting properties. Moreover, we can assume the existence of open disks inside  $W_k^{ss}(p(h))$  containing  $h^{-j}(x)$  converging in the  $C^1$ -topology to the local strong stable manifold of  $\tilde{p}(h)$ , when  $j$  goes to infinity.

Thus, we choose a large positive integer  $N_1$  such that  $T_{h^{-N_1}(x)}(W_k^{ss}(p(h)))$  is  $\varepsilon$ -close to  $E_k^{ss}(\tilde{p}(h))$ , and such that for every  $n \geq N_1$ , there are disks  $D(n) \subset W_k^{ss}(p(h))$  containing  $h^{-n}(x)$  which are  $C^1$ -close to the local strong stable manifold of  $\tilde{p}(h)$ . See Figure 1(b). Also, recall that  $h^{-n}(x)$  belongs to  $V$ , for every  $n \geq N_1$ , since  $x \in W_k^{uu}(\tilde{p}(h))$ .

Since the local strong stable manifold of  $\tilde{p}(h)$  has two components outside  $V'$  and since  $h$  restricted to this local strong stable manifold is a contraction, we can take a point  $y \in (D(N_1 + l) \cap V') \subset W_k^{ss}(p(h))$  such that  $h^{-1}(y) \notin \text{cl}(V')$  and  $h^j(y) \in V'$  for  $0 \leq j \leq l$ . Once again, using Lemma 2.8, we can find a diffeomorphism  $\tilde{g} \in \mathcal{U}''$  arbitrary  $C^1$ -close to  $h$ , and  $\tilde{y}$  a point arbitrary close to  $y$  such that

- $\tilde{g}^{-1}(\tilde{y}) \notin \text{cl}(V')$  and  $\tilde{g}^j(\tilde{y}) \in V'$  for  $0 \leq j \leq l$ ;
- $\tilde{y} \in W_k^{ss}(p(\tilde{g})) \cap W_k^{uu}(p(\tilde{g}))$ ; and
- $T_{\tilde{g}^l(\tilde{y})}W_k^{ss}(p(\tilde{g}))$  is  $\varepsilon$ -close to  $T_{\tilde{p}(\tilde{g})}W_k^{ss}(\tilde{p}(\tilde{g}))$ .

Where the last item comes from the continuous variation with respect to the diffeomorphism of the  $k$ -strong stable manifold of  $p$  and  $\tilde{p}$  in compact parts.

Let us remark that, for  $\tilde{g}$ , it is possible that there is no more strong connection between  $W_k^{ss}(\tilde{g}(p))$  and  $W_k^{uu}(\tilde{g}(\tilde{p}))$ . However, as we can see in the remainder of the proof, this is unnecessary.

Considering the isotropic subspace  $T_{\tilde{y}}W_k^{uu}(p(g))$  in  $T_{\tilde{y}}M$ , by the choice of  $\delta$  and  $\gamma > 0$ , there exists a linear symplectic map  $\tilde{A}$ ,  $\delta$ -close to the identity map, such that  $\text{ang}(\tilde{A}(T_{\tilde{y}}W_k^{uu}(p(g))), E_{\tilde{y}} \oplus F_{\tilde{y}}) > \gamma$ . In particular,  $A = \tilde{A} \circ D\tilde{g}(\tilde{g}^{-1}(\tilde{y}))$  is a symplectic linear map  $\delta$ -close to  $D\tilde{g}(\tilde{g}^{-1}(\tilde{y}))$ . Thus, since  $\tilde{g} \in \mathcal{U}'' \subset \mathcal{U}'$  and  $\tilde{g}^{-1}(\tilde{y}) \notin \text{cl}(V')$  by Lemma 2.3, we can find a diffeomorphism  $g \in \mathcal{U}$  such that

- $\tilde{y} \in W_k^{ss}(p(g)) \cap W_k^{uu}(p(g))$ ;
- $T_{\tilde{g}^l(\tilde{y})}W_k^{ss}(p(g))$  is  $\varepsilon$ -close to  $T_{\tilde{p}(g)}W_k^{ss}(\tilde{p}(g))$ ;

- $g(x) = \tilde{g}(x)$  for every  $x \in V'$ ; and
- $\text{ang}(T_{\tilde{y}}W_k^{uu}(p(g)), E_{\tilde{y}} \oplus F_{\tilde{y}}) > \gamma$ . In particular,  $T_{\tilde{y}}W_k^{uu}(p(g)) \subset C_\alpha(\tilde{y})$ .

Thus, since  $\tilde{g} \in \mathcal{U}''$  and  $g$  coincides with  $\tilde{g}$  on  $V'$ , by the choice of  $l$ , we have  $Dg^l(T_{\tilde{y}}W_k^{uu}(p(g))) \in C_\varepsilon(\tilde{y})$ , which implies that it is  $\varepsilon$ -close to  $E_k^{uu}(\tilde{p}(g))$ . Finally, by continuity of the partially hyperbolic splitting, we also have  $T_{g^l(\tilde{y})}W_k^{uu}(p(g))$  is  $\varepsilon$ -close to  $T_{\tilde{p}(g)}W_k^{uu}(\tilde{p}(g))$ . See Figure 1(c). The lemma is proved. ■

## 6. Proof of Proposition 4.5

To prove Proposition 4.5, first we use Lemmas 5.1 and 5.2 to perturb a symplectic diffeomorphism  $f \in \mathcal{PH}_\omega^1(m)$  in order to find a symplectic  $(2d - 2m)$ -dimensional surface containing a hyperbolic periodic point  $p$  and an interval of homoclinic points in the intersection of the strong stable and strong unstable manifolds of  $p$ . After, we use arguments in spirit of those ones present in [7], to find a nice hyperbolic set by means of Newhouse's snake perturbations.

**Proof of Proposition 4.5:** Let  $f \in \text{int}(\mathcal{PH}_\omega^1(m))$  and  $p$  be a diagonalizable hyperbolic periodic point of  $f$ . In order to simplify notation, let us suppose that  $p$  is a fixed point.

Fixing an arbitrary  $\varepsilon > 0$ , by Lemma 5.1, after a perturbation, we can suppose there exists a hyperbolic periodic point  $\tilde{p}$  of  $f$  having  $d - m + 1$ -strong stable and unstable manifolds such that  $\text{ang}(E_{d-m+1}^{ss}(\tilde{p}), E_{d-m+1}^{uu}(\tilde{p})) < \varepsilon/2$ . Thus, since we have defined  $(d - m + 1)$ -strong manifolds for  $p$ , we can use Lemma 5.2, to find a diffeomorphism  $f_1$   $C^1$ -close to  $f$ , such that  $p(f_1)$  has a  $(d - m + 1)$ -strong quasi-transversal homoclinic point  $q \in W_{d-m+1}^{ss}(p(f_1)) \cap W_{d-m+1}^{uu}(p(f_1))$ , such that

$$\text{ang}(T_q W_{d-m+1}^{ss}(p(f_1)), T_q W_{d-m+1}^{uu}(p(f_1))) < \frac{2\varepsilon}{3}.$$

Since  $f_1$  is arbitrary  $C^1$ -close to  $f$ , we can suppose  $p(f_1)$  still is a diagonalizable hyperbolic fixed point.

Now, we use a Pasting lemma of Arbieto and Matheus [1] to linearize the diffeomorphism in a small neighbourhood  $V$  of  $p(f_1)$ . More precisely, we can find  $f_2$   $C^1$ -close to  $f_1$  such that  $p(f_1) = p(f_2)$  and  $f_2 = Df_1(p)$  in  $V$  (in local coordinates). We remark that after this perturbation, we could have no more a strong quasi-transversal intersection between  $W_{d-m+1}^{ss}(p(f_2))$  and  $W_{d-m+1}^{uu}(p(f_2))$  near  $q$ . However, provided that these submanifolds vary continuously in compact parts with respect to the diffeomorphism, this intersection could be recovered after a local perturbation of  $f_2$  in a neighbourhood of  $q$ .

Let  $T_x V = E(x) \oplus F(x) \oplus G(x)$  be a continuous extension (not necessarily invariant) of the local linear coordinates  $\mathbb{R}^{2d} = E^{ss}(p) \oplus E^c(p) \oplus E^{uu}(p)$  induced by  $Df_2(p) = Df_1(p)$ , i.e.  $E(p) = E^{ss}(p)$ ,  $F(p) = E^c(p)$ , and  $G(p) = E^{uu}(p)$ , with  $F(x)$  symplectic and  $E(x)$ ,  $G(x)$  isotropic.

For  $1 < m \leq d$ , we consider  $\tilde{E}^c(q) \subset E^c(q)$  a  $(2m - 2)$ -dimensional symplectic subspace having trivial intersection with  $T_q W_{d-m+1}^{ss} \oplus T_q W_{d-m+1}^{uu}$ . We set  $\tilde{E}^c(f_2^j(q)) := Df_2^j(\tilde{E}^c(q)) \subset E^c(f_2^j(q))$ .

We remark now that the local strong stable and unstable manifolds of  $p$  coincide with their strong directions restrict to  $V$ , since  $f_2$  is linear in this neighbourhood. That is, the local strong stable (resp. unstable) manifold of  $p$  is  $E_{d-m+1}^{ss}(p) \cap V$  (resp.  $E_{d-m+1}^{uu}(p) \cap V$ ). Thus, since  $q$  is a strong homoclinic point, for any large positive integer  $k$ ,

$$f_2^k(q) \in E_{d-m+1}^{ss}(p) \cap V \text{ and } f_2^{-k}(q) \in E_{d-m+1}^{uu}(p) \cap V.$$

In the reminder of this proof by abuse of notation, we denote by  $p$  all its continuations with respect to nearby diffeomorphisms and we denote the  $(d-m+1)$ -strong directions and manifolds of  $p$  only by  $E^*(p)$  and  $W^*(p)$ ,  $*$  =  $ss$ ,  $uu$ , respectively. The same for the  $(2m-2)$ -central direction  $E^c(p)$ . ■

**Lemma 6.1:** *There exists a symplectic diffeomorphism  $f_3$   $C^1$ -close to  $f_2$ , a positive integer  $K$ , a neighbourhood  $V' \subset V$  of  $p$ , and small neighbourhoods  $U_{-K}$ ,  $U_K \subset V'$  of  $f_3^{-K}(q)$  and  $f_3^K(q)$ , respectively, such that*

- $f_3 = Df_3(p) = Df_1(p)$  is still linear on  $V'$  (in local coordinates);
- $f_3^{2K}((T_{f_3^{-K}(q)}W^{ss}(p, f_3) \oplus T_{f_3^{-K}(q)}W^{uu}(p, f_3)) \cap U_{-K}) \subset (T_{f_3^K(q)}W^{ss}(p, f_3) \oplus T_{f_3^K(q)}W^{uu}(p, f_3)) \cap U_K$ .

**Proof:** First, we remark that after a local perturbation, if necessary, we can suppose that, for any large positive integer  $k$ ,

$$Df_2^{-k}(T_qW^{ss}(p)) \cap (\tilde{E}^c(f_2^{-k}(q)) \oplus T_{f_2^{-k}(q)}W^{uu}(p)) = 0,$$

and

$$Df_2^k(T_qW^{uu}(p)) \cap (T_{f_2^k(q)}W^{ss}(p) \oplus \tilde{E}^c(f_2^k(q))) = 0.$$

It follows from the dominated splitting properties that  $Df_2^{-k}(T_qW^{ss}(p))$  (resp.  $Df_2^k(T_qW^{uu}(p))$ ) converges to  $E^{ss}(p)$  (resp.  $E^{uu}(p)$ ) when  $k$  goes to infinity. Hence, we can choose a large positive integer  $K$  such that both  $Df_2^{-K}(T_qW^{ss}(p))$  and  $Df_2^K(T_qW^{uu}(p))$  are close enough to  $E(f_2^{-K}(q))$  and  $G(f_2^K(q))$ , respectively. Since for any hyperbolic periodic point of a symplectic map its stable (resp. unstable) manifold is a Lagrangian submanifold, see Remark 2.4, we have that  $T_xW^s(p)$  (resp.  $T_xW^u(p)$ ) is a Lagrangian subspace for any  $x \in W^s(p)$  (resp.  $W^u(p)$ ). In particular,  $T_xW^{ss}(p)$  (resp.  $T_xW^{uu}(p)$ ) is an isotropic subspace, see Remark 2.5, for any  $x \in W^{ss}(p)$  (resp.  $W^{uu}(p)$ ).

Hence, from Lemma 2.1, there is a symplectic linear map  $\tilde{B}$  on  $\mathbb{R}^{2d}$ ,  $C^1$ -close to  $Id$ , such that

$$\tilde{B}(Df_2^{-K}(T_qW^{ss}(p))) = E(f_2^{-K}(q)) \text{ and } \tilde{B}(\tilde{E}^c(f_2^{-K}(q)) \oplus T_{f_2^{-K}(q)}W^{uu}(p))) = Id.$$

Thus, taking  $B = Df_2 \circ \tilde{B}^{-1}$ , we can use Franks lemma to perform a local  $C^1$ -perturbation  $f_{2,1}$  of  $f_2$  in a neighbourhood  $U_{2,1}$  of  $f_2^{-K}(q)$ , if necessary, such that  $f_{2,1} = f_2$  in  $\{f_2^{-K}(q)\} \cup$

$(M \setminus U_{2,1})$  and

$$Df_{2,1}^{-K}(T_q W^{ss}(p)) = E(f_{2,1}^{-K}(q)) = E(f_2^{-K}(q)).$$

Moreover, this perturbation does not change the action of  $Df_2$  over  $\tilde{E}^c(O(q)) \oplus T_{O(q)} W^{uu}(q)$ .

On the other hand, as before, we can perform a local  $C^1$ -perturbation  $f_{2,2}$  of  $f_{2,1}$  in a neighbourhood  $U_{2,2}$  of  $f_{2,1}^{K-1}(q) = f_2^{K-1}(q)$ , if necessary, such that  $f_{2,2} = f_{2,1} = f_2$  in  $\{f_2^{K-1}(q)\} \cup (M \setminus U_{2,2})$  and

$$Df_{2,2}^{2K}(T_{f_2^{-K}(q)} W^{uu}(p)) = G(f_{2,2}^K(q)),$$

and keeps the action of  $Df_{2,1}$  over  $\tilde{E}^c(O(q)) \oplus T_{O(q)} W^{ss}(q, f_{2,1})$ . In particular,  $E(f_2^{-K}(q)) = T_{f_{2,2}^{-K}(q)} W^{ss}(p)$ . We take  $V' \subset V$ , neighbourhood of  $p$  such that  $f_{2,2}^{-K}(q), f_{2,2}^K(q) \in V'$  and  $U_{2,1} \cup U_{2,2} \cap V = \emptyset$ . Thus,  $f_{2,2}$  is still linear on  $V'$ , which implies

$$Df_{2,2}^{2K}(T_{f_{2,2}^{-K}(q)} W^{ss}(p) \oplus T_{f_{2,2}^{-K}(q)} W^{uu}(p)) = T_{f_{2,2}^K(q)} W^{ss}(p) \oplus T_{f_{2,2}^K(q)} W^{uu}(p).$$

Notice that the above perturbations do not change neither the orbit of  $q$  nor of  $p$ .

Finally, we use the Pasting lemma to get a symplectic diffeomorphism  $f_3$  arbitrarily close to  $f_{2,2}$  that linearizes  $f_{2,2}^{2K}$  in a small neighbourhood  $U \subset V'$  of  $f_{2,2}^{-K}(q)$  such that

$$f_3^{2K}((E(f_3^{-K}(q)) \oplus T_{f_3^{-K}(q)} W^{uu}(p)) \cap U) = (T_{f_3^K(q)} W^{ss}(p) \oplus G(f_3^K(q))) \cap V'.$$

To finish the proof of the lemma, we take  $U_{-K} = U$  and  $U_K = f_3^{2K}(U)$ . ■

**Lemma 6.2:** *There is a symplectic diffeomorphism  $f_4$ ,  $C^1$ -close to  $f_3$ , such that  $f_4^{2K}(T_{f_4^{-K}(q)} W^{ss}(p, f_3)) \cap T_{f_4^K(q)} W^{uu}(p, f_4)$  contains a segment of line  $I$ .*

**Proof:** Since strong invariant manifolds vary continuously in compact parts with respect to the diffeomorphism, we have  $\text{ang}(T_q W^{ss}(p, f_3), T_q W^{uu}(p, f_3)) < \varepsilon$  whenever  $f_3$  is sufficiently  $C^1$ -close to  $f_1$ . Let  $u \in T_q W^{ss}(p, f_3)$  and  $w \in T_q W^{uu}(p, f_3)$  be unit vectors such that  $\text{ang}(u, w) < \varepsilon$  and define  $E$  and  $W$  the one-dimensional subspaces generated by  $u$  and  $w$ , respectively. In particular,  $E$  and  $W$  are near-isotropic subspaces. Also, recall that  $T_q W^{ss}(p, f_3), T_q W^{uu}(p, f_3)$  is a symplectic space (see Section 4).

Thus, it follows from Lemma 2.1 that there is a symplectic linear map  $A$  defined on  $T_q W^{ss}(p, f_3) \oplus T_q W^{uu}(p, f_3)$  such that  $A(W) = E$ . Notice that  $A$  is  $C^1$ -close to  $\text{Id}|(T_q W^{ss}(p, f_3) \oplus T_q W^{uu}(p, f_3))$ , whenever  $\varepsilon$  is sufficiently small. By Lemma 2.2, there is a symplectic linear map  $L: T_q M \rightarrow T_q M$  such that  $L|(T_q W^{ss}(p, f_3) \oplus T_q W^{uu}(p, f_3)) = A$  and  $L|\tilde{E}^c(q) = \text{Id}$ . Then, using Franks lemma, we make a local perturbation of  $f_3$  in a neighbourhood of  $f_3^{-1}(q)$  to get a symplectic diffeomorphism  $f_4$   $C^1$ -close to  $f_3$  (and so to  $f_2$ ) such that  $Df_4(f_4^{-1}(q)) = L \circ Df_3(f_3^{-1}(q))$ , which implies that  $T_q W^{ss}(p, f_4) \cap T_q W^{uu}(p, f_4)$  is non-trivial. We stress that the above local perturbation keeps unchanged the orbits of  $p$  and  $q$ .

Since  $T_q W^{ss}(p, f_4) \cap T_q W^{uu}(p, f_4)$  is non-trivial, there is an interval of strong homoclinic points, that is,  $f_4^{2K}(T_{f_4^{-K}(q)} W^{uu}(p, f_4)) \cap U \cap T_{f_4^K(q)} W^{ss}(p, f_4)$  contains a segment of line  $I$ . See Figure 2 The proof is finished. ■

Let  $f_4$  be  $C^1$ -close to  $f_3$  (and so to  $f_2$ ) as given by Lemma 6.2. After a perturbation, if necessary, we assume that  $T_q W^{ss}(p, f_4) \cap T_q W^{uu}(p, f_4)$  is a one-dimensional subspace. Hence, if  $I$  is sufficiently small, this holds for every  $x \in I$ :  $T_x W^{ss}(p, f_4) \cap T_x W^{uu}(p, f_4)$  is a one-dimensional subspace for every  $x \in I$ .

What follows is the construction of a local perturbation of the identity map in  $U_K$  to finish the proof of Proposition 4.5. In order to simplify notation, let us set  $\tilde{E}^{ss}(f_4^K(q)) := T_{f_4^K(q)} W^{ss}(p, f_3)$ ,  $\tilde{E}^{uu}(f_4^K(q)) := T_{f_4^K(q)} W^{uu}(p, f_3)$ . In  $U_K$  we consider linear local coordinates given by the extension of partial hyperbolic decomposition of  $\tilde{E}^{ss}(f_4^K(q)) \oplus \tilde{E}^c(f_4^K(q)) \oplus \tilde{E}^{uu}(f_4^K(q))$ . That is, for every  $x \in U_K$ , there exists  $x^s \in \tilde{E}^{ss}(f_4^K(q))$ ,  $x^c \in \tilde{E}^c(f_4^K(q))$ , and  $x^u \in \tilde{E}^{uu}(f_4^K(q))$  such that  $x = (x^s, x^c, x^u)$ . By construction, in these coordinates,  $f_4^K(q) = (0, 0, 0)$ .

Now, let  $E_1^s = Df_3^K(E)$  ( $E$  as in the proof of Lemma 6.2) be the one-dimensional subspace of  $\tilde{E}^{ss}(f_4^K(q))$  containing the interval  $I$  and denote  $E_2^s = Df_3^K(E^s(q))$ . So,  $\tilde{E}^{ss}(f_4^K(q)) = E_1^s \oplus E_2^s$ . Similarly, we denote  $E_1^u = Df_3^K(W)$  and  $E_2^u = Df_3^K(E^u(q))$ . Thus,  $\tilde{E}^{uu}(f_4^K(q)) = E_1^u \oplus E_2^u$  and  $E_1^s \oplus E_1^u$  is a symplectic subspace. According to these last direct sums, for  $x^s \in \tilde{E}^{ss}(f_4^K(q))$  and  $x^u \in \tilde{E}^{uu}(f_4^K(q))$ , we write  $x^s = (x_1^s, x_2^s)$  and  $x^u = (x_1^u, x_2^u)$ .

Let  $N$  be a large positive integer and  $\delta > 0$  be an arbitrary small real number. Using the Pasting lemma of [1], we find a symplectic diffeomorphism  $\Theta_N: M \rightarrow M$ ,  $\delta$ - $C^1$ -close to  $Id$ ,  $\Theta_N = Id$  in the complement of  $U_K$ , and such that, for  $r > 0$  small enough and  $x \in B(f_4(q), r) \subset U_K$ ,

$$\Theta_N(x_1^s, x_2^s, x^c, x_1^u, x_2^u) = \left( x_1^s, x_2^s, x^c, x_1^u + A \cos \frac{\pi x_1^s N}{2r}, x_2^u \right),$$

where  $A = \frac{2Rr\delta}{\pi N}$ ,  $R$  a constant depending only on the symplectic coordinate on  $U_K$ . Notice that  $\Theta_N(I) \cap I$  contains  $N$  distinct points and that  $\Theta_N(\tilde{E}^{ss}(f_4^K(q)) \oplus \tilde{E}^{uu}(f_4^K(q)) \cap U_K) \subset \tilde{E}^{ss}(f_4^K(q)) \oplus \tilde{E}^{uu}(f_4^K(q)) \cap U_K$ .

Now, we use  $\Theta_N$  to get a symplectic perturbation of  $f_4$ . We set  $f_{4,N} = \Theta_N \circ f_4$  which is  $\delta$ - $C^1$ -close to  $f_4$  and satisfies

- $f_{4,N} = f_4$  in the complement of  $f_4^{-1}(U_K)$ ;
- $f_{4,N}^{2K}((T_{f_4^{-K}(q)} W^{ss}(p, f_3) \oplus T_{f_4^{-K}(q)} W^{uu}(p, f_3)) \cap U_{-K}) \subset (T_{f_4^K(q)} W^{ss}(p, f_3) \oplus T_{f_4^K(q)} W^{uu}(p, f_3)) \cap U_K$ ;
- $f_{4,N}^{2K}(T_{f_4^{-K}(q)} W^{uu}(p, f_4) \cap U_{-K}) \cap (T_{f_4^K(q)} W^{ss}(p, f_4) \cap U_K)$  contains at least  $N$  distinct points.

By construction,  $T_{f_4^{-K}(q)} W^{uu}(p, f_4) \cap U_{-K} \subset E^{uu}(p)$  and  $T_{f_4^K(q)} W^{ss}(p, f_4) \cap U_K \subset E^{ss}(p)$ . Since we are considering  $V'$  in local coordinates where  $f_4$  is linear, these sets belong to  $W^{uu}(p)$  and  $W^{ss}(p)$  for  $f_{4,N}$ , respectively.

This perturbation is a kind of Newhouse's snake perturbation for higher dimensions, i.e., it destroys the interval of homoclinic intersections and creates  $N$  transversal homoclinic points for  $p$  inside  $U_K$ . See Figure 3.

After we have found these strong homoclinic points in  $U_K$ , we follow the arguments developed in step 3 of the proof of [7, Proposition 3.1], in order to find a hyperbolic set  $\Lambda$  satisfying the proposition. Here  $\Lambda \cap V' \subset (E^{ss}(p) \oplus E^{uu}(p))$ , and this is a key point. For completeness, we sketch how we use the arguments in [7].

First, we choose a positive integer  $t$  and a rectangle  $D_t$  in  $(E^{ss}(p) \oplus E^{uu}(p)) \cap U_K$  containing the  $N$  transversal homoclinic points obtained above and also that  $g^t(D_t) \cap D_t$  has  $N$  disjoint connected components. We also require that  $t$  is the smallest possible such that  $D_t$  is  $(A/2)$ - $C^1$ -close to  $E^{ss}(p) \cap U_K$  and  $f_{4,N}^t(D_t)$  is  $(A/2)$ - $C^1$  close to the connected component of  $W^{uu}(p) \cap U_K$  containing the  $N$  transversal homoclinic points. Notice,  $t$  goes to infinity when  $N$  goes to infinity.

Therefore, the maximal invariant set in the orbit of  $D_t$  is a hyperbolic set  $\Lambda(p, N)$  is conjugated to a product of shift maps, with topological entropy

$$h_{\text{top}}(f_{4,N} | \Lambda(p, N)) = \frac{1}{t} \log N.$$

Since the dynamics of  $f_{4,N}$  is linear on  $V'$  and  $\Lambda(p, N)$  belongs to  $V' \subset E^{ss}(p) \oplus E^{uu}(p)$ , it follows straightforward from the proof of [7, Lemma 4.2] an upper estimate for  $A$ , as follows.

**Lemma 6.3:** *For  $A$  and  $t$  defined as before, there exists a positive integer  $K_1$  independent of  $A$ , such that*

$$A < K_1 \max\{\|Df_{4,N}^{-t}|E^{uu}(p)\|, \|Df_{4,N}^t|E^{ss}(p)\|\}.$$

Finally, given a positive integer  $k$ , we can choose  $N$  large enough such that by the choice of  $A$  and Lemma 6.3,

$$\frac{1}{t} \log N > \min \left\{ \frac{1}{t} \log \|Df_{4,N}^{-t}|E^{uu}(p)\|^{-1}, \frac{1}{t} \log \|Df_{4,N}^t|E^{ss}(p)\|^{-1} \right\} - \frac{1}{2k}. \quad (11)$$

Since  $f_{4,N} = f_4$  in  $V'$  for every  $N$ , when  $t$  goes to infinity, the minimum in (11) goes to the smallest positive Lyapunov exponent of  $p$  restrict to  $E^{ss}(p) \oplus E^{uu}(p)$ . As  $\dim(E^{ss} \oplus E^{uu}) = 2(d - m + 1)$ , the smallest positive Lyapunov exponent of  $p$  restricts to this subspace is equal to  $\log \sigma(Df_{4,N}|E_m^c(p))$ . Thus, taking  $g = f_{4,N}$  for  $N$  large enough, we have that

$$h_{\text{top}}(g | \Lambda(p, N)) > \log \sigma(Dg|E_m^c(p)) - \frac{1}{n}.$$

In the general case, when  $p$  is a hyperbolic periodic point, we also can create a strong homoclinic intersection between  $W^{ss}(p)$  and  $W^{uu}(p)$ , as before, and thus repeating the above arguments for  $\tilde{f} = f^{\tau(p,f)}$ , we can find a nice hyperbolic set  $\tilde{\Lambda}(p, N)$  of a symplectic diffeomorphism  $\tilde{g} = g^{\tau(p,f)}$ , such that  $\tilde{g}$  is  $C^1$ -close to  $f$ , satisfying

$$h_{\text{top}}(\tilde{g} | \tilde{\Lambda}(p, N)) > \log \sigma(D\tilde{g}|E_m^c(p)) - \frac{1}{n}.$$

Therefore, the proposition is proved, since the hyperbolic set

$$\Lambda(p, N) = \bigcup_{0 \leq i < \tau(p, f)} g^i(\tilde{\Lambda}(p, N))$$

of  $g$  has topological entropy:

$$h_{\text{top}}(g|\Lambda(p, N)) = \frac{1}{\tau(p, f)} h_{\text{top}}(\tilde{g}|\tilde{\Lambda}(p, N)).$$

The proof is complete.

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