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Unexpected behavior of Caputo fractional derivative

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Abstract This paper discusses the modeling via mathematical methods based on fractional calculus, using Caputo fractional derivative. From the fractional models associated with harmonic oscillator, logistic equation and Malthusian growth, an unexpected behavior of the Caputo fractional derivative is discussed. The difference between the rate of variation and the order of the Caputo fractional derivative is explained.

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1 Introduction

The obtaining of a differential equation whose solution describes well the reality brings great difficulty. In Albert Einstein words "One thing I have learned in a long life: that all our science, measured against reality, is primitive and childlike—and yet it is the most precious thing we have". Usually, the closer we are to perfectly describe a real problem, the bigger are the number of variables involved and the complexity of the equations.

In this sense, the Non-Integer Order Calculus, traditionally known as Fractional Calculus (FC),[1] which is the branch of mathematics that deals with the study of integrals and derivatives of non-integer orders, has played an outstanding role (Machado et al. 2011). Several mathematicians and applied researchers have obtained important results and generalizations from modeling real processes using FC (Arafa et al. 2016; Camargo et al. 2009a, 2012; Camargo and de Oliveira 2015; Debnath 2003; Mainardi 2009; Ortigueira and Machado 2015; Podlubny 1999; Soubhia et al. 2010).

Considering a differential equation that describes a specific phenomenon, a common way to use fractional modeling is to replace the integer order derivatives by a non-integer derivatives, usually with order lower than or equal to the order of the original derivatives, so that the usual solution may be recovered as a particular case (Camargo and de Oliveira 2015).

Although there is no trivial physical and geometrical interpretation for the fractional derivative and the fractional integral (Podlubny 2002; Tavassoli et al. 2013), fractional order differential equations are naturally related to systems with memory, since the fractional derivatives are usually not local operators, i.e., calculating time-fractional derivative at some time requires all the previous (Camargo and de Oliveira 2015; Podlubny 1999). Processes with memory exist in many biological systems (Arafa et al. 2012, 2016; Diethelm et al. 2005; Elsadany and Matouk 2014; Matouk et al. 2015; El-Sayed et al. 2009, 2007). Besides, fractional differential equation can help us to reduce the errors arising from the neglected parameters in modeling real life phenomena (Arafa et al. 2016; Gutierrez et al. 2010; Mainardi 2009).

In engineering, there are several applications of fractional calculus (Sabatier et al. 2007), for example, in the study of control and dynamical systems (Matigon 1996; Matouk 2010, 2015). Moreover, in physics, there are several potential applications of fractional derivative (Hilfer 2000), for example, in the generalization of the classical equations (Camargo et al. 2008, 2009a, b, c).

In medicine, it has been deduced that the membranes of cells of biological organism have fractional order electrical conductance and then, they are classified in groups of non-integer order models. Fractional derivatives embody essential features of cell rheological behavior and have enjoyed greatest success in the field of rheology (Arafa et al. 2016). Some mathematical models in HIV shown that fractional models are more approximate than their integer order form (Arafa et al. 2016; Diethelm et al. 2005).

With the aim of solving fractional partial differential equations and generalize results, several definitions to the "fractional derivative" (FD) have been proposed (de Oliveira and

[1] In fact, the name Fractional Calculus is not accurate, since the order of an integral and a derivative can be real and also complex.

Machado 2014). In a recent paper (Khalia et al. 2014) a new definition of FD, called conformable FD, has been proposed. Another example is the so-called local FD (Kolwankar and Gangal 1996). Furthermore, Grünwald–Letnikov, Riemann–Liouville and Caputo fractional derivatives and Riesz potential can be considered. A natural question arises: "What is a fractional derivative?" In a paper, whose title is exactly this question (Ortigueira and Machado 2015), Ortigueira and Tenreiro Machado set a criterion called *Wide Sense Criterion* (WSC) which establishes when an operator is a FD and showed that the well-known definitions of Grünwald–Letnikov, Riemann–Liouville, Caputo and Riesz satisfy the WSC.

Fractional calculus modeling (FCM), using Caputo derivative (Sabatier et al. 2007), has been recently used to generalize the logistic equation (Verhulst 1838). The solution of the corresponding fractional differential equation provides a suitable description for the growth of certain types of cancer tumor (Varalta et al. 2014).

Probably, the best known example of the efficiency of FCM, with Caputo derivative, is the fractional harmonic oscillator (Gutierrez et al. 2010; Li et al. 2011). Indeed, replacing the derivative of order two, present in the model of the simple harmonic oscillator, by a Caputo fractional derivative with order $1 < \alpha \leq 2$ the damped harmonic oscillator is obtained as solution (Li et al. 2011). The physical interpretation usually given to this fact is that different friction, presented in the system, leads to a decrease in the rate of variation. In this sense, when we consider the order of the rate of variation as a number between one and two, we would be putting in the order of the derivative the effect of all the friction of the system; therefore, to more accurately model a harmonic motion instead of determining each of coefficients of friction presented in the system, is enough to determine the order of the derivative that is best suited.

In this paper, the FCM, with Caputo derivative is analyzed. After introducing some preliminary concepts and results about FC, the fractional harmonic oscillator and the fractional logistic equation are presented. After that, following the same logic and physical interpretation of FD, the fractional generalization of the differential equation of Malthus for the growth of a population in an ideal environment is presented and solved (Verhulst 1838). The results obtained for the harmonic oscillator and logistic equation are consistent with the usual intuition, that is, the lower the order of the derivative, the slower is the growth, but for Malthus fractional differential equation, the result is opposite, and then, the difference between the value of the order of the rate of variation and the value of the rate of variation is explained.

2 Preliminary concepts

In this section, we consider the gamma function to introduce the so-called Riemann–Liouville fractional integral (Camargo and de Oliveira 2015).

Definition 1 Let $n \in \mathbb{N}$ and $\nu \notin \mathbb{Z}$ Gel'fand–Shilov function is defined by Mainardi (2009):

$$\phi_n(t) := \begin{cases} \dfrac{t^{n-1}}{(n-1)!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad \text{and} \quad \phi_\nu(t) := \begin{cases} \dfrac{t^{\nu-1}}{\Gamma(\nu)} & \text{if } t \geq 0. \\ 0 & \text{if } t < 0 \end{cases}$$

As a result, the Laplace transform of the Gel'fand–Shilov function is given by Mainardi and Gorenflo (2000),

$$\mathfrak{L}[\phi_\nu(t)] = \int_0^\infty \mathrm{e}^{-st} \frac{t^{\nu-1}}{\Gamma(\nu)} \mathrm{d}t = \frac{s^{-\nu}}{\Gamma(\nu)} \int_0^\infty \mathrm{e}^{-a} a^{\nu-1} \mathrm{d}a = s^{-\nu}. \tag{1}$$

2.1 Mittag-Leffler functions

Definition 2 The classical Mittag-Leffler function is defined as Mittag-Leffler (1903):

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad z \in \mathbb{C} \quad \text{and} \quad \mathrm{Re}(\alpha) > 0. \tag{2}$$

Taking $\alpha = 1$ the exponential function is recovered, $E_1(z) = \mathrm{e}^z$. A two parameter generalization has been proposed by Wiman (1905) as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C} \quad \text{and} \quad \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta) > 0. \tag{3}$$

For $\beta = 1$ the classical Mittag-Leffler function is recovered, i.e. $E_{\alpha,1}(z) = E_\alpha(z)$.

The Laplace transform of the two parameter Mittag-Leffler function is written as (Camargo and de Oliveira 2015; Podlubny 1999)

$$\mathfrak{L}[t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}. \tag{4}$$

2.2 Riemann–Liouville fractional integral

There are several ways to introduce the fractional integral (Camargo and de Oliveira 2015; Mainardi 2009). Here, we introduce it through a generalization of the integer order integral operator.

Definition 3 Let $n \in \mathbb{N}$ and $f(t) : \mathbb{R} \to \mathbb{R}$ a real integrable function. The integral operator with order 1 and n, denoted by I and I^n, are, respectively, defined as:[2]

$$I f(t) = \int_0^t f(t_1)\,\mathrm{d}t_1 \qquad \text{and} \qquad I^n f(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_n)\,\mathrm{d}t_n\,\mathrm{d}t_{n-1} \ldots \mathrm{d}t_2\,\mathrm{d}t_1.$$

Theorem 1 Let $f(t) : \mathbb{R} \to \mathbb{R}$ be an integrable function. The integrate of order n may be written as (Camargo and de Oliveira 2015):

$$I^n f(t) = \phi_n(t) * f(t) = \int_0^t \phi_n(t-\tau) f(\tau)\,\mathrm{d}\tau = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\,\mathrm{d}\tau, \tag{5}$$

where $*$ denotes the Laplace convolution and $\phi_n(t)$ is the Gel'fand–Shilov function.

Definition 4 Let $f(t)$ be an integrable function. The Riemann–Liouville fractional integral of order ν of $f(t)$, with $\mathrm{Re}(\nu) > 0$, is given by:

$$I^\nu f(t) = \phi_\nu(t) * f(t) = \int_0^t \frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)} f(\tau)\,\mathrm{d}\tau. \tag{6}$$

2.3 Caputo fractional derivative

Definition 5 Caputo's fractional derivative is defined as the Riemann–Liouville fractional Integral of the usual derivative, in such a way that the law of exponents holds. That is, let

[2] For convenience, is defined that $I^0 f(t) = f(t)$.

$f(t)$ be a differentiable and integrable function and $m \in \mathbb{N}$ such that $m-1 < \mathrm{Re}(\alpha) \leq m$, the Caputo Fractional Derivative is defined as:[3]

$$D^{\alpha} f(t) = I^{m-\alpha} D^{m} f(t) = \phi_{m-\alpha}(t) * D^{m} f(t). \tag{7}$$

2.4 Laplace transform

From the definition of Riemann–Liouville Integral, the convolution theorem and the Eq. (7), is obtained that Varalta et al. (2014)

$$\mathfrak{L}[D^{\alpha} f(t)] = \mathfrak{L}[\phi_{m-\alpha} * D^{m} f(t)] = \mathfrak{L}[\Phi_{m-\alpha}(t)]\, \mathfrak{L}[D^{m} f(t)] = s^{\alpha-m}\, \mathfrak{L}[D^{m} f(t)]. \tag{8}$$

3 Fractional harmonic oscillator

In this section, we present the known result of the fractional harmonic oscillator and its usual physical interpretations (Gutierrez et al. 2010; Li et al. 2011).

It is known, by Newton's second law of motion applied to systems that repeat in time, the equation

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) + \mu \frac{\mathrm{d}}{\mathrm{d}t} x(t) + k\, x(t) = g(t),$$

describes the displacement (elongation) of a body of mass m, in time t, from the equilibrium position, subject to Hooke's Law, $-kx(t)$, a damping force $-\mu \frac{\mathrm{d}}{\mathrm{d}_t} x(t)$ and to an external force $g(t)$, where μ and k are positive constants.

Consider the particular case of this equation in which there are no friction or external forces acting on the system, i.e.

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) + \omega_0^2\, x(t) = 0, \tag{9}$$

where $\omega_0^2 = k/m$ and subject to the initial conditions $x(0) = x_0$ and $x'(0) = 0$. The Eq. (9) can be rewritten as follows:

$$x(t) = x(0) + t x'(0) - \omega_0^2 \int_0^t \int_0^v x(u) \mathrm{d}u \mathrm{d}v = x(0) + t x'(0) - \omega_0^2\, I^2 x(t). \tag{10}$$

In order to consider a more realistic model, i.e., a model involving friction, a fractional model may be introduced with the following argument (Diethelm 2010): " the presence of friction will lead to a decrease in rate of variation of space-time. As a result, instead of introducing the various types of friction into the equation, the rate of variation of order two presented in this equation may be replaced by another one with order $1 < \alpha \leq 2$" (Li et al. 2011; Malinowska et al. 2015), so:[4]

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} x(t) + \omega_0^2 x(t) = 0 \quad \Leftrightarrow \quad x(t) = x(0) + t x'(0) - \omega^{\alpha}\, I^{\alpha} x(t), \tag{11}$$

[3] It follows from the definition that if $\alpha = m$, so $D^m f(t) = I^{m-m} D^m f(t) = I^0 D^m f(t) = D^m f(t)$, that is, the usual derivative is a particular case.

[4] Indeed, this argument is not accurate, and it will be explained in the last section.

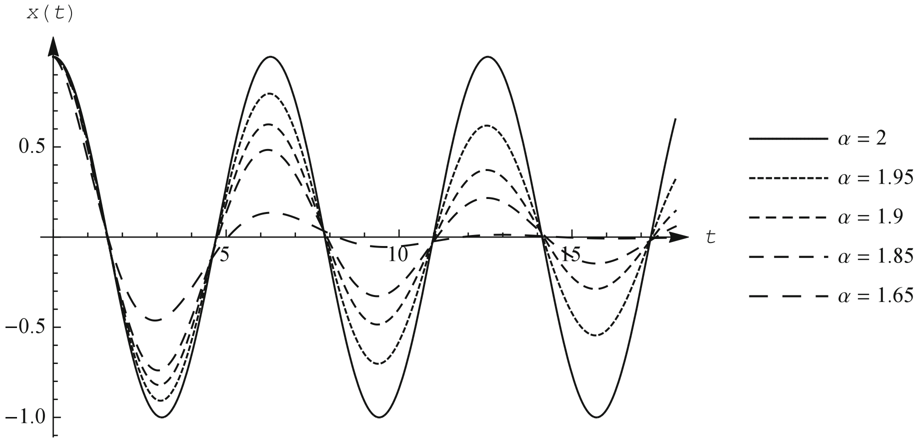

Fig. 1 Graphics of Eq. (13), for different values of α, between 1 and 2

where $\omega_0^2 = \omega^\alpha$. Applying the Laplace transform to the last equation, using the definition (6), the convolution theorem of Laplace and the property (1), we can write:[5]

$$X(s) = x(0)\frac{s^{-1}}{1+\omega^\alpha s^{-\alpha}} + x'(0)\frac{s^{-2}}{1+\omega^\alpha s^{-\alpha}} = x(0)\frac{s^{\alpha-1}}{s^\alpha+\omega^\alpha} + x'(0)\frac{s^{\alpha-2}}{s^\alpha+\omega^\alpha} \quad (12)$$

where $X(s)$ is the Laplace transform of $x(t)$. Since $x'(0) = 0$, applying the inverse Laplace transform on the Eq. (12) and using the Eq. (4), it follows that

$$x(t) = x_0\, E_\alpha(-\omega^\alpha\, t^\alpha). \quad (13)$$

In Fig. 1, we show graphics for some values of α with $x_0 = 1$ and $\omega = 1$.

From Fig. 1 and applying the limit $\alpha \to 2$ in Eq. (13), we conclude that the integer case, i.e., the simple harmonic oscillator is a particular case of the fractional solution. Indeed, $\lim_{\alpha\to 2} x(t) = x_0\, E_2(-\omega^\alpha\, t^\alpha) = x_0 \cos(\omega\, t)$.

4 Fractional logistic equation

In reference (Varalta et al. 2014) is presented and solved a fractional version of the logistic equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = kN(t)\left(1 - \frac{N(t)}{r}\right), \quad (14)$$

where $N(t)$ is the number of individuals in time t, k is the intrinsic growth rate and r is carrying capacity. Without loss of generality, taking $r = 1$, and the variable change $v(t) = 1/N(t)$, the following linear and separable equation is obtained:

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = k[1 - v(t)], \quad (15)$$

whose solution is given by

$$v(t) = 1 + \frac{1}{c}\mathrm{e}^{-kt} \quad \Rightarrow \quad v(0) = 1 + \frac{1}{c}\cdot$$

[5] Since for $1 < \alpha \leq 2$, in Eq. (8), $m = 2$.

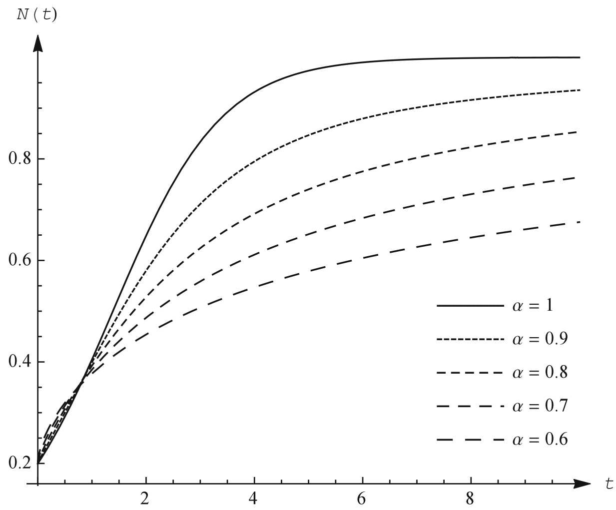

Fig. 2 Graphics of Eq. (18), for different values of α

Since $N(t) = v(t)^{-1}$, we can write $1/c = 1/N(0) - 1$, as a result:

$$N(t) = \frac{1}{1 + \left[\frac{1}{N(0)} - 1\right] \mathrm{e}^{-kt}}. \tag{16}$$

The fractional version of Eq. (14) has been *numerically* solved by El-Sayed et al. (2007), and the fractional version of Eq. (15) has been *analytic* solved by Varalta et al. (2014) both with order $0 < \alpha \leq 1$. Although the solutions are similar, they are not the same. Here, for convenience, we present the fractional version of Eq. (15) with order $0 < \alpha \leq 1$, i.e.,

$$\frac{\mathrm{d}^\alpha v(t)}{\mathrm{d}t^\alpha} = D^\alpha v(t) = k[1 - v(t)]. \tag{17}$$

Applying the Laplace transform, solving the transformed equation and taking $v(t) = 1/N(t)$, we obtain:

$$N(t) = \frac{1}{1 + \left[\frac{1}{N(0)} - 1\right] E_\alpha(-kt^\alpha)}. \tag{18}$$

Note that,

$$\lim_{\alpha \to 1} N(t) = \frac{1}{1 + \left[\frac{1}{N(0)} - 1\right] \mathrm{e}^{-kt}},$$

i.e. the solution of integer order is a particular case of the fractional solution.

In Fig. 2, we present some graphics corresponding to the solution of Eq. (17), taking $N(0) = 0.2$, $k = 1$ and carrying capacity $r = 1$, for different values of α.

Since $\lim_{t \to \infty} E_\alpha(-kt^\alpha) = 0$ for all values $0 < \alpha \leq 1$, we have:

$$\lim_{t \to \infty} N(t) = \lim_{t \to \infty} \frac{1}{1 + \left[\frac{1}{N(0)} - 1\right] E_\alpha(-kt^\alpha)} = 1,$$

that is, the all the values considered converge to the value of the carrying capacity.

In Varalta et al. (2014), is shown the convenience of this fractional model to describe the growth of certain types of cancer tumor. For the propose of this article, it is important to note that, as it happens in the fractional harmonic oscillator, the lower the order of the derivative, the slower is the growth.

5 Exponential growth of Malthus

The model proposed by Malthus to describe growth in an ideal environment of a population of $P(t)$ individuals at the instant t is based on the hypothesis that, in these circumstances, the rate of variation of the number of individuals will be proportional to the population itself, in other words (Varalta et al. 2014),

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t) = kP(t) \qquad \Rightarrow \qquad P(t) = P(0)\mathrm{e}^{kt}. \tag{19}$$

Let us introduce the fractional version of the this model using the same type of reasoning done to the harmonic oscillator and the logistic equation, i.e., "since in a non-ideal situation, there are a number of inhibiting factors such as competition for vital resources, instead of considering such factors into the equation, we take into account that those inhibiting factors lead to a decrease in rate of variation and replace the derivative of order one, in this model, by a derivative of order $0 < \alpha \leq 1$" (Debnath 2003), i.e.,:

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}P(t) = kP(t). \tag{20}$$

By applying the Laplace transform on the previous equation, we can write from the Eq. (8), that

$$s^\alpha F(s) - s^{\alpha-1}P(0) = KF(s) \quad \Rightarrow \quad F(s) = P(0)\,\frac{s^{\alpha-1}}{s^\alpha - k}, \tag{21}$$

where $F(s)$ is the Laplace transform of $P(t)$. Applying inverse Laplace transform and using the Eq. (4) we have

$$P(t) = P(0)E_\alpha(k\,t^\alpha). \tag{22}$$

Figure 3 shows some solutions of Eq. (20), taking $P(0) = 1$ (scale 1 to 1000) and $k = 1$.

Note that, the lower the order of derivative of fractional equation (20), greater is the rate of variation of $P(t)$, which is an opposite result than the expected one; by our initial considerations, the growth of $P(t)$ was supposed to be smaller when we decrease the order of the derivative.

Since the result of the Eq. (20) was not the expected, we will solve it considering the order of the derivative as $1 < \beta \leq 2$, i.e.,:

$$\frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}P(t) = kP(t). \tag{23}$$

Applying the Laplace transform, solving the transformed equation and applying the inverse Laplace transform, follow that:

$$P(t) = P(0)\,E_\beta(k\,t^\beta) \;+\; P'(0)\,t\,E_{\beta,2}(k\,t^\beta). \tag{24}$$

In Fig. 4, it is presented the graphic of the solution of Eq. (23), taking $P(0) = 1 = P'(0)$ (scale 1 to 1000) and $k = 1$.

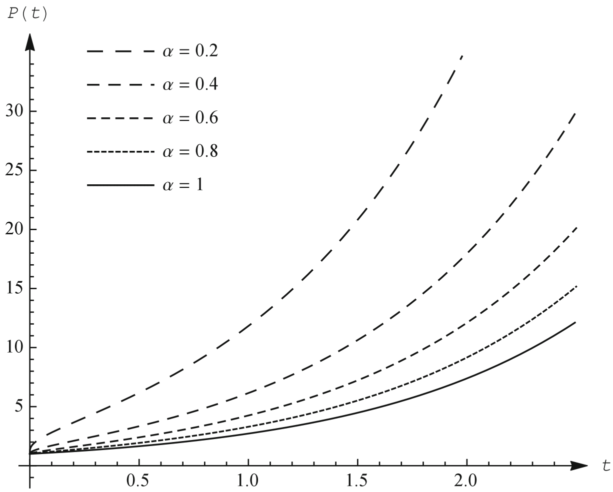

Fig. 3 Graphics of Eq. (22), for values of α, between 0 and 1

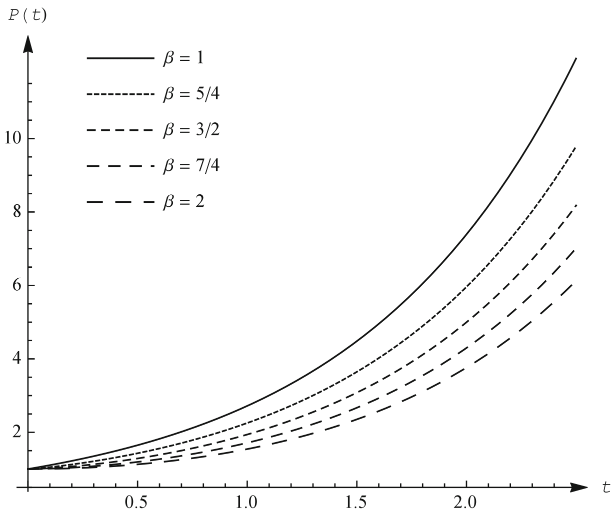

Fig. 4 Graphics of Eq. (24), for different values of β, between 1 and 2

Note that also in the solution of the Eq. (23), the lower the order of fractional derivative, the greater is the growth of $P(t)$. However, the solution of Eq. (23) is more convenient to describe a non-ideal growth of a population than the solution of Eq. (20).

6 Discussions and conclusions

The fractional modeling has been widely used to generalize and make more precise the usual modeling. The most common reason found for this type of generalization is that "when modeling a particular phenomenon is common to make some simplifications, usually those simplifications, if considered in the model, lead to a decrease in the rate of variation of the phenomenon. Thus, instead of considering several factors in the equation, their influence in the order of the derivative can be embedded".

This sort of argument, proved to be valid for many problems (Debnath 2003; Podlubny 1999) such as, to the harmonic oscillator (Li et al. 2011), presented here, and the logistic growth (Varalta et al. 2014). However, when analyzing the exponential model of growth, we observe an opposite behavior, i.e., the lower the order of the derivative, the faster is the growth.

This work shows that there is not a general relation between the fractional order of the dynamic model and its state variable. Although, it presents a similar behavior in some problems, this is not always checked. Naturally, this type of observation can be useful in the study of the physical interpretation of fractional derivative.

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