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TECHNICAL PAPER

Applications of the Lie symmetries to complete solution of a bead on a rotating wire hoop

Cláudio H. C. Costa Basquerotto[1] · **Edison Righetto**[1] · **Samuel da Silva**[1]

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Abstract

The existence of Lie symmetries in differential equations can generate transformations in the dependent and independent variables and obtain new equations that may be easier to integrate. In particular, in some situations, one can reduce the order and it is possible to obtain first integrals. Thus, this article presents the application of the fundamental Lie theorem to obtain the complete solution of a classical nonlinear problem of the dynamics of mechanical systems: the bead on a rotating wire hoop. From the first integral obtained with the Lie symmetry generators, the exact solution can be found with the aid of the Jacobi elliptic functions.

Keywords Lie symmetries · Classical mechanics · Jacobi elliptic functions

1 Introduction

Many problems found in engineering can be modeled through differential equations [1–3]. This type of equation provides detailed information regarding the distributions or changes of the dependent variable as a function of independent variables [4]. Several methods can be used to obtain solutions of these equations, both numerically and with analytical approaches. The ability to find a solution, or else, the integrability of the system, is possible if there is a sufficient number of invariants associated with symmetries [5, 6].

The Norwegian Sophus Lie was the first to use continuous group as a way to produce a transformation to leave a differential equation invariant [7–10]. The procedure proposed by Lie has allowed to join many usual techniques to solve differential equations that were known until the nineteen century. These coordinate transformations in time or spatial domain involve Lie symmetries that can reduce the order, decouple the variables or to make integration easier [9, 11, 12]. If the number of symmetries is equal or bigger than the number of degree of freedom, the system is completely integrable [10]. A symmetry of a system of differential equations is a transformation that maps any solution to another solution [5, 13]. Such transformations are groups that depend on continuous parameters and consist of transformations (point symmetries), acting in the system space of independent and dependent variables, as well as in all the first derivatives of the dependent variables [14, 15]. Elementary examples of Lie groups include: translations, rotations, and scaling [6, 8].

The Lie symmetries are already used to solve different kinds of problems in mechanics, as for instance, vortex fluid dynamics [16], heat equation [17–19], wave equation [20, 21], continuum mechanics involving plasticity [22, 23] and elasticity problems of rod, beams and plates [22, 24, 25], beyond others. However, the extensive application of the Lie theory to solve engineering problems is not common yet. Thus, the main goal of the present paper is to illustrate how to use the Lie symmetries to find the analytical solution of a classical dynamic problem: a bead on a rotating wire hoop. Here we describe in full detail to the reader unfamiliar with the Lie theory, that the authors believe that is the case of the most part of the

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Cláudio H. C. Costa Basquerotto
cbasquerotto@ymail.com

Edison Righetto
edisonrighetto72@gmail.com

Samuel da Silva
samuel@dem.feis.unesp.br

[1] Departamento de Engenharia Mecânica, Universidade Estadual Paulista, UNESP, Faculdade de Engenharia de Ilha Solteira, Ilha Solteira, São Paulo 15385-000, Brazil

readers of Journal of the Brazilian Society of Mechanical Sciences and Engineering. The original point in our article is to present the solution of the bead on a rotating wire hoop and the order reduction using Lie symmetries extracted directly using the motion equation.

This paper is organized in four sections as follows. First, a brief introduction about the model used in this work and the use of Lie symmetries in dynamics. Next, the Lie symmetries of the bead on a rotating wire hoop are illustrated. Finally, the application of the use of Lie symmetries to reduce and to solve the motion equation of bead on a rotating wire hoop and the concluding remarks are presented.

2 Bead on a rotating wire hoop

Figure 1 shows a point mass m (bead) sliding freely without friction in a hoop of radius ℓ spinning with constant angular velocity ω in a constant gravitational field with acceleration g. Thus, this bead moves on a surface of a sphere of radius ℓ. The reference frame $\{x, y, z\}$ used is solidary to the circular hoop. The generalized coordinate used to describe this motion is given by θ (measured from the negative z-axis). In these conditions, the kinetic energy of the bead is:

$$\mathcal{T} = \frac{1}{2} m\ell^2(\dot{\theta}^2 + \omega^2 \sin^2\theta), \tag{1}$$

and the potential energy is given by:

$$\mathcal{V} = mg\ell(1 - \cos\theta), \tag{2}$$

where we chose the zero potencial energy point at $\theta = 0$ (see Fig. 1). Thus, the lagrangian $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is:

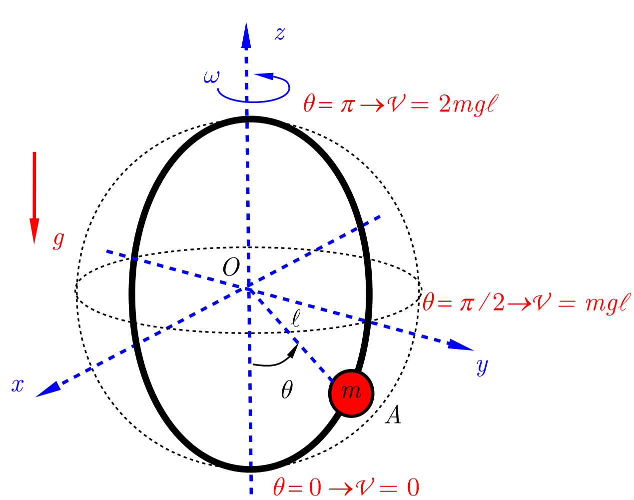

Fig. 1 Bead on a rotating wire hoop

$$\mathcal{L} = \frac{1}{2} m\ell^2(\dot{\theta}^2 + \omega^2 \sin^2\theta) + mg\ell\cos\theta - mg\ell, \tag{3}$$

By applying in the Euler–Lagrange equation [26, 27]:

$$\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = 0, \tag{4}$$

is obtained the equation of motion described by:

$$\ddot{\theta} + \left(\frac{g}{\ell} - \omega^2\cos\theta\right)\sin\theta = 0, \tag{5}$$

where $\theta(t)$ is the position angle varying in time t, the dot upper represents the derivative with respect to the time, ω is the wire constant angular velocity, g is the acceleration of gravity and ℓ is the wire radius.

The analytical solution of this motion equation is already known, using other methods [28–30]. However, several transformations could be applied to keep it invariant and thus to solve it. The procedure proposed by Lie is a powerful procedure to integrate differential equation. First of all, it is necessary to find an infinitesimal generator for all variables. After that, the prolongations are computed to describe the higher order derivatives. By applying the Lie theorem, the determining equations are obtained and can be solved to find the symmetry generators [8, 31–34]. Next section describes all above steps in full detail for the beginning readers.

3 Lie symmetries

This section shows a brief introduction to Lie symmetries and how to apply them in a dynamic problem described by an ordinary differential equation to reduce the order and to obtain an analytical solution.

3.1 Infinitesimal generators

A group transformation involving t and θ to a continuous parameter $\varepsilon \in \mathbb{R}$ can be taken from [8, 32, 33]:

$$\bar{t} = \psi(t, \theta, \varepsilon), \quad \bar{\theta} = \phi(t, \theta, \varepsilon), \tag{6}$$

where ψ and ϕ are analytic functions that perform possible transformation. One can expand $\bar{t}$ and $\bar{\theta}$ with MacLaurin series close to ε using:

$$\bar{t} \approx t + \varepsilon\left(\frac{\partial \psi}{\partial \varepsilon}\bigg|_{\varepsilon \to 0}\right), \quad \bar{\theta} \approx \theta + \varepsilon\left(\frac{\partial \phi}{\partial \varepsilon}\bigg|_{\varepsilon \to 0}\right), \tag{7}$$

since $\varepsilon \to 0$ constitutes the identity of the group. By defining new functions called by infinitesimals:

$$\xi(t,\theta)=\left.\frac{\partial\psi}{\partial\varepsilon}\right|_{\varepsilon\to0},\quad \eta(t,\theta)=\left.\frac{\partial\phi}{\partial\varepsilon}\right|_{\varepsilon\to0}. \tag{8}$$

So the Eq. (7) becomes:

$$\bar{t}\approx t+\varepsilon\xi(t,\theta),\quad \bar{\theta}\approx\theta+\varepsilon\eta(t,\theta). \tag{9}$$

One can define the tangent vector field, $\gamma=\{\xi\,\eta\}^T$, of the group at the point (t,θ), where η and ξ are unknown and need to be found to perform the symmetry transformation. However, in addition to changes in the variables t and θ, an extension for the derivatives $\dot{\theta}$ and $\ddot{\theta}$ must be obtained before.

3.2 Prolongation of transformations and their generators

The last section was concentrated on transforming the dependent and independent variables by one-parameter point transformations and to find their infinitesimal generators. However, if one wants to apply a point transformation to this differential equation, the changes of the derivatives should also be known [35]. Moreover, one needs to extend the infinitesimal generators. By obtaining $\dot{\bar{\theta}}$:

$$\dot{\bar{\theta}}=\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{t}} \tag{10}$$

where $\mathrm{d}\bar{\theta}=\mathrm{d}\theta+\varepsilon\mathrm{d}\eta$ and $\mathrm{d}\bar{t}=\mathrm{d}t+\varepsilon\mathrm{d}\xi$. Differentiating these terms with respect to t:

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t}=\dot{\theta}+\varepsilon\mathcal{D}_{\mathrm{t}}(\eta),\frac{\mathrm{d}\bar{t}}{\mathrm{d}t}=1+\varepsilon\mathcal{D}_{\mathrm{t}}(\xi),$$

since that $\varepsilon\to0$ and applying the binomial rule[1], Eq. (10) becomes:

$$\begin{aligned}\dot{\bar{\theta}}&=\frac{\dot{\theta}+\varepsilon\mathcal{D}_{\mathrm{t}}(\eta)}{1+\varepsilon\mathcal{D}_{\mathrm{t}}(\xi)}\approx\{\dot{\theta}+\varepsilon\mathcal{D}_{\mathrm{t}}(\eta)\}\{1-\varepsilon\mathcal{D}_{\mathrm{t}}(\xi)\}\\&=\dot{\theta}-\dot{\theta}\varepsilon\mathcal{D}_{\mathrm{t}}(\xi)+\varepsilon\mathcal{D}_{\mathrm{t}}(\eta)-\varepsilon^2\mathcal{D}_{\mathrm{t}}(\eta)\mathcal{D}_{\mathrm{t}}(\xi)\approx\dot{\theta}+\varepsilon\beta^{(1)},\end{aligned} \tag{11}$$

where $\mathcal{D}_{\mathrm{t}}$ is the total derivative given by:

$$\mathcal{D}_{\mathrm{t}}=\frac{\partial}{\partial t}+\dot{\theta}\frac{\partial}{\partial\theta}+\ddot{\theta}\frac{\partial}{\partial\dot{\theta}}, \tag{12}$$

and the high order terms were disregarded (if $\varepsilon\approx0$, so $\varepsilon^2=0$), thus:

$$\beta^{(1)}=\mathcal{D}_{\mathrm{t}}(\eta)-\dot{\theta}\mathcal{D}_{\mathrm{t}}(\xi)=\frac{\partial\eta}{\partial t}+\left(\frac{\partial\eta}{\partial\theta}-\frac{\partial\xi}{\partial t}\right)\dot{\theta}-\frac{\partial\xi}{\partial\theta}(\dot{\theta})^2, \tag{13}$$

hence, this is the first prolongation [11].

In systems of differential equations with higher order terms as in the cases described by Eq. (5), should extend the extension for these variables. One can write $\ddot{\bar{\theta}}$ due to the vector field $\gamma=\{\xi\,\eta\}^T$ and ε with the same procedure as above from:

$$\begin{aligned}\ddot{\bar{\theta}}&=\frac{d\dot{\bar{\theta}}}{d\bar{t}}=\frac{\mathcal{D}_{\mathrm{t}}(\dot{\theta}+\varepsilon\beta^{(1)})}{\mathcal{D}_{\mathrm{t}}(t+\varepsilon\xi)}=\frac{\ddot{\theta}+\varepsilon\mathcal{D}_{\mathrm{t}}(\beta^{(1)})}{1+\varepsilon\mathcal{D}_{\mathrm{t}}(\xi)}\\&\approx\{\ddot{\theta}+\varepsilon\mathcal{D}_{\mathrm{t}}(\beta^{(1)})\}\{1-\varepsilon\mathcal{D}_{\mathrm{t}}(\xi)\}\approx\ddot{\theta}+\varepsilon\beta^{(2)},\end{aligned} \tag{14}$$

where [32]:

$$\begin{aligned}\beta^{(2)}=\mathcal{D}_{\mathrm{t}}(\beta^{(1)})-\ddot{\theta}\mathcal{D}_{\mathrm{t}}(\xi)&=\frac{\partial^2\eta}{\partial t^2}+\left(2\frac{\partial^2\eta}{\partial t\partial\theta}-\frac{\partial^2\xi}{\partial t^2}\right)\dot{\theta}\\&+\left(\frac{\partial^2\eta}{\partial\theta^2}-2\frac{\partial^2\xi}{\partial t\partial\theta}\right)(\dot{\theta})^2-\frac{\partial^2\xi}{\partial\theta^2}(\dot{\theta})^3\\&+\left(\frac{\partial\eta}{\partial\theta}-2\frac{\partial\xi}{\partial t}\right)\ddot{\theta}-3\frac{\partial\xi}{\partial\theta}\dot{\theta}\ddot{\theta},\end{aligned} \tag{15}$$

is the second prolongation and where $\mathcal{D}_{\mathrm{t}}(\beta^{(1)})$ is:

$$\mathcal{D}_{\mathrm{t}}(\beta^{(1)})=\frac{\partial\beta^{(1)}}{\partial t}+\dot{\theta}\frac{\partial\beta^{(1)}}{\partial\theta}+\ddot{\theta}\frac{\partial\beta^{(1)}}{\partial\dot{\theta}}.$$

Finally, the generalized procedure for high order prolongation is given by [32]:

$$\beta^{(\mathrm{k})}(t,\theta,\dot{\theta},\ddot{\theta},\ldots,\overset{\mathrm{k}}{\theta})=(\mathcal{D}_{\mathrm{t}})^{\mathrm{k}}\eta-\sum_{j=1}^{\mathrm{k}}\frac{\mathrm{k}!}{(\mathrm{k}-\mathrm{j})!\mathrm{j}!}\overset{(\mathrm{k}-\mathrm{j}+1)}{\theta}(\mathcal{D}_{\mathrm{t}})^j\xi, \tag{16}$$

where $\mathrm{k}=1,2,3,\ldots$ is the number of prolongation.

3.3 Lie theorem

Once known, the vector field $\gamma=\{\xi\,\eta\}^T$, an infinitesimal generator of symmetry can be obtained from:

$$\mathcal{X}=\begin{Bmatrix}\xi\\\eta\end{Bmatrix}\cdot\begin{Bmatrix}\frac{\partial}{\partial t}\\\frac{\partial}{\partial\theta}\end{Bmatrix}=\xi\frac{\partial}{\partial t}+\eta\frac{\partial}{\partial\theta}. \tag{17}$$

By applying the second order operator $\mathcal{U}''$:

$$\mathcal{U}''=\beta^{(1)}\frac{\partial}{\partial\dot{\theta}}+\beta^{(2)}\frac{\partial}{\partial\ddot{\theta}}. \tag{18}$$

By rewriting Eq. (5), in the following form:

$$\mathcal{F}\left(t,\theta,\dot{\theta},\ddot{\theta}\right)\equiv\ddot{\theta}+\left(\frac{g}{\ell}-\omega^2\cos\theta\right)\sin\theta=0. \tag{19}$$

Such that, the Lie condition is:

[1] $(1+x)^\lambda=1+\lambda x+\frac{\lambda(\lambda-1)}{2!}x^2+\cdots$

$$(\mathcal{U}'' + \mathcal{X})\mathcal{F} = 0 \Rightarrow (\mathcal{U}'' + \mathcal{X})\mathcal{F} = \xi\frac{\partial \mathcal{F}}{\partial t} + \eta\frac{\partial \mathcal{F}}{\partial \theta} + \beta^{(1)}\frac{\partial \mathcal{F}}{\partial \dot{\theta}} + \beta^{(2)}\frac{\partial \mathcal{F}}{\partial \ddot{\theta}} = 0. \tag{20}$$

The infinitesimal invariance criterion, described by Eq. (20), involves t and θ, and the derivatives of θ with respect t, such that ξ and η and their partial derivatives with respect to t and θ. After eliminating all dependencies through the derivatives involving θ, can be equated to the coefficients of the remaining partial derivatives from θ to zero. This gives a large number of partial differential equations to determine the functions ξ and η. These equations are known as determining equations for the symmetry group of a given system [8]. To find the solutions of the determining equations, some symbolic manipulation packages can be used, such as wxMaxima, Mathematica, Maple, MathLie [33], Sym [36], and others.

3.4 The Lie symmetries of the bead on a rotating wire hoop

After applying the Lie condition, Eq. (20), into Eq. (5), one obtains the following determining equations:

$$\frac{\partial^2 \xi}{\partial \theta^2} = 0, \tag{21}$$

$$\frac{\partial^2 \eta}{\partial \theta^2} - 2\frac{\partial^2 \xi}{\partial \theta \partial t} = 0, \tag{22}$$

$$3\sin\theta\left(\frac{g}{\ell} - \omega^2\cos\theta\right)\frac{\partial \xi}{\partial \theta} + 2\frac{\partial^2 \eta}{\partial \theta \partial t} - \frac{\partial^2 \xi}{\partial t^2} = 0 \tag{23}$$

$$2\sin\theta\frac{\partial \xi}{\partial t}\left(\frac{g}{\ell} - \omega^2\cos\theta\right) - \sin\theta\left(\frac{g}{\ell} - \omega^2\cos\theta\right)\frac{\partial \eta}{\partial \theta} - \eta\left(-\omega^2\sin^2\theta - \left(\frac{g}{\ell} - \omega^2\cos\theta\right)\cos\theta\right) + \frac{\partial^2 \eta}{\partial t^2} = 0. \tag{24}$$

Equations (21) to (24) can be simplified to obtain the follow equations:

$$\eta = 0, \tag{25}$$

$$\frac{\partial \xi}{\partial t} = 0, \tag{26}$$

$$\frac{\partial \xi}{\partial \theta} = 0. \tag{27}$$

Finally, solving Eqs. (25) to (27) produces the follow infinitesimal generator:

$$\mathcal{X}_1 = \frac{\partial}{\partial t}.$$

So the infinitesimal functions are:

$$\xi = 1 \quad \text{and} \quad \eta = 0,$$

that corresponds to the temporal translation, that is the classic invariant that shows the energy conservation, since no dissipation is assumed in the model.

It should be noted that in this example only one generator is obtained and hence a single first integral. Since the system has a degree of freedom, $\theta(t)$, this indicates that the system is completely integrable. Note that in some equations of motion it is possible to have more than one symmetry generator, for example, in the harmonic oscillator [37, 38].

3.5 Order reduction of the ODE

We can introduce new coordinates $\{\bar{\theta}(t,\theta), \bar{t}(t,\theta)\}$ supposing that $\mathcal{X}$ is a vector field not vanishing at a point. The change of variables is constructed using the methods for finding group invariants. This implies that $\mathcal{X}$ is transformed into the form $\partial/\partial\bar{t}$ provided $\bar{\theta}$ and $\bar{t}$ satisfy the linear partial differential equations [8]:

$$\mathcal{X}_1(\bar{\theta}) = \xi\frac{\partial \bar{\theta}}{\partial t} + \eta\frac{\partial \bar{\theta}}{\partial \theta} = 0, \qquad \mathcal{X}_1(\bar{t}) = \xi\frac{\partial \bar{t}}{\partial t} + \eta\frac{\partial \bar{t}}{\partial \theta} = 1. \tag{28}$$

To satisfy the condition (28) must be chosen $\bar{\theta} = \theta$ and $\bar{t} = t$. Expanding this condition to $\dot{\bar{t}}$ e $\ddot{\bar{t}}$, one obtains:

$$\dot{\bar{t}} = \frac{d\bar{t}}{d\bar{\theta}} = \frac{1}{\dot{\theta}} \quad \Rightarrow \quad \ddot{\bar{t}} = \frac{d\dot{\bar{t}}}{d\bar{\theta}} = -\frac{\ddot{\theta}}{\dot{\theta}^3} \tag{29}$$

The new coordinates are used to rewrite Eq. (5):

$$-\frac{\ddot{\bar{t}}}{\dot{\bar{t}}^3} + \left(\frac{g}{\ell} - \omega^2\cos\bar{\theta}\right)\sin\bar{\theta} = 0. \tag{30}$$

A new projection $\kappa = \dot{\bar{t}}$ and $\dot{\kappa} = \ddot{\bar{t}}$ is performed and the Eq. (30) yields to:

$$-\frac{\dot{\kappa}}{\kappa^3} + \left(\frac{g}{\ell} - \omega^2\cos\bar{\theta}\right)\sin\bar{\theta} = 0. \tag{31}$$

After solving Eq. (31):

$$\kappa = \frac{\sqrt{\omega^2\ell^2\sin^2\bar{\theta} + 2g\ell\cos\bar{\theta} + \mathcal{C}\ell^2}}{\omega^2\ell\sin^2\bar{\theta} + 2g\cos\bar{\theta} + \mathcal{C}\ell}, \tag{32}$$

where $\mathcal{C}$ is a constant of integration.

The previous coordinates can be returned:

$$\dot{\bar{t}} = \frac{d\bar{t}}{d\bar{\theta}} = \frac{\sqrt{\omega^2\ell^2\sin^2\bar{\theta} + 2g\ell\cos\bar{\theta} + \mathcal{C}\ell^2}}{\omega^2\ell\sin^2\bar{\theta} + 2g\cos\bar{\theta} + \mathcal{C}\ell}, \tag{33}$$

so:

$$\bar{t} = \int \frac{d\bar{\theta}}{\sqrt{\omega^2\sin^2\bar{\theta} + 2\frac{g}{\ell}\cos\bar{\theta} + \mathcal{C}}}. \tag{34}$$

Equation (34) cannot be solved using trigonometric functions. To solve this integral, the Jacobi elliptical integrals can be more useful [29, 39].

Defining the functions

$$\mathrm{sn}(\mathrm{u},\mathrm{k}) \equiv \sin\theta = \sin(\mathrm{am}\,\mathrm{u}),$$

$$\mathrm{cn}(\mathrm{u},\mathrm{k}) \equiv \cos\theta = \cos(\mathrm{am}\,\mathrm{u}),$$

$$\mathrm{dn}(\mathrm{u},\mathrm{k}) \equiv \sqrt{1-\mathrm{k}^2\mathrm{sn}^2\mathrm{u}} = \frac{\mathrm{d}(\mathrm{am}\,\mathrm{u})}{\mathrm{du}},$$

where am is the amplitude function and k is its modulus, sn is the Jacobi's elliptic sine function and cn is the elliptic cosine function. From the definitions about Jacobi elliptic and trigonometric functions, it is possible to obtain the following identities [29, 40]:

$$\mathrm{sn}^2\mathrm{u} + \mathrm{cn}^2\mathrm{u} = 1,$$

$$\mathrm{dn}^2\mathrm{u} + \mathrm{k}^2\mathrm{sn}^2\mathrm{u} = 1,$$

$$\mathrm{cn}^2\mathrm{u} + (1-\mathrm{k}^2)\mathrm{sn}^2\mathrm{u} = \mathrm{dn}^2\mathrm{u}.$$

Thus, Eq. (34) can be solved using the argument (u, k) with attention in the transformation, thus:

$$\theta(t) = 2\arctan\left\{\sqrt{a}\mathrm{dn}\left[\frac{1}{2}\omega\mathrm{t}\sqrt{|\,\mathrm{p}\,|\,\mathrm{a}}, \sqrt{\frac{\mathrm{a}-\mathrm{b}}{\mathrm{a}}}\right]\right\}, \tag{35}$$

where $a \equiv \frac{1-\cos\theta}{1+\cos\theta}$, $b \equiv \frac{1+\cos\theta-2\cos\theta}{1-\cos\theta+2\cos\theta}$ and $p \equiv \mathcal{C} - 1 \pm 2\cos\theta$.

Equation (35) represents the analytical solution of the angle displacement of the bead on a rotating wire with constant angular velocity ω and is the same obtained in the paper [29]. This specific problem is similar to the example presented, however, in that paper Lie symmetries are not used. Other examples are found, but the conditions are quite different to allow a comparison [28, 30]. The first one, the example is used to investigate the three-dimensional bifurcation set of a system with two degrees of freedom depending on a single bifurcation parameter and the second one it is shown a new approach for creating a one-dimensional gravitational ponderomotive trap.

4 Final remarks

The present paper presented the Lie symmetries to solve analytically the motion equation of a bead on a rotating wire hoop. From that, one proposes alternative ways to reduce the order of the original motion equation to simplify the integration of the new equations using the Lie theorem. The Lie symmetries obtained can be effectively used to perform it. The symmetry groups of differential equations or variational problems have all been local transformation group acting "geometrically" on the space of independent and dependent variable. It can be seen that by applying the Lie symmetries, the order is easily reduced and is possible to obtain first integrals that can be solved using some classes of special functions, such as Jacobi elliptic functions. The method presented here can be extended to several equations of motion founded in dynamic systems. Besides this, the Lie theorem can be useful to explain when a system cannot be integrated analytically or missing the necessary symmetries, for example in Navier-Stokes equations.

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Compliance with ethical standards

Conflict of Interests The authors declare that there is no conflict of interests regarding the publication of this article.

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