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Functional RG approach to the Potts model

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Abstract. The critical behavior of the $(n + 1)$ -states Potts model in d -dimensions is studied with functional renormalization group techniques. We devise a general method to derive β -functions for continuous values of d and n and we write the flow equation for the effective potential (LPA') when instead n is fixed. We calculate several critical exponents, which are found to be in good agreement with Monte Carlo simulations and ϵ -expansion results available in the literature. In particular, we focus on Percolation ($n \rightarrow 0$) and Spanning Forest ($n \rightarrow -1$) which are the only non-trivial universality classes in $d = 4, 5$ and where our methods converge faster.

Keywords: critical exponents and amplitudes

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1. Introduction

The most important problem of statistical field theory is the classification of universality classes in arbitrary dimension and for a general symmetry group. Universality not only works as an unifying principle that organizes in equivalence classes the large number of definable models, but crucially highlights and explains which properties, both qualitative and quantitative, can be observed in experiments and more generally in nature. While the most simple cases in which the order parameter field φ is single component and is invariant under the group \mathbb{Z}_2 have been studied thoroughly and describe an infinite family of unitary⁵ universality classes, the first of which is the well known Ising universality class (see [2, 4–6] for an analysis in arbitrary dimension);

⁵ In the non-unitary case the \mathbb{Z}_2 -symmetry is replaced by \mathcal{PT} -symmetry [1–3].

more general *discrete* symmetries have received much less attention and the general properties of their theory spaces are still quite unknown in dimensions higher than two, both qualitatively and quantitatively. Remaining in the realm of finite groups, the two principal generalizations of the \mathbb{Z}_2 Ising model are the \mathbb{Z}_n family of abelian clock models and the S_{n+1} family of Potts models. While the first is always characterizable by a two dimensional order parameter field, the implementation of the permutation group symmetry requires an n -component field.

In this work we will begin a general study of the Potts_q ($q \equiv n + 1$) universality classes in arbitrary dimensions by extending the functional RG (FRG) approach to field theories with global S_q -symmetry. This family of universality classes includes a great variety of interesting cases, but in particular we will be interested in the cases of Percolation ($q = 1$) and Spanning Forest ($q = 0$), the only non-trivial cases, apart Ising ($q = 2$), in dimension greater than two. One of the main virtues of the FRG approach is its simple adaptation to any field theory without any restrictions on d . In this respect the method is one of the few analytical tools that are general enough to allow a systematic study of universality and our work is the first of a series intended at the exploration of field theories with discrete symmetries in $d > 2$.

Since its introduction in 1952 [7], the Potts model has attracted an increasingly amount of attention stimulating both theoretical and experimental research: despite a simple definition, the model exhibits a rich critical behaviour reflecting a very different class of physical situations. The three state version of the model can describe the transition of a liquid crystal from its nematic to its isotropic phase [8, 9], the transition of a cubic crystal into a tetragonal phase [10], as well as the deconfinement phase transition in QCD at finite temperature [11–15]. The two state version is of course the Ising model. The Potts model with a single state can describe the critical behaviour of bond percolation [16–18], while the limit of zero states is related both to the electrical resistor network and the spanning forest problem [19, 20]. The Potts model has also been territory of controversy and debate: according to Landau’s phenomenological theory, the presence of the nonzero third order term in the corresponding Lagrangian implies that it undergoes a first order phase transition in any dimension [21, 22]. This opened the long and entangled problem on the nature of the phase transition in the Potts model. Baxter [19] proved rigorously in 1973 that in two dimensions it undergoes a second order phase transition for $q \leq 4$ and a first order one for $q > 4$ and still is the only case known exactly. Despite a satisfactory picture for q_c in $d \geq 2$ is still missing, numerical simulations performed mainly in the 70s [23–28] and RG analysis [21, 29–33, 35, 36] suggest for example that in three dimensions $2 < q_c < 3$ while in $d = 3$ Potts_3 undergoes a first order phase transition. (The existence of a critical value of q_c should correspond to a collapse of fixed points in the RG formalism [31]). It is worth to mention here that quantum fluctuations can change the nature of the phase transition. This is the case for the quantum phase transition of gapless Dirac fermions coupled to a \mathbb{Z}_3 symmetric order parameter within a Gross–Neveu–Yukawa model in $2 + 1$ dimensions, suitable to describe the Kekulé transition in honeycomb lattice materials. It has been proven recently from an RG and fRG analysis that quantum fluctuations of the massless Dirac fermions at zero temperature can render the putative first-order character of the transition continuous [37–39].

The main open problems concerning the Potts_q family of universality classes are the precise quantitative determination of the critical properties of Spanning Forest

and Percolation in $d \geq 3$ and the determination of the critical q_c in $d \geq 2$ at which the phase transition ceases to be continuous. In this work we will address mainly the first of the above problems by giving estimates for the critical exponents for Spanning Forest and Percolation in $d = 4, 5$ (and preliminary results in $d = 3$) while we will postpone to a future work the question related to the critical q_c separating discontinuous from continuous phase transitions, for which, in any case, we do the preparatory work obtaining the flow equation for the effective potential (LPA') for the three-states case ($n = 2$), the flow of which can, in principle, be used to determine the location of a first order phase transition as a function of d .

In section 2 we review and generalize the construction of the Potts field theory, i.e. the field theory for an n -component scalar with discrete global symmetry S_{n+1} . Then in section 3 we implement the construction of the exact functional RG equation for the Potts field theory and develop two approximations: an algorithm to compute the beta functions of power interactions for arbitrary n and the LPA' for fixed n , both in arbitrary dimension. In section 4 we use the beta functions so derived to the study of the critical properties of the Potts_{n+1} universality classes. After a preliminary study which allows us to make connection with the ϵ -expansion, we push our approach to obtain accurate estimates for the critical exponents in $d = 4, 5$. While in these two dimensions we are able to achieve, respectively, almost full or full converging estimates for the critical exponents within the truncation employed, in $d = 3$ we obtain only preliminary results.

2. Potts field theory

2.1. Potts model

Originally proposed by Potts [7] as a generalization of the Ising model, the Potts model consists of a statistical model of interacting spins, where at each site of a lattice there is a variable σ_i that takes q discrete values, $\sigma_i = 1, 2, \dots, q$. In this model two adjacent spins have an interaction energy given by $J\delta(\sigma_i, \sigma_j)$, so that it assumes one value when two nearest-neighbor spins are different and another when they are the same, namely

$$\delta(\sigma_i, \sigma_j) = \begin{cases} 1 & \text{if } \sigma_i = \sigma_j \\ 0 & \text{if } \sigma_i \neq \sigma_j. \end{cases} \quad (2.1)$$

The model is ferromagnetic when $J > 0$ and anti-ferromagnetic when $J < 0$ and the Hamiltonian reads

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta(\sigma_i, \sigma_j). \quad (2.2)$$

This expression is invariant under the group S_q of the permutations of q objects. It is clear that the nature of the values taken by the spins is completely inessential: instead of the q values listed above, one can consider other q distinct numbers or variables of other nature, for example q different colors.

The model can be alternatively formulated to reflect its full symmetry in a $n = q - 1$ dimensional space [30, 40]. This is achieved by writing

$$\delta(\alpha, \beta) = \frac{1}{q} [1 + e^\alpha \cdot e^\beta], \quad (2.3)$$

where e^α are the q -vectors pointing in the q -symmetric directions of a *simplex* in $n = q - 1$ dimensions (sometimes referred to as hyper-tetrahedron). Geometrically the symmetries of the Potts model are thus those of an n -simplex (see figure 1).

2.2. Universality classes

The q -states Potts model, in its continuous formulation, describes the universality classes associated to the spontaneous breaking of the permutation symmetry of q -colors. Baxter [41] proved that in two dimensions the transition is continuous for $q \leq 4$. Nevertheless near two dimensions the critical value $q_c(d)$ below which the transition is second order, decreases rapidly as a function of d . It is known from a variational RG analysis [21] that q_c is already lower than three in $d \simeq 2.32$ and therefore the 3-states Potts model undergoes a first order phase transition in $d = 3$. For $q < 2$ instead the transition is continuous in all the critical range $2 \leq d \leq 6$. The obvious question of the meaning of S_q symmetry for a non-integer q arises spontaneously. Long time ago Fortuin and Kasteleyn [16, 17] introduced the Random Cluster Model (RCM) as a model for phase transitions and other phenomena in lattice systems, or more generally in systems with a graph structure, where formal use of the symmetry unambiguously leads to final expressions containing q as a parameter which could be varied continuously.

The following is a summary of the present knowledge about the universality classes with S_q symmetry:

- **Potts _{q}** . In their seminal paper Fortuin and Kasteleyn showed that the Potts partition function $Z = \sum_{\{\sigma\}} e^{-\mathcal{H}}$ can be written, up to an inessential constant, as

$$Z = \sum_{G \subseteq \mathcal{L}} p^{n_b} (1-p)^{\bar{n}_b} q^{N_c} \quad (2.4)$$

where G is a graph obtained putting n_b bonds on the lattice \mathcal{L} , each one with weight $p = 1 - e^{-J} \in [0, 1]$ (\bar{n}_b is the number of absent bonds in \mathcal{L}) and N_c is the number of clusters in G . The partition function (2.4) defines the RCM. The probability measure for the graph G depends on q through the factor q^{N_c} and it is well defined for any real positive q . In the thermodynamic limit the random cluster model undergoes a phase transition associated to the appearance of a non-zero probability of finding an infinite cluster. By specifying the values of q to $0, 1, 2, 3, \dots$ the model is able to capture at once the phase transition of well known statistical models, the most important of which are listed below.

- **Spanning Forest = Potts₀**. In combinatorics the multivariate generating polynomial $Z_G(q, p)$ that enumerates the spanning subgraphs (i.e. a subgraph containing all vertices) of G according to their precise edge content (with weight p for the edge) and their number of connected components (with weight q for each component) is called multivariate Tutte polynomial and is known in statistical physics as the partition function of the q -state Potts model in the form (2.4).

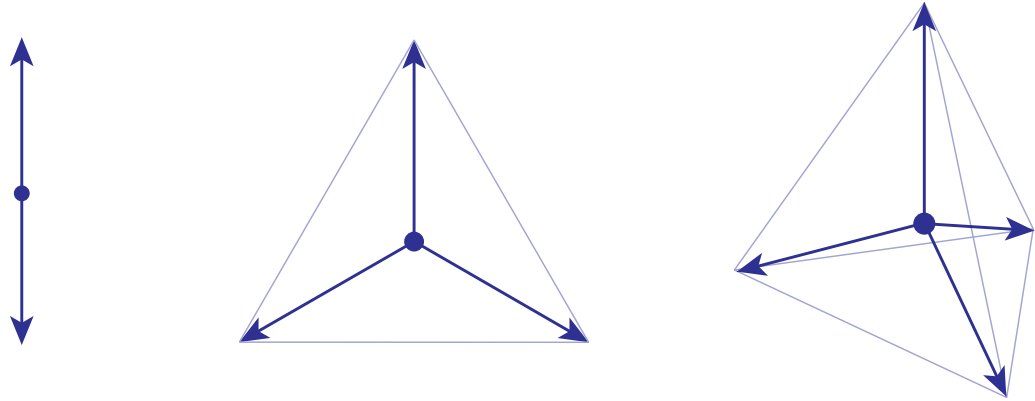


Figure 1. The discrete symmetries characterizing the Potts_{n+1} universality classes are those of the n -simplex, here shown for $n = 1, 2, 3$.

The limit of $q \rightarrow 0$ with p/q fixed selects the generating polynomial of spanning forests/trees as well as the one of electrical networks introduced a long time ago by Kirchhoff [42, 43].

- $\text{Percolation} = \text{Potts}_1$. The limit $q \rightarrow 1$ which eliminates the factor q^{N_c} in (2.4), describes ordinary bond percolation. This formulation, after its introduction, has been revisited and extended to site percolation in 1978 by Wu [44].
- $\text{Ising} = \text{Potts}_2$ thanks to the group isomorphism $\mathbb{Z}_2 \cong S_2$.
- Potts_3 . It is related to the \mathbb{Z}_3 model since $S_3 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$ and in $d = 2$ it has the same central charge ($c = \frac{4}{5}$) as the Tricritical universality class. The three states version of the model has various connections from nematic to isotropic phase in liquid crystals [8, 22] to the deconfinement phase transition of mesons and baryons in QCD in two dimensions [11–15].
- Potts_4 . The 4-states Potts model can describe the deconfinement of baryons and mesons as in the case $q = 3$ as well as tetraquark confined states which are allowed for $q = 4$ only. Moreover the Ashkin-Teller model is a \mathbb{Z}_4 -symmetric model which is represented by the following Hamiltonian

$$\mathcal{H}_{\text{AT}} = -J \sum_{\langle ij \rangle} (\sigma_i \sigma_j + \tau_i \tau_j) - J_4 \sum_{\langle ij \rangle} (\sigma_i \sigma_j \tau_i \tau_j)$$

and since σ and τ variables are equal to ± 1 , but also $\sigma\tau = \pm 1$, when $J = J_4$ we can rephrase the problem to have $4 - 1 = 3$ invariants. This particular point is explicitly S_4 symmetric and can be described by a 4-states Potts model [45].

In the next section we are going to introduce the continuous formulation of the Potts model (Potts field theory) which describes the scaling limit of the RCM for $q \in \mathbb{R}$.

2.3. Action and invariants

As was shown by Golner [29] and Zia and Wallace [30], the critical behavior of the $(n+1)$ -state Potts model in d -dimensions can be studied by a n -component bosonic field ϕ_i ($i = 1, \dots, n$) carrying a representation of the S_{n+1} -symmetry. This representation involves the set of the $n+1$ vectors e_i^α ($\alpha = 1, \dots, n+1$) pointing to the vertices of the n -simplex, so that the underlying symmetry of the model, arising from the equivalence of its $(n+1)$ -states, is reflected by the fact that the indices $\alpha, \beta, \gamma, \dots$ can be permuted amongst each other without changing the physics. The corresponding action is therefore symmetric under the discrete group which maps the n -dimensional hyper-tetrahedron on itself and this group isomorphic to S_{n+1} [32, 35, 46].

2.3.1. Simplex. In geometry a *simplex* is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Specifically, a regular k -simplex is a k -dimensional polytope which is the convex hull of its $k+1$ vertices. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on (see figure 1). We can embed the regular n -dimensional simplex in \mathbb{R}^n by writing directly its cartesian components. This can be achieved with the help of the following two properties:

1. the distances of the vertices to the center are all equal
2. the angle subtended by any two vertices through its center is $\arccos(-\frac{1}{n})$.

These properties allow the explicit construction of the vectors e_i^α that we will use in the definition of the S_{n+1} -invariants. Explicit manipulations of expressions involving e_i^α can be performed using the following relations:

$$\begin{aligned} \sum_{i=1}^n e_i^\alpha e_i^\beta &= (n+1)\delta^{\alpha\beta} - 1 \\ \sum_{\alpha=1}^{n+1} e_i^\alpha &= 0 \\ \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha &= (n+1)\delta_{ij}. \end{aligned} \tag{2.5}$$

Note that in terms of the rules (2.5), the vectors e^α are normalised such that $e^\alpha \cdot e^\alpha = n$. This choice turns out to be useful in order to be able to take the limit $n \rightarrow 0$ later on. The rules (2.5) are the basic relations that we will use in section 3 to reduce the traces involved in the computation of the beta functions.

2.3.2. Invariants. A simple way to understand how to construct invariants under the permutation group is to approach the problem geometrically. Consider a n -component scalar ϕ_i in \mathbb{R}^n as our fluctuating field. First we can construct the projections along the vectors of the hyper-tetrahedron defining the $(n+1)$ -fields $\psi^\alpha \equiv e_i^\alpha \cdot \phi_i$. To aid the intuition consider a regular triangle in the plane as in figure 2. Invariants are easily constructed in terms of the fields ψ^α : any symmetric polynomial will be invariant. In particular we consider the *power sum symmetric polynomials*, which are a type of basic building block for symmetric polynomials, in the sense that every symmetric polynomial

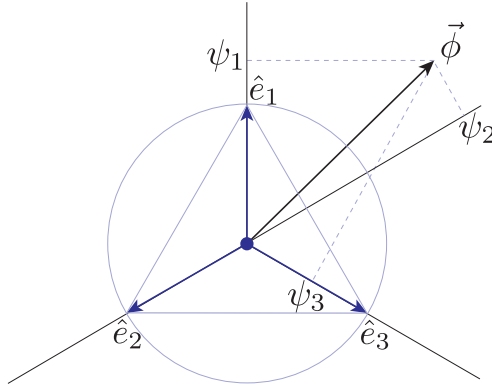


Figure 2. Relation between the fields ϕ_i and ψ^α in the $n = 2$ case when S_3 is the symmetry group of an equilateral triangle in the plane.

with rational coefficients can be expressed as a sum and difference of products of power sum symmetric polynomials with rational coefficients. These are defined as

$$P_k = \sum_{\alpha=1}^{n+1} (\psi^\alpha)^k. \quad (2.6)$$

Note that $\sum_{\alpha} e_i^\alpha = 0$ in (2.5) implies $P_1 = \sum_{\alpha} \psi^\alpha = 0$, because obviously not all the fields ψ^α are independent. Since any symmetric polynomial in $\psi^1, \dots, \psi^{n+1}$ can be expressed as a polynomial expression with rational coefficients in the power sum symmetric polynomials, then it is evident that invariants are monomials in the variables $\{P_2, \dots, P_{n+1}\}$. Once the field power p has been fixed the number of invariants at each order is easily seen to be given by $N(p) = P(p) - P(p-1)$, where $P(p)$ is the partition of p objects. Starting from $p = 2$, the number of invariants is then given by the sequence 1, 1, 2, 2, 4, 4, 7, 8, 12, 14, ... More specifically, for $p = 1$ and $p = 2$ we have P_2 and P_3 ; for $p = 4$ we have P_4 and P_2^2 ; for $p = 5$ the invariants are P_5 and $P_2 P_3$; while for $p = 6$ there are four possibilities: P_6 , P_3^2 , $P_2 P_4$ and P_2^3 ; and so on.

The basic invariants P_k can be expressed back in terms of the fields ϕ_i . For example

$$P_2 = \sum_{\alpha} (\psi^\alpha)^2 = \sum_{\alpha} e_i^\alpha e_j^\alpha \phi_i \phi_j = (n+1) \delta_{ij} \phi_i \phi_j,$$

where we used the last rules in (2.5). Similarly

$$P_3 = \sum_{\alpha} (\psi^\alpha)^3 = \sum_{\alpha} e_i^\alpha e_j^\alpha e_k^\alpha \phi_i \phi_j \phi_k,$$

and more generally

$$P_k = \sum_{\alpha} e_{i_1}^\alpha \dots e_{i_k}^\alpha \phi_{i_1} \dots \phi_{i_k}. \quad (2.7)$$

At this point it is convenient to define the tensors

$$T_{i_1 \dots i_p}^{(p,1)} \equiv \sum_{\alpha} e_{i_1}^\alpha \dots e_{i_p}^\alpha \quad p \geq 3, \quad (2.8)$$

Table 1. Tensor invariants $T^{(p,m)}$ defined in the text with their relative number $N(p)$. The index structure of the invariants follows the general structure of equation (2.8).

$T^{(p,m)}$	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$N(p)$
$p = 2$	δ				1
$p = 3$	$\sum eee$				1
$p = 4$	$\sum eeee$	$\delta\delta$			2
$p = 5$	$\sum eeeee$	$\delta \sum eee$			2
$p = 6$	$\sum eeeeee$	$\sum eee \sum eee$	$\delta \sum eeee$	$\delta\delta\delta$	4

with the exception $T_{i_1 i_2}^{(2,1)} \equiv \delta_{i_1 i_2}$ when $p = 2$ (we factor away an inessential $n + 1$). When an invariant is the product of two or more power polynomials it can be reduced to a product of the tensors just defined. For example

$$\begin{aligned}
 P_2 P_4 &= \sum_{\alpha} (\psi^{\alpha})^2 \sum_{\beta} (\psi^{\beta})^4 = \sum_{\alpha} e_{i_1}^{\alpha} e_{i_2}^{\alpha} \sum_{\beta} e_{i_3}^{\beta} e_{i_4}^{\beta} e_{i_5}^{\beta} e_{i_6}^{\beta} \phi_{i_1} \cdots \phi_{i_6} \\
 &= (n + 1) \delta_{i_1 i_2} \sum_{\beta} e_{i_3}^{\beta} e_{i_4}^{\beta} e_{i_5}^{\beta} e_{i_6}^{\beta} \phi_{i_1} \cdots \phi_{i_6} = (n + 1) \left\{ T_{i_1 i_2}^{(2,1)} T_{i_2 i_3 i_4 i_5}^{(4,1)} \right\} \phi_{i_1} \cdots \phi_{i_6},
 \end{aligned}$$

or

$$\begin{aligned}
 P_3^2 &= \sum_{\alpha} (\psi^{\alpha})^3 \sum_{\beta} (\psi^{\beta})^3 = \left(\sum_{\alpha} e_{i_1}^{\alpha} e_{i_2}^{\alpha} e_{i_3}^{\alpha} \right) \left(\sum_{\beta} e_{i_4}^{\beta} e_{i_5}^{\beta} e_{i_6}^{\beta} \right) \phi_{i_1} \cdots \phi_{i_6} \\
 &= \left\{ T_{i_1 i_2 i_3}^{(3,1)} T_{i_4 i_5 i_6}^{(3,1)} \right\} \phi_{i_1} \cdots \phi_{i_6},
 \end{aligned}$$

and similarly for all other possible cases. We remark here that a symmetrization over all indexes is intended when needed. Thus at any order p we can define $N(p)$ tensors $T_{i_1 \dots i_p}^{(p,m)}$ with $m = 1, \dots, N(p)$ as shown in table 1, that when contracted with p -fields ϕ_i constitute a basis for non-derivative invariants.

2.3.3. Action. We can now construct the general action invariant under the permutation group containing non-derivative interactions. Introducing the convenient notation $T_{i_1 \dots i_p}^{(p)}$ for the tensor coupling of p -fields we can compactly write

$$\begin{aligned}
 S[\phi] &= \int_x \left\{ \frac{1}{2} \partial^{\mu} \phi_i \partial_{\mu} \phi_i + \frac{1}{2} T_{i_1 i_2}^{(2)} \phi_{i_1} \phi_{i_2} + \frac{1}{3!} T_{i_1 i_2 i_3}^{(3)} \phi_{i_1} \phi_{i_2} \phi_{i_3} \right. \\
 &\quad \left. + \frac{1}{4!} T_{i_1 i_2 i_3 i_4}^{(4)} \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} + \frac{1}{5!} T_{i_1 i_2 i_3 i_4 i_5}^{(5)} \phi_{i_1} \phi_{i_2} \phi_{i_3} \phi_{i_4} \phi_{i_5} + \dots \right\}, \quad (2.9)
 \end{aligned}$$

which implicitly defines the dimensionful couplings $\bar{\lambda}_{p,m}$ (clearly one for each invariant of table 1),

Table 2. Explicit form of the invariants of S_{n+1} for $n = 1, 2, 3$. The first n invariants constitute a basis with rational coefficients upon which all other invariants defined in (2.11) can be expressed.

	$n = 1$	$n = 2$	$n = 3$
I_2	φ_1^2	$\varphi_1^2 + \varphi_2^2$	$\varphi_1^2 + \varphi_2^2 + \varphi_3^2$
I_3	0	$\frac{3}{\sqrt{2}}\varphi_2(\varphi_2^2 - 3\varphi_1^2)$	$\frac{4}{\sqrt{3}}(\sqrt{2}\varphi_1(\varphi_1^2 - 3\varphi_2^2) - 3(\varphi_1^2 + \varphi_2^2)\varphi_3 + 2\varphi_3^3)$
$I_{4,1}$	$2I_2^2$	$\frac{9}{2}I_2^2$	$(8(\varphi_1^4 + \varphi_2^4 + \frac{7}{6}\varphi_3^4 + \varphi_1^2(\varphi_3^2 - \frac{2\sqrt{2}}{3}\varphi_1\varphi_3) + 2\varphi_2^2(\varphi_1^2 + \sqrt{2}\varphi_1\varphi_3 + \frac{1}{2}\varphi_3^2)))$
$I_{4,2}$	I_2^2	I_2^2	I_2^2
$I_{5,1}$	0	$\frac{5}{2}I_2I_3$	$\frac{10}{3}I_2I_3$
$I_{5,2}$	0	I_2I_3	I_2I_3
$I_{6,1}$	$2I_2^3$	$\frac{27}{4}I_2^3 + I_3^2$	$\frac{1}{3}I_3^2 + 3I_2I_{4,1} - 8I_2^3$
$I_{6,2}$	$2I_2^3$	$\frac{9}{2}I_2^3$	$I_2I_{4,1}$
$I_{6,3}$	I_2^3	I_2^3	I_2^3
$I_{6,4}$	0	I_3^2	I_3^2

$$\begin{aligned}
T_{i_1 i_2 i_3}^{(3)} &= \bar{\lambda}_3 T_{i_1 i_2 i_3}^{(3,1)} \\
T_{i_1 i_2 i_3 i_4}^{(4)} &= \bar{\lambda}_{4,1} T_{i_1 i_2 i_3 i_4}^{(4,1)} + \bar{\lambda}_{4,2} T_{i_1 i_2 i_3 i_4}^{(4,2)} \\
T_{i_1 i_2 i_3 i_4 i_5}^{(5)} &= \bar{\lambda}_{5,1} T_{i_1 i_2 i_3 i_4 i_5}^{(5,1)} + \bar{\lambda}_{5,2} T_{i_1 i_2 i_3 i_4 i_5}^{(5,2)} \\
T_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6)} &= \bar{\lambda}_{6,1} T_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6,1)} + \bar{\lambda}_{6,2} T_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6,2)} + \bar{\lambda}_{6,3} T_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6,3)} + \bar{\lambda}_{6,4} T_{i_1 i_2 i_3 i_4 i_5 i_6}^{(6,4)}.
\end{aligned} \quad (2.10)$$

Finally, we define the following invariants through contraction

$$I_{p,m} \equiv T_{i_1 \dots i_p}^{(p,m)} \phi_{i_1} \cdots \phi_{i_p}, \quad (2.11)$$

which are clearly related to the invariants P_k of the preceding subsection, but now function of the fields ϕ_i , and re-write the action (2.9) as

$$S[\phi] = \int_x \left\{ \frac{1}{2} \partial^\mu \phi_i \partial_\mu \phi_i + V(\phi_1, \dots, \phi_n) \right\}, \quad (2.12)$$

where the potential is constructed from the invariants (2.11) as follows

$$V(\phi_1, \dots, \phi_n) = \sum_{p=2}^{\infty} \frac{1}{p!} \sum_{m=1}^{N(p)} \bar{\lambda}_{p,m} I_{p,m} = \frac{1}{2} \bar{\lambda}_2 I_2 + \frac{1}{3!} \bar{\lambda}_3 I_3 + \frac{1}{4!} (\bar{\lambda}_{4,1} I_{4,1} + \bar{\lambda}_{4,2} I_{4,2}) + \dots \quad (2.13)$$

But, as expected, only the first n invariants are independent, as can be seen from the explicitly construct reported in table 2 in the cases $n = 1, 2, 3$. The examples of table 2 also show that indeed rational coefficients are needed to express the dependent invariants in terms of the independent ones.

2.4. Explicit construction for $n = 2$

To facilitate the understanding we give an explicit construction of the Potts field theory in the S_3 case. The coordinates of the vertices of a regular triangle in the plane, as shown in figure 2, are

$$e^1 = \sqrt{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e^2 = \sqrt{2} \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \quad e^3 = \sqrt{2} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}. \quad (2.14)$$

The matrix representation of the $|S_3| = 3! = 6$ elements of S_3 , which leave invariant the triangle, are

$$\begin{aligned} I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & R &= \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & R^{-1} &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \mu_1 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \mu_2 &= \begin{pmatrix} +\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & \mu_3 &= \begin{pmatrix} +\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned} \quad (2.15)$$

Clearly I is the identity, R the (counter-clockwise) rotation of $\frac{2}{3}\pi$ and R^{-1} its inverse, while μ_i for $i = 1, 2, 3$ are the reflections along the axis passing through the i th vertex of the triangle. The two $n = 2$ invariants are, from table 2, the following

$$\rho \equiv I_2 = 3(\varphi_1^2 + \varphi_2^2) \quad \tau \equiv I_3 = \frac{3}{\sqrt{2}}\varphi_2(\varphi_2^2 - 3\varphi_1^2).$$

We can check explicitly that these invariants are indeed so under the transformations (2.15). Consider for example the rotation R

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \mapsto \begin{pmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{pmatrix} = R \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\varphi_1 - \frac{\sqrt{3}}{2}\varphi_2 \\ \frac{\sqrt{3}}{2}\varphi_1 - \frac{1}{2}\varphi_2 \end{pmatrix}.$$

Its easy to check the invariance of ρ and τ

$$\begin{aligned} \tilde{\rho} &= 3(\tilde{\varphi}_1^2 + \tilde{\varphi}_2^2) \\ &= \frac{3}{4}\varphi_1^2 + \frac{9}{4}\varphi_2^2 + \frac{3\sqrt{3}}{2}\varphi_1\varphi_2 + \frac{3}{4}\varphi_2^2 + \frac{9}{4}\varphi_1^2 - \frac{3\sqrt{3}}{2}\varphi_1\varphi_2 \\ &= 3(\varphi_1^2 + \varphi_2^2) = \rho \\ \tilde{\tau} &= \frac{3}{\sqrt{2}}\tilde{\varphi}_2(\tilde{\varphi}_2^2 - 3\tilde{\varphi}_1^2) \\ &= \frac{3}{\sqrt{2}} \left(\frac{\sqrt{3}}{2}\varphi_1 - \frac{1}{2}\varphi_2 \right) \left[\left(\frac{\sqrt{3}}{2}\varphi_1 - \frac{1}{2}\varphi_2 \right)^2 - 3 \left(-\frac{1}{2}\varphi_1 - \frac{\sqrt{3}}{2}\varphi_2 \right)^2 \right] \\ &= \frac{3}{\sqrt{2}}\varphi_2(\varphi_2^2 - 3\varphi_1^2) = \tau. \end{aligned}$$

Similarly one can check all the other transformations in (2.15). So we showed explicitly that the theory with potential of the form (2.13)

$$V(\varphi_1, \varphi_2) = \frac{\bar{\lambda}_2}{2} 3 (\varphi_1^2 + \varphi_2^2) + \frac{\bar{\lambda}_3}{3!} \frac{3}{\sqrt{2}} \varphi_2 (\varphi_2^2 - 3\varphi_1^2) + \dots \quad (2.16)$$

is invariant under the action of the elements of S_3 .

3. Functional RG for Potts

3.1. Flow equation

The functional renormalization group (FRG) approach to quantum field theory is based on the exact flow equation satisfied by the scale dependent effective action Γ_k (for a general review see e.g. [47]; while for a self-contained introduction with particular attention to statistical physics see [48]). This is a scale-dependent functional which includes fluctuations between a given microscopic UV scale Λ down to a running scale $k < \Lambda$. The effective action interpolates smoothly between the bare UV action $S = \Gamma_{k=\Lambda}$ and the full effective action, or free energy, $\Gamma = \Gamma_{k=0}$ for $k \rightarrow 0$ so that all fluctuations are summed over. The scale dependence of the effective action Γ_k on the RG time $t := \log k$ is governed by the exact flow equation [49], which for an n -component scalar $\varphi_i \equiv \langle \phi_i \rangle$ reads

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \text{Tr} \left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta \varphi_i \delta \varphi_j} + R_{k,ij} \right)^{-1} \partial_t R_{k,ji}. \quad (3.1)$$

Here R_k is a proper infrared regulator function which suppresses the propagation of the infrared modes (of momentum smaller than k) by directly modifying the bare propagator of the theory. The Wetterich equation (3.1) is the starting point of all our subsequent analysis.

3.1.1. Local potential approximation. Despite its simplicity, the Wetterich equation (3.1) is difficult to solve and one should rely on approximations based on non-perturbative truncations, which amounts to project the RG flow on a subset of suitable functionals. One of these truncations is called improved local potential approximation (LPA') and consists in considering the following ansatz for the effective-action

$$\Gamma_k[\varphi] = \int_x \left\{ \frac{1}{2} Z_k \partial^\mu \varphi_i \partial_\mu \varphi_i + V_k(\varphi_1, \dots, \varphi_n) \right\}, \quad (3.2)$$

where the whole theory space is projected into the infinite dimensional functional space of effective potentials V_k . At first order of the derivative expansion, also called local potential approximation (LPA), one neglects the running and the field dependence of the wave function renormalization, $Z_k \equiv 1$. In the improved local potential approximation (LPA') we consider throughout, Z_k is a non-vanishing field-independent but scale-dependent running wavefunction renormalization constant, directly related to the anomalous dimension $\eta_k = -\partial_t \log Z_k$.

We can obtain a flow equation for the effective potential by inserting the ansatz (3.2) in the Wetterich equation (3.1). The first thing to do is to compute the Hessian which, dropping for the moment Z_k , reads

$$\frac{\delta^2 \Gamma_k}{\delta \varphi_i \delta \varphi_j} = -\partial^2 \delta_{ij} + V_{k,ij} =: -\partial^2 (\mathbb{I})_{ij} + (\mathbb{V})_{ij}, \quad (3.3)$$

where we introduced matrix notation for clarity. When we insert (3.3) in the flow equation (3.1), with the choice $R_{k,ij} = \delta_{ij} R_k$ for the matrix structure of the cutoff, we find

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{\partial_t R_k (-\partial^2)}{(-\partial^2 + R_k (-\partial^2)) \mathbb{I} + \mathbb{V}}. \quad (3.4)$$

After choosing constant field configuration so that $\partial_t \Gamma_k = (\int d^d x) \partial_t V_k$ and performing the angular integrations, we obtain the following expression for the flow of the potential

$$\partial_t V_k = \frac{1}{2} \frac{1}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^\infty dz z^{\frac{d}{2}-1} \text{tr} \frac{\partial_t R_k(z)}{(z + R_k(z)) \mathbb{I} + \mathbb{V}}. \quad (3.5)$$

We will adopt now the linear cutoff $R_k(z) = (k^2 - z)\theta(k^2 - z)$, where θ is the standard Heaviside step function, that allows a simple explicit evaluation of the integral in (3.5). With this choice we find the following form for the flow equation of the potential

$$\partial_t V_k = c_d k^{d+2} \text{tr} \frac{1}{k^2 \mathbb{I} + \mathbb{V}}, \quad (3.6)$$

where we defined the constant $c_d^{-1} \equiv (4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2} + 1)$. This expression is the general form for the LPA of an n -component scalar in d -dimensions and as such it is the generating function of all beta functions of the couplings $\bar{\lambda}_{p,m}$ of non-derivative interactions entering (2.13).

Unfortunately, for general n , it is not possible to obtain a closed form for the inverse of the matrix $k^2 \mathbb{I} + \mathbb{V}$. Therefore, since our main interest is the study of the limits $n \rightarrow 0$ (Percolation) and $n \rightarrow -1$ (Spanning Forest) (which we remember are the only non-trivial cases apart Ising in $d > 2$) we are forced to *truncate* the effective potential (2.13) in order to convert (3.6) in a set of coupled beta functions (explicitly n -dependent) for a *finite* set of couplings, by expanding the rhs of (3.6) in powers of \mathbb{V} and thus reducing the problem to the evaluation of traces of these powers. This approach is presented in section 3.2, where the appropriate ‘trace machinery’ will be developed and where we report the truncation up to φ^6 . A different approach is based on the fact that the inversion of the matrix in (3.6) is instead possible whenever n is a given, conceivably small, positive integer (and thus not applicable in the Percolation and Spanning Forest cases) and in this case we are able to write the explicit form of the LPA’. This is presented in section 3.4 for the $n = 1, 2$ cases.

3.2. Beta functions for general n

The aim of this section is to provide a general framework to extract beta functions for any value of $n \in \mathbb{R}$. Defining $\mathbb{V} = \bar{\lambda}_2 \mathbb{I} + \mathbb{M}$ we proceed expanding the inverse propagator as

$$\mathrm{tr} \frac{1}{k^2 \mathbb{I} + \mathbb{V}} = \mathrm{tr} \frac{k^{-2}}{(1 + \lambda_2) \mathbb{I} + \mathbb{M}/k^2} = \sum_{m=0}^{\infty} (-1)^m \frac{k^{-2-2m}}{(1 + \lambda_2)^{m+1}} \mathrm{tr} \mathbb{M}^m, \quad (3.7)$$

where from now on we absorb the factor c_d in the definition of the effective potential and of the field $V_k \rightarrow c_d V_k$, $\varphi_i \rightarrow c_d^{1/2} \varphi_i$. Inserting (3.7) into the flow equation of the effective potential (3.6) gives

$$\partial_t V_k = \sum_{m=0}^{\infty} (-1)^m \frac{k^{d-2m}}{(1 + \lambda_2)^{m+1}} \mathrm{tr} \mathbb{M}^m. \quad (3.8)$$

Any field dependence on the rhs of equation (3.8) is encoded in $\mathrm{tr} \mathbb{M}^m$ since

$$(\mathbb{M})_{ab} = T_{abi_1}^{(3)} \varphi_{i_1} + \frac{1}{2} T_{abi_1 i_2}^{(4)} \varphi_{i_1} \varphi_{i_2} + \frac{1}{3!} T_{abi_1 i_2 i_3}^{(5)} \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} + \dots \quad (3.9)$$

and once the traces $\mathrm{tr} \mathbb{M}^m$ are computed with the help of rules (2.5) and expressed in terms of the invariants (2.11), the rhs of the flow equation (3.6) assumes the form

$$\partial_t V_k = \frac{1}{2} \bar{\beta}_2 I_2 + \frac{1}{3!} \bar{\beta}_3 I_3 + \frac{1}{4!} (\bar{\beta}_{4,1} I_{4,1} + \bar{\beta}_{4,2} I_{4,2}) + \frac{1}{5!} (\bar{\beta}_{5,1} I_{5,1} + \bar{\beta}_{5,2} I_{5,2}) + \dots, \quad (3.10)$$

from which we can extract the beta functions of all the couplings included in the truncation of the potential (2.13). Before presenting our results to order φ^6 , in the following subsection we are going to work out the simple φ^3 example to see how this general procedure works at hand.

3.2.1. Example: φ^3 . When we truncate the expansion (2.13) at order $p = 3$, only the trilinear coupling and the mass terms are present in the corresponding effective action Γ_k . Accordingly, the non-diagonal part of the Hessian is just $\mathbb{M} = \bar{\lambda}_3 T_{abi}^{(3)} \varphi_i$ and therefore we have to consider the contributions $\mathrm{tr} \mathbb{M}$, $\mathrm{tr} \mathbb{M}^2$, $\mathrm{tr} \mathbb{M}^3$ since $\mathrm{tr} \mathbb{M}^m \sim \varphi^{i \geq m}$. In the following, with the help of the rules (2.5), we show explicitly how the computation of the traces works.

Apart the linear trace which is zero,

$$\mathrm{tr} \mathbb{M} = \bar{\lambda}_3 T_{aai}^{(3)} \varphi_i = \bar{\lambda}_3 \sum_{\alpha} e_a^{\alpha} e_a^{\alpha} e_i^{\alpha} \varphi_i = n \bar{\lambda}_3 \sum_{\alpha} e_i^{\alpha} = 0, \quad (3.11)$$

non-trivial contributions come from the trace of the square

$$\begin{aligned} \mathrm{tr} \mathbb{M}^2 &= T_{aj i_1}^{(3,1)} \varphi_{i_1} T_{jai_2}^{(3,1)} \varphi_{i_2} \\ &= \bar{\lambda}_3^2 \left[\sum_{\alpha\beta} e_a^{\alpha} e_j^{\alpha} e_{i_1}^{\alpha} e_j^{\beta} e_a^{\beta} e_{i_2}^{\beta} \right] \varphi_{i_1} \varphi_{i_2} \\ &= \bar{\lambda}_3^2 \left[(n+1) \sum_{\alpha} e_j^{\alpha} e_{i_1}^{\alpha} e_j^{\alpha} e_{i_2}^{\alpha} - \sum_{\alpha\beta} e_j^{\alpha} e_{i_1}^{\alpha} e_j^{\beta} e_{i_2}^{\beta} \right] \varphi_{i_1} \varphi_{i_2} \\ &= \bar{\lambda}_3^2 \left[n(n+1)^2 \delta_{i_1 i_2} - (n+1) \sum_{\alpha} e_{i_1}^{\alpha} e_{i_2}^{\alpha} - \sum_{\alpha} e_{i_1}^{\alpha} \sum_{\beta} e_{i_2}^{\beta} \right] \varphi_{i_1} \varphi_{i_2} \\ &= \bar{\lambda}_3^2 (n+1)^2 (n-1) I_2 \end{aligned} \quad (3.12)$$

and from the trace of the cube

$$\begin{aligned}
\text{tr } \mathbb{M}^3 &= T_{abi_1}^{(3,1)} \varphi_{i_1} T_{bci_2}^{(3,1)} \varphi_{i_2} T_{cai_3}^{(3,1)} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 \sum_{\alpha\beta\gamma} e_a^\alpha e_b^\alpha e_{i_1}^\alpha e_b^\beta e_c^\beta e_{i_2}^\beta e_c^\gamma e_a^\gamma e_{i_3}^\gamma \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 \left[(n+1) \sum_{\alpha\beta} e_b^\alpha e_{i_1}^\alpha e_b^\beta e_c^\beta e_{i_2}^\beta e_c^\alpha e_{i_3}^\alpha - \sum_{\alpha\beta\gamma} e_b^\alpha e_{i_1}^\alpha e_b^\beta e_c^\beta e_{i_2}^\beta e_c^\gamma e_{i_3}^\gamma \right] \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 \left[(n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_c^\alpha e_{i_2}^\alpha e_c^\alpha e_{i_3}^\alpha - (n+1) \sum_{\alpha\beta} e_{i_1}^\alpha e_c^\beta e_{i_2}^\beta e_c^\alpha e_{i_3}^\alpha \right. \\
&\quad \left. - (n+1) \sum_{\alpha\gamma} e_{i_1}^\alpha e_c^\alpha e_{i_2}^\gamma e_c^\gamma e_{i_3}^\gamma + \sum_{\alpha\beta\gamma} e_{i_1}^\alpha e_c^\beta e_{i_2}^\beta e_c^\gamma e_{i_3}^\gamma \right] \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 \left[n(n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\alpha - (n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\alpha \right. \\
&\quad \left. + (n+1) \sum_{\alpha\beta} e_{i_1}^\alpha e_{i_2}^\beta e_{i_3}^\alpha - (n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\alpha + (n+1) \sum_{\alpha\beta\gamma} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\gamma \right] \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 \left[n(n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\alpha - 2(n+1)^2 \sum_{\alpha} e_{i_1}^\alpha e_{i_2}^\alpha e_{i_3}^\alpha \right] \varphi_{i_1} \varphi_{i_2} \varphi_{i_3} \\
&= \bar{\lambda}_3^3 (n+1)^2 (n-2) I_3.
\end{aligned} \tag{3.13}$$

Inserting these traces into (3.8) and comparing with the flow equation in the form (3.10) we can immediately read off the (dimensionful) beta functions

$$\bar{\beta}_2 = k^{d-4} \frac{2(n-1)(n+1)^2}{(1+\lambda_2)^3} \bar{\lambda}_2^2 \quad \bar{\beta}_3 = -k^{d-6} \frac{6(n-2)(n+1)^2}{(1+\lambda_2)^4} \bar{\lambda}_3^3.$$

3.2.2. Explicit beta functions. It is clear from the previous example that the computation of the general trace $\text{tr } \mathbb{M}^m$ can be performed along the same lines but rapidly becomes unfeasible by hand, and for a given truncation order p in the expansion (2.13) we need to expand the inverse propagator up to $\text{tr } \mathbb{M}^p$ and keep all contributions up to order p . To tackle the ‘trace machinery’ involved in the general computation we have used symbolic manipulation software (the `xTensor` package for `Mathematica` [50]) for which we have written an explicit code.

In this work we consider the expansion (2.13) up to order φ^6 for a total of ten dimensionless couplings: $\{\lambda_2, \lambda_3, \lambda_{4,1}, \lambda_{4,2}, \lambda_{5,1}, \lambda_{5,2}, \lambda_{6,1}, \lambda_{6,2}, \lambda_{6,3}, \lambda_{6,4}\}$. The beta functions of these couplings can be written in terms of the dimensionful ones as⁶

$$\beta_{m,i} = \left[m \left(\frac{d}{2} - 1 + \frac{\eta}{2} \right) - d \right] \lambda_{m,i} + k^{m(\frac{d}{2}-1+\frac{\eta}{2})-d} \bar{\beta}_{m,i}, \tag{3.14}$$

⁶ We just differentiate both sides of $\lambda_{m,i} = k^{m(d/2-1+\eta/2)-d} \bar{\lambda}_{m,i}$.

and their explicit form can be extracted once the reduction of the traces up to $\text{tr } \mathbb{M}^6$ has been performed. The final result is the following set of ten beta functions, valid for arbitrary d and n , and it is the main achievement of the present work:

$$\beta_2 = (-2 + \eta)\lambda_2 + \frac{2(n-1)(n+1)^2\lambda_3^2}{(1+\lambda_2)^3} - \frac{n(n+1)\lambda_{4,1} + \frac{1}{3}(n+2)\lambda_{4,2}}{(1+\lambda_2)^2} \quad (3.15)$$

$$\beta_3 = \frac{1}{2}(d+3\eta-6)\lambda_3 + \frac{6\left(\frac{2}{3}\lambda_{4,2} + (n-1)(n+1)\lambda_{4,1}\right)\lambda_3}{(1+\lambda_2)^3} - \frac{n\lambda_{5,1} + \frac{1}{10}(n+6)\lambda_{5,2}}{(1+\lambda_2)^2} - \frac{6(n-2)(n+1)^2\lambda_3^3}{(1+\lambda_2)^4} \quad (3.16)$$

$$\begin{aligned} \beta_{4,1} = & (d+2\eta-4)\lambda_{4,1} + \frac{6\left(\frac{4}{3}\lambda_{4,2}\lambda_{4,1} + (n^2-1)\lambda_{4,1}^2\right)}{(1+\lambda_2)^3} \\ & + \frac{6\frac{4}{15}(n+1)(3\lambda_{5,2} + 5(n-1)\lambda_{5,1})\lambda_3}{(1+\lambda_2)^3} \\ & - \frac{24\left(\frac{3}{2}(n-2)(n+1)^2\lambda_{4,1} + (n+1)\lambda_{4,2}\right)\lambda_3^2}{(1+\lambda_2)^4} \\ & - \frac{15n\lambda_{6,1} + (n+8)\lambda_{6,3} + 9(n+1)\lambda_{6,2}}{15(1+\lambda_2)^2} \\ & + \frac{24(n-3)(n+1)^3\lambda_3^4}{(1+\lambda_2)^5} \end{aligned} \quad (3.17)$$

$$\begin{aligned} \beta_{4,2} = & (d+2\eta-4)\lambda_{4,2} - \frac{24\left(\frac{3}{2}(n+1)^3\lambda_{4,1} + \frac{1}{2}(n-3)(n+1)^2\lambda_{4,2}\right)\lambda_3^2}{(1+\lambda_2)^4} \\ & + \frac{6(n+1)^2\lambda_{4,1}^2 + 6\frac{2}{3}n(n+1)\lambda_{4,1}\lambda_{4,2} + 6\frac{1}{9}(n+8)\lambda_{4,2}^2}{(1+\lambda_2)^3} \\ & + \frac{\frac{12}{5}(n-3)(n+1)^2\lambda_3\lambda_{5,2}}{(1+\lambda_2)^3} \\ & + \frac{3(n+1)^2\lambda_{6,2} - 2n(n+1)\lambda_{6,3} - (n+4)\lambda_{6,4}}{5(1+\lambda_2)^2} \\ & + \frac{48(n+1)^4\lambda_3^4}{(1+\lambda_2)^5} \end{aligned} \quad (3.18)$$

$$\begin{aligned}
\beta_{5,1} = & \frac{1}{2}(3d + 5\eta - 10)\lambda_{5,1} + \frac{80(n+1)^2(3(n+1)(n-3)\lambda_{4,1} + 2\lambda_{4,2})\lambda_3^3}{(1+\lambda_2)^5} \\
& - \frac{6(n+1)(15(n+1)(n-2)\lambda_{4,1}^2 + 20\lambda_{4,1}\lambda_{4,2})\lambda_3}{(1+\lambda_2)^4} \\
& - \frac{12(n+1)^2(5(n-2)\lambda_{5,1} + 3\lambda_{5,2})\lambda_3^2}{(1+\lambda_2)^4} \\
& + \frac{\frac{4}{3}(10\lambda_{4,2}\lambda_{5,1} + 9(n+1)\lambda_{4,1}\lambda_{5,2} + 15(n+1)(n-1)\lambda_{4,1}\lambda_{5,1})}{(1+\lambda_2)^3} \\
& + \frac{\frac{2}{3}(n+1)(8\lambda_{6,3} + 15(n-1)\lambda_{6,1} + 9(n+1)\lambda_{6,2})\lambda_3}{(1+\lambda_2)^3} \\
& - \frac{120(n-4)(n+1)^4\lambda_3^5}{(1+\lambda_2)^6}
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\beta_{5,2} = & \frac{1}{2}(3d + 5\eta - 10)\lambda_{5,2} - \frac{20(9(n+1)^2\lambda_{4,1}^2 + 3(n^2 - 2n - 3)\lambda_{4,2}\lambda_{4,1} + 4\lambda_{4,2}^2)\lambda_3}{(1+\lambda_2)^4} \\
& - \frac{6(n+1)^2(10\lambda_{5,1} + (4n-19)\lambda_{5,2})\lambda_3^2}{(1+\lambda_2)^4} \\
& + \frac{\frac{4}{3}(6\lambda_{6,4} + (3n^2 - 4n - 7)\lambda_{6,3} + 3(n-4)(n+1)^2\lambda_{6,2})\lambda_3}{(1+\lambda_2)^3} \\
& + \frac{2(n+1)(10\lambda_{5,1} + (4n-9)\lambda_{5,2})\lambda_{4,1} + (26\lambda_{5,2} + n(10\lambda_{5,1} + \lambda_{5,2}))\lambda_{4,2}}{(1+\lambda_2)^3} \\
& + \frac{80(n+1)^2(9(n+1)\lambda_{4,1} + (n-6)\lambda_{4,2})\lambda_3^3}{(1+\lambda_2)^5} - \frac{600(n+1)^4\lambda_3^5}{(1+\lambda_2)^6}
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\beta_{6,1} = & (2d + 3\eta - 6)\lambda_{6,1} - \frac{600(n+1)^3(3(n^2 - 3n - 4)\lambda_{4,1} + 2\lambda_{4,2})\lambda_3^4}{(1+\lambda_2)^6} \\
& + \frac{24(n+1)^2(60\lambda_{4,1}\lambda_{4,2} + 45(n^2 - 2n - 3)\lambda_{4,1}^2)\lambda_3^2}{(1+\lambda_2)^5} \\
& + \frac{96(n+1)^3(5(n-3)\lambda_{5,1} + 3\lambda_{5,2})\lambda_3^3}{(1+\lambda_2)^5} \\
& - \frac{24(n+1)(3(n+1)(5(n-2)\lambda_{5,1} + 3\lambda_{5,2})\lambda_{4,1} + 10\lambda_{4,2}\lambda_{5,1})\lambda_3}{(1+\lambda_2)^4} \\
& - \frac{90(n+1)((n^2 - n - 2)\lambda_{4,1} + 2\lambda_{4,2})\lambda_{4,1}^2}{(1+\lambda_2)^4} \\
& - \frac{6(n+1)^2(15(n-2)\lambda_{6,1} + 8\lambda_{6,3} + 9(n+1)\lambda_{6,2})\lambda_3^2}{(1+\lambda_2)^4} \\
& + \frac{20\lambda_{4,2}\lambda_{6,1} + 4(n+1)\lambda_{5,1}(5(n-1)\lambda_{5,1} + 6\lambda_{5,2})}{(1+\lambda_2)^3} \\
& + \frac{2(n+1)(15(n-1)\lambda_{6,1} + 8\lambda_{6,3} + 9(n+1)\lambda_{6,2})\lambda_{4,1}}{(1+\lambda_2)^3} \\
& + \frac{720(n-5)(n+1)^5\lambda_3^6}{(1+\lambda_2)^7}
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
\beta_{6,2} = & (2d + 3\eta - 6) \lambda_{6,2} + \frac{20 (5\lambda_{4,2}\lambda_{6,2} + \lambda_{4,1} (-4\lambda_{6,3} + 3(n^2 - 1) \lambda_{6,2}))}{5(1 + \lambda_2)^3} \\
& + \frac{100\lambda_{5,1}^2 + 20n\lambda_{5,1}\lambda_{5,2} + (n + 30)\lambda_{5,2}^2}{5(1 + \lambda_2)^3} \\
& - \frac{1200(n + 1)^2 (3(n + 1)\lambda_{4,1} - \lambda_{4,2}) \lambda_3^4}{(1 + \lambda_2)^6} \\
& + \frac{48(n + 1)^2 (10\lambda_{5,1} + (n - 8)\lambda_{5,2}) \lambda_3^3}{(1 + \lambda_2)^5} \\
& + \frac{16 (15(n + 1) (3(n + 1)\lambda_{4,1} - 4\lambda_{4,2}) \lambda_{4,1} + 5 (9(n + 1)^2 \lambda_{4,1}^2 + 2\lambda_{4,2}^2)) \lambda_3^2}{(1 + \lambda_2)^5} \\
& - \frac{6 (-30\lambda_{4,1}^2 \lambda_{4,2} + 6(n + 1) (10\lambda_{5,1} + (n - 4)\lambda_{5,2}) \lambda_3 \lambda_{4,1})}{(1 + \lambda_2)^4} \\
& - \frac{6 (3(n + 1)^2 (2n - 7) \lambda_3 \lambda_{6,2} - 8 ((n + 1) \lambda_3 \lambda_{6,3} - 2\lambda_{4,2} \lambda_{5,2})) \lambda_3}{(1 + \lambda_2)^4} \\
& + \frac{2160(n + 1)^4 \lambda_3^6}{(1 + \lambda_2)^7} \tag{3.22}
\end{aligned}$$

$$\begin{aligned}
\beta_{6,3} = & (2d + 3\eta - 6) \lambda_{6,3} + \frac{16(n + 1) (60(n + 1)^2 \lambda_{5,1} - 63\lambda_{5,2}) \lambda_3^3}{(1 + \lambda_2)^5} \\
& + \frac{16(n + 1) (9n^3 - 45n^2 - 117n) \lambda_3^3 \lambda_{5,2}}{(1 + \lambda_2)^5} \\
& + \frac{16(n + 1) (180(n + 1)^2 \lambda_{4,1}^2 + 15 (3n^2 - 11n - 14) \lambda_{4,1} \lambda_{4,2} + 50\lambda_{4,2}^2) \lambda_3^2}{(1 + \lambda_2)^5} \\
& - \frac{6(n + 1) (7 (n^2 - 2n - 3) \lambda_{6,3} - 18(n + 1)^2 \lambda_{6,2}) \lambda_3^2}{(1 + \lambda_2)^4} \\
& - \frac{18(n + 1) (5(n + 1) \lambda_{6,1} + 4\lambda_{6,4}) \lambda_3^2}{(1 + \lambda_2)^4} \\
& - \frac{30 (9(n + 1)^2 \lambda_{4,1}^2 + 3 (n^2 - 1) \lambda_{4,2} \lambda_{4,1} + 8\lambda_{4,2}^2) \lambda_{4,1}}{(1 + \lambda_2)^4} \\
& - \frac{12(n + 1) (2 (5(n - 3) \lambda_{5,1} + 9\lambda_{5,2}) \lambda_{4,2} + 3(n + 1) (10\lambda_{5,1} + 3(n - 5) \lambda_{5,2}) \lambda_{4,1}) \lambda_3}{(1 + \lambda_2)^4} \\
& + \frac{18(n + 1) (10(n - 3) \lambda_{5,1} + 9\lambda_{5,2}) \lambda_{5,2}}{15(1 + \lambda_2)^3} + \frac{10 (15n\lambda_{6,1} + (n + 38) \lambda_{6,3} + 9(n + 1) \lambda_{6,2}) \lambda_{4,2}}{15(1 + \lambda_2)^3} \\
& + \frac{30 (15(n + 1) \lambda_{6,1} + (7n^2 + n - 6) \lambda_{6,3} + 12\lambda_{6,4} - 18(n + 1)^2 \lambda_{6,2}) \lambda_{4,1}}{15(1 + \lambda_2)^3} \\
& - \frac{600(n + 1)^3 (12(n + 1) \lambda_{4,1} + (n - 7) \lambda_{4,2}) \lambda_3^4}{(1 + \lambda_2)^6} + \frac{4320(n + 1)^5 \lambda_3^6}{(1 + \lambda_2)^7} \tag{3.23}
\end{aligned}$$

$$\begin{aligned}
\beta_{6,4} = & (2d + 3\eta - 6) \lambda_{6,4} + \frac{600(n+1)^4 (3(n+1)\lambda_{4,1} - 4\lambda_{4,2}) \lambda_3^4}{(1 + \lambda_2)^6} \\
& + \frac{8(n+1)^2 (-90(n+1)^2 \lambda_{4,1}^2 + 72(n+1)^2 \lambda_3 \lambda_{5,2}) \lambda_3^2}{(1 + \lambda_2)^5} \\
& + \frac{8(n+1)^2 (150(n+1)\lambda_{4,1}\lambda_{4,2} + 5(3n-23)\lambda_{4,2}^2) \lambda_3^2}{(1 + \lambda_2)^5} \\
& - \frac{108(n+1)^2 (6(n+1)\lambda_{4,1} + (n-7)\lambda_{4,2}) \lambda_3 \lambda_{5,2}}{3(1 + \lambda_2)^4} \\
& - \frac{54(n+1)^2 (2(n+1)\lambda_{6,3} + (n-5)\lambda_{6,4} + 3(n+1)^2 \lambda_{6,2}) \lambda_3^2}{3(1 + \lambda_2)^4} \\
& - \frac{10 (-27(n+1)^3 \lambda_{4,1}^3 + 27(n+1)^2 \lambda_{4,1}^2 \lambda_{4,2} + 9n(n+1)\lambda_{4,1}\lambda_{4,2}^2 + (n+26)\lambda_{4,2}^3)}{3(1 + \lambda_2)^4} \\
& + \frac{2 (6(n+1)^2 \lambda_{4,1}\lambda_{6,3} + 9(n+1)^3 \lambda_{4,1}\lambda_{6,2} + 14\lambda_{4,2}\lambda_{6,4})}{(1 + \lambda_2)^3} \\
& + \frac{2 (3n(n+1)\lambda_{4,1}\lambda_{6,4} + 2n(n+1)\lambda_{4,2}\lambda_{6,3} + n\lambda_{4,2}\lambda_{6,4} - 3(n+1)^2 \lambda_{4,2}\lambda_{6,2})}{(1 + \lambda_2)^3} \\
& + \frac{9(n-7)(n+1)^2 \lambda_{5,2}^2}{5(1 + \lambda_2)^3} - \frac{1440(n+1)^6 \lambda_3^6}{(1 + \lambda_2)^7}. \tag{3.24}
\end{aligned}$$

The anomalous dimension η entering these expressions will be computed in the next subsection.

While the beta functions (3.15)–(3.25) are written using the linear cutoff, it is easy to shift to a general cutoff $R_k(z)$ by the substitution

$$\frac{1}{(1 + \lambda_2)^m} \rightarrow k^{2m-2-d} \frac{d}{4} \int_0^\infty dz z^{d/2-1} G_k^m(z) \partial_t R_k(z).$$

In this way it is straightforward to obtain beta functions for an arbitrary cutoff as needed for the study of cutoff dependence and for the optimization of convergence. We leave this task to a future study, since, as we will show in section 4.2, within the $p = 6$ truncation convergence of critical exponents is fully achieved only in $d = 5$ while larger truncations are needed in $d = 4$ and in $d = 3$.

3.3. Anomalous dimension

The computation of the anomalous dimension η_k requires the computation of the flow of the wavefunction Z_k since $\eta_k = -\partial_t \log Z_k$. It is clear from (3.2) that Z_k corresponds to the term in Γ_k which is quadratic in the fields and in the momentum

$$\partial_t Z_k \delta_{ij} = \lim_{p^2 \rightarrow 0} \frac{d^2}{dp^2} \frac{\delta^2}{\delta \varphi_i(p) \delta \varphi_j(-p)} \partial_t \Gamma_k[\varphi] \Big|_{\varphi=0}. \tag{3.25}$$

The flow of Z_k is therefore related to that of the two-point function whose flow equation reads

$$[\partial_t \Gamma_k^{(2)}(p^2)]_{ij} = -\frac{1}{2} \int_q [\Gamma_k^{(4)}(q, p, -p, -q)]_{aija} G_k^2(q^2) \partial_t R_k(q^2) \\ + \int_q [\Gamma_k^{(3)}(q, p, -q-p)]_{aib} G_k((q+p)^2) [\Gamma_k^{(3)}(q+p, -p, -q)]_{bja} G_k^2(q^2) \partial_t R_k(q^2), \quad (3.26)$$

where we introduced the regularized propagator (at $\varphi = 0$)

$$[G_k(q^2)]^{-1} = \Gamma_k^{(2)}(q^2) + R_k(q^2) = Z_k q^2 + \bar{\lambda}_2 + R_k(q^2).$$

Equation (3.27) depends on the three- and four-point functions, but the only contribution proportional to p^2 comes from the integral involving the first. Thus without loss of generality we can consider the effective action (3.2) where the potential (2.13) is truncated at order $p = 3$ finding

$$\partial_t Z_k \delta_{ij} = \bar{\lambda}_3^2 T_{aib}^{(3)} T_{ajb}^{(3)} \int_q G_k((q+p)^2) G_k^2(q^2) \partial_t R_k(q^2) \Big|_{p^2}. \quad (3.27)$$

Employing the linear cutoff (including now the wavefunction) $R_k(z) = Z_k(k^2 - z)\theta(k^2 - z)$ gives

$$\int_q G_k((q+p)^2) G_k^2(q^2) \partial_t R_k(q^2) \Big|_{p^2} = -Z_k^2 c_d \left(1 - \frac{\eta_k}{d+2}\right) \frac{k^{d+2}}{(Z_k k^2 + \bar{\lambda}_2)^4}. \quad (3.28)$$

At this point we are left with the task of computing the trace $T_{aib}^{(3)} T_{ajb}^{(3)}$ which, on the other hand, we already computed in section 3.2.1 when calculating $\text{tr } \mathbb{M}^2$ in (3.12). Equation (3.28) then becomes

$$\partial_t Z_k = -Z_k^2 c_d \left(1 - \frac{\eta_k}{d+2}\right) \frac{k^{d+2}}{(Z_k k^2 + \bar{\lambda}_2)^4} (n-1)(n+1)^2 \bar{\lambda}_3^2.$$

Finally, switching to dimensionless variables $\bar{\lambda}_m = Z_k^{m/2} k^{d-m(d/2-1)} \lambda_m$ and after reabsorbing the factor c_d in a field redefinition as before, we find the following simple expression for the anomalous dimension

$$\eta_k = (n-1)(n+1)^2 \frac{\lambda_3^2}{(1+\lambda_2)^4} \left(1 - \frac{\eta_k}{d+2}\right). \quad (3.29)$$

Even if the anomalous dimension term on the r.h.s of this equation stems from the non-perturbative part of the flow equation, we will omit it in the following since its contributions turns out to be negligible in $d = 4, 5$ since η is small in that range of dimensions. We remark here that in the LPA' the anomalous dimension receives contributions only from the three-point function at $\varphi = 0$, i.e. from the trilinear coupling λ_3 , irrespectively of the number of LPA' couplings considered: we therefore expect no further corrections to (3.29) when the potential (2.13) is truncated at higher orders. Beyond the LPA' approximation scheme instead, the expression for the anomalous dimension would receive additional contributions, but this will not be considered in this work.

3.4. LPA' at fixed n

The flow equation (3.6) can be expressed exactly, in the lowest cases, using the Cayley–Hamilton theorem to compute $(k^2\mathbb{I} + \mathbb{V})^{-1}$. For $n = 1, 2, 3$ we find the following explicit forms

$$\partial_t V_k = k^{d+2} \frac{1}{k^2 + \text{tr} \mathbb{V}} \quad n = 1 \quad (3.30)$$

$$\partial_t V_k = k^{d+2} \frac{2k^2 + \text{tr} \mathbb{V}}{k^4 + k^2 \text{tr} \mathbb{V} + \det \mathbb{V}} \quad n = 2 \quad (3.31)$$

$$\partial_t V_k = k^{d+2} \frac{3k^4 + 2k^2 \text{tr} \mathbb{V} - \frac{1}{2} (\text{tr}(\mathbb{V}^2) - (\text{tr} \mathbb{V})^2)}{k^6 + k^4 \text{tr} \mathbb{V} - \frac{1}{2} k^2 (\text{tr}(\mathbb{V}^2) - (\text{tr} \mathbb{V})^2) + \det \mathbb{V}} \quad n = 3. \quad (3.32)$$

Up to now these are just the flow equations for an n -component scalar φ_i since the symmetry has not been imposed yet. The information relative to the discrete symmetry S_{n+1} enters through the form of the invariants which the potential is function of. In general, the LPA' for S_{n+1} depends on n independent invariants, as can be seen explicitly for the $n = 1, 2, 3$ cases in table 2. In what follows any k dependence is intended.

In the `Ising` case $n = 1$ the only invariant is $\rho = \varphi_1^2$ so we define $U(\rho) \equiv V(\varphi_1)$. Since by change of variables, the second functional derivative of the effective potential w.r.t. φ_1 reads $V_{11} = 2U_\rho + 4\rho U_{\rho\rho}$, the LPA' (3.30) takes the well known form

$$\partial_t U = \frac{k^{d+2}}{k^2 + 2U_\rho + 4\rho U_{\rho\rho}}. \quad (3.33)$$

Standard `Ising` beta functions are retrieved once we consider, for example, the following φ^6 truncation

$$U(\rho) = \frac{1}{2} \bar{g}_2 \rho + \frac{1}{4!} \bar{g}_4 \rho^2 + \frac{1}{6!} \bar{g}_6 \rho^3.$$

Inserting this expression into equation (3.33), switching to dimensionless variables and comparing equal powers of the fields gives

$$\begin{aligned} \beta_2 &= (-2 + \eta) g_2 - \frac{g_4}{(1 + g_2)^2} \\ \beta_4 &= (d + 2\eta - 4) g_4 + \frac{6g_4^2}{(1 + g_2)^3} - \frac{g_6}{(1 + g_2)^2} \\ \beta_6 &= (2d + 3\eta - 6) g_6 - \frac{90g_4^3}{(1 + g_2)^4} + \frac{30g_6 g_4}{(1 + g_2)^3}. \end{aligned} \quad (3.34)$$

We can compare these beta functions against the general beta functions of section 3.2.2 when $n = 1$. Having in mind the result of table 2 we find the following mapping between the two representations

$$g_2 = \lambda_2 \quad g_4 = 2\lambda_{4,1} + \lambda_{4,2} \quad g_6 = 2\lambda_{6,1} + 2\lambda_{6,3} + \lambda_{6,4}.$$

If we now make the linear combinations of the general beta functions of section 3.2.2 according to this mapping and set $n = 1$, we will find that the rhs of these linear

combinations correctly become functions of only the couplings g_1, g_2, g_3 and exactly match (3.34). This is a non-trivial check of the correctness of our formalism.

In the same fashion we turn to the $n=2$ case where, accordingly, we have two invariants

$$\rho = \varphi_1^2 + \varphi_2^2 \quad \tau = \frac{3}{\sqrt{2}}\varphi_2(\varphi_2^2 - 3\varphi_1^2)$$

and the potential is a function of them $U(\rho, \tau) \equiv V(\varphi_1, \varphi_2)$. We can therefore express the flow equation for a two component scalar field in terms of the S_3 invariants (ρ, τ) . The derivatives needed to evaluate (3.31) are

$$\begin{aligned} V_{11} &= 2U_\rho + 4U_{\rho\rho}\varphi_1^2 - 9\varphi_2 \left(\sqrt{2}U_\tau + 2\varphi_1^2 \left(2\sqrt{2}U_{\rho\tau} - 9U_{\tau\tau}\varphi_2 \right) \right) \\ V_{22} &= 2U_\rho + \frac{81}{2}(\varphi_1^2 - \varphi_2^2)^2 U_{\tau\tau} + \varphi_2 \left(4\varphi_2 U_{\rho\rho} - 18\sqrt{2}(\varphi_1^2 - \varphi_2^2) U_{\rho\tau} + 9\sqrt{2}U_\tau \right) \end{aligned}$$

and

$$V_{12} = V_{21} = -\varphi_1\varphi_2 \left(81(\varphi_2^2 - \varphi_1^2) U_{\tau\tau} - 4U_{\rho\rho} \right) - 9\sqrt{2}\varphi_1(\varphi_1^2 + \varphi_2^2) U_{\rho\tau} + 9\sqrt{2}U_\tau.$$

Note that these relations are not yet expressed solely in terms of the invariants. Only the terms $\text{tr } \mathbb{V}$ and $\det \mathbb{V}$ turn out to be entirely expressible in terms of ρ and τ

$$\text{tr } \mathbb{V} = V_{11} + V_{22} = \frac{81\rho^2 U_{\tau\tau}}{2} + 4\rho U_{\rho\rho} + 4U_\rho + 12\tau U_{\rho\tau},$$

$$\begin{aligned} \det \mathbb{V} &= V_{11}V_{22} - (V_{12})^2 \\ &= 4U_\rho^2 + U_\rho \left(81\rho^2 U_{\tau\tau} + 8\rho U_{\rho\rho} + 24\tau U_{\rho\tau} \right) - 6U_{\rho\rho} \left(3(2\tau^2 - 9\rho^3) U_{\tau\tau} + 4\tau U_\tau \right) \\ &\quad - 18(9\rho^3 - 2\tau^2) U_{\rho\tau}^2 - 324\rho^2 U_{\rho\tau} U_\tau - 81\rho U_\tau (3\tau U_{\tau\tau} + 2U_\tau). \end{aligned}$$

We can now insert these expressions into the flow equation (3.31) for $V(\varphi_1, \varphi_2)$ to get the flow equation for $U(\rho, \tau)$

$$\begin{aligned} \partial_t U_k &= \left\{ 2k^2 + 4U_\rho + 4\rho U_{\rho\rho} + 12\tau U_{\rho\tau} + \frac{81}{2}\rho^2 U_{\tau\tau} \right\} \left\{ k^4 + k^2 \left(4U_\rho + 4\rho U_{\rho\rho} + 12\tau U_{\rho\tau} + \frac{81}{2}\rho^2 U_{\tau\tau} \right) \right. \\ &\quad + 4U_\rho^2 + U_\rho (8\rho U_{\rho\rho} + 24\tau U_{\rho\tau} + 81\rho^2 U_{\tau\tau}) - 18(9\rho^3 - 2\tau^2) (U_{\rho\tau}^2 - U_{\tau\tau} U_{\rho\rho}) \\ &\quad \left. - 162\rho U_\tau^2 - 3U_\tau (8\tau U_{\rho\rho} + 108\rho^2 U_{\rho\tau} + 81\rho\tau U_{\tau\tau}) \right\}^{-1}. \end{aligned} \quad (3.35)$$

This is the explicit form of the $n=2$ LPA'. As an application we can take advantage of the flow equation in this form to extract the beta functions of the corresponding S_3 -symmetric potential. Considering a φ^6 truncation

$$U(\rho, \tau) = \frac{1}{2!}\bar{g}_2\rho + \frac{1}{3!}\bar{g}_3\tau + \frac{1}{4!}\bar{g}_4\rho^2 + \frac{1}{5!}\bar{g}_5\rho\tau + \frac{1}{6!}(\bar{g}_{6,1}\rho^3 + \bar{g}_{6,2}\tau^2),$$

leads to the following dimensionless beta functions

$$\begin{aligned}
\beta_2 &= (-2 + \eta)g_2 + \frac{18g_3^2}{(1+g_2)^3} - \frac{4g_4}{3(1+g_2)^2} \\
\beta_3 &= \frac{1}{2}(d + 3\eta - 6)g_3 + \frac{4g_3g_4}{(1+g_2)^3} - \frac{4g_5}{5(1+g_2)^2} \\
\beta_4 &= (d + 2\eta - 4)g_4 + \frac{972g_3^4}{(1+g_2)^5} - \frac{216g_3^2g_4}{(1+g_2)^4} + \frac{648g_3g_5 + 100g_4^2}{15(1+g_2)^3} \\
&\quad - \frac{12g_{6,1} + 27g_{6,2}}{10(1+g_2)^2} \\
\beta_5 &= \frac{1}{2}(3d + 5\eta - 10)g_5 + \frac{720g_3^3g_4}{(1+g_2)^5} - \frac{216g_3^2g_5}{(1+g_2)^4} \\
&\quad + \frac{56g_4g_5 - 240g_3g_4^2 + 24g_3g_{6,1} + 189g_3g_{6,2}}{3(1+g_2)^3} \\
\beta_{6,1} &= (2d + 3\eta - 6)g_{6,1} + \frac{131220g_3^6}{(1+g_2)^7} - \frac{48600g_3^4g_4}{(1+g_2)^6} + \frac{4680g_3^2g_4^2 + 11664g_3^3g_5}{(1+g_2)^5} \\
&\quad - \frac{729g_3^2(4g_{6,1} + 9g_{6,2}) - 560g_4^3 - 7776g_3g_4g_5}{6(1+g_2)^4} \\
&\quad + \frac{5g_4(32g_{6,1} + 27g_{6,2}) + 324g_5^2}{5(1+g_2)^3} \\
\beta_{6,2} &= (2d + 3\eta - 6)g_{6,2} + \frac{160g_3^2g_4^2}{(1+g_2)^5} - \frac{96g_3g_5g_4}{(1+g_2)^4} + \frac{100g_4g_{6,2} + 32g_5^2}{5(1+g_2)^3}.
\end{aligned} \tag{3.36}$$

We can compare this LPA' result against the general beta functions of section 3.2.2, now evaluated for $n = 2$. Having in mind the results of table 2 in the case $n = 2$, we get the following mapping between couplings

$$\begin{aligned}
g_2 &= \lambda_2 & g_3 &= \lambda_3 & g_4 &= \frac{9}{2}\lambda_{4,1} + \lambda_{4,2} \\
g_5 &= \frac{5}{2}\lambda_{5,1} + \lambda_{5,2} & g_{6,1} &= \frac{27}{4}\lambda_{6,1} + \frac{9}{2}\lambda_{6,3} + \lambda_{6,4} & g_{6,2} &= \frac{1}{3}\lambda_{6,1} + \lambda_{6,2}.
\end{aligned}$$

Again, matching the beta functions of section 3.2.2 for $n = 2$ with those of (3.36) according to this mapping works out perfectly, giving us a further strong confirmation of our results.

Finally, it is clear that for any integer n it is possible to find corresponding expressions for the relative LPA', but these flow equations become extremely complicated as n increases.

4. Applications

4.1. Cubic interaction

As a first application of the FRG formalism to S_{n+1} -symmetric theories, we consider the simplest case where only a mass term and the trilinear coupling are present. Within

this two couplings truncation we can perform a complete analytical analysis for arbitrary n and d .

Referring to the general system of dimensionless beta functions of section 3.2.2 we switch off all couplings but λ_2 and λ_3 so that the corresponding dimensionless beta functions (3.15) and (3.16) become

$$\begin{aligned}\beta_2 &= -2\lambda_2 + \eta\lambda_2 + 2\frac{(n-1)(n+1)^2}{(1+\lambda_2)^3}\lambda_3^2 \\ \beta_3 &= \frac{1}{2}\lambda_3(d-6+3\eta) - 6\frac{(n-2)(n+1)^2}{(1+\lambda_2)^4}\lambda_3^3.\end{aligned}$$

Fixed point solutions $\beta_i = 0$ give a snapshot of the critical behavior of such theories in the corresponding theory space. Apart the trivial Gaussian fixed point ($\lambda_2^* = 0$, $\lambda_3^* = 0$) we find the following non-trivial fixed point

$$\lambda_2^* = -\frac{(d-6)(n-1)}{d(n-1)-18n+30} \quad \lambda_3^* = \pm \frac{24\sqrt{3}\sqrt{(d-6)(n-2)(n-2)}}{\sqrt{(n+1)^2(d(n-1)-18n+30)^4}}, \quad (4.1)$$

which, as expected, shrinks to the Gaussian one in $d=6$. Note that in $d < 6$ and $n > 2$ the fixed point is imaginary and as a consequence the theory is non-unitary, as in the case of the Lee-Yang universality class which also has $d_c = 6$ [1–3]. But for $n < 2$ the fixed point is real and so is the corresponding Landau–Ginzburg action.

We can now linearize the RG flow around this non-trivial fixed point to acquire a qualitative understanding of the flow and to extract the critical exponents. The stability matrix $M_{ij} = \left. \frac{\partial \beta_i}{\partial \lambda_j} \right|_*$ around the fixed point has the following components,

$$\begin{aligned}M_{11} &= -\frac{(d(n-1)-8)(12(n-3)+d(n-1))}{(3n-7)(d-(d-24)n-48)} \\ M_{12} &= \frac{2(6-d)^{3/2}(d-(d-24)n-48)}{3(d(n-1)-12n+20)\sqrt{21-9n(n+1)}} \\ M_{21} &= \frac{24\sqrt{3}(d(n-1)-12n+20)\sqrt{(d-6)(3n-7)}(n^2-1)}{(d-(d-24)n-48)^2} \\ M_{22} &= 6-d,\end{aligned}$$

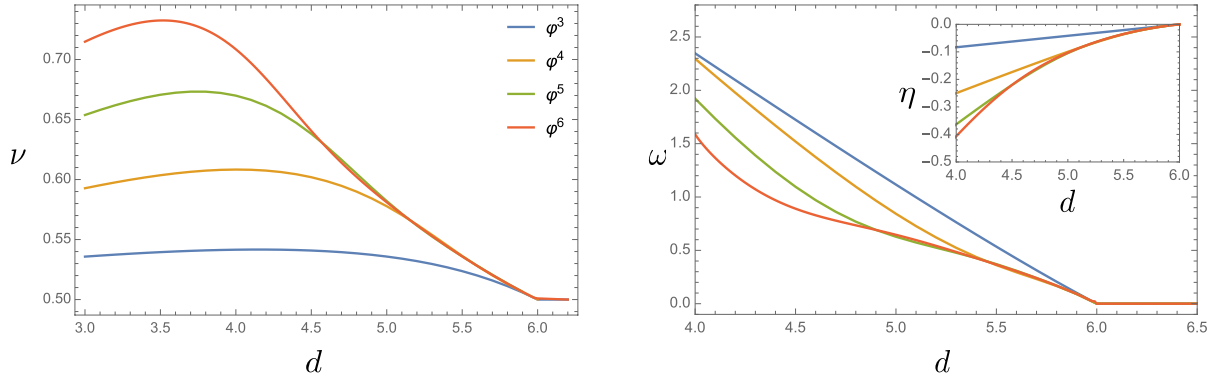
with corresponding eigenvalues

$$\theta_{\pm} = \frac{1}{2} \left(M_{11} + M_{22} - \sqrt{M_{11}^2 - 2M_{22}M_{11} + M_{22}^2 + 4M_{12}M_{21}} \right).$$

At this point one can obtain analytical expressions for the critical exponents simply considering that the correlation length critical exponent ν is related to the inverse of the negative eigenvalue $\nu = -\theta_-^{-1}$ while the correction-to-scaling critical exponent ω can be obtained as the first positive eigenvalue of the stability matrix which in this case is $\omega = \theta_+$. The anomalous dimension instead is obtained once the fixed point values (4.1) are substituted into equation (3.29), obtaining

$$\eta = \frac{(d-6)(n-1)}{3(7-3n)}. \quad (4.2)$$

Percolation



Spanning Forest

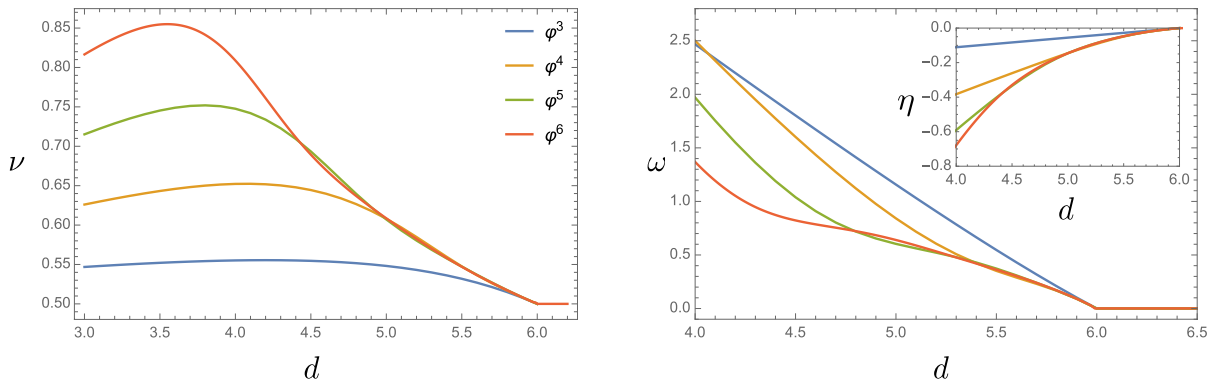


Figure 3. Percolation (upper plots) and Spanning Forest (lower plots) critical exponents ν (left), ω (right) and η (inset) as a function of d for increasing order of the truncation. For both universality classes, convergence is evident down to $d = 5$ while in $d = 4$ it is close. In lower dimensions, as in $d = 3$, convergence is expected only at higher orders in the truncation.

The critical exponents obtained in this approximation furnish the first curves reported in figure 3 in the next subsection.

We conclude this section specialising the analysis to $d = 6 - \epsilon$ dimensions to make contact with the ϵ -expansion. Expanding to first order in ϵ our results we find

$$\nu = \frac{1}{2} + \frac{5(n-1)}{12(3n-7)}\epsilon + O(\epsilon^2) \quad \eta = \frac{n-1}{3(7-3n)}\epsilon + O(\epsilon^2) \quad (4.3)$$

and $\omega = \epsilon + O(\epsilon^2)$ independently of n [34]. These relations are in agreement with [35] after a brief manipulation, furnishing a further check of our formalism. Notice that setting $n = 1$ here gives mean field exponents consistently with the fact that Ising has upper critical dimension $d_c = 4$. The explicit results for Percolation are [36, 51]

$$\nu = \frac{1}{2} + \frac{5}{84}\epsilon + O(\epsilon^2) \quad \eta = -\frac{\epsilon}{21} + O(\epsilon^2), \quad (4.4)$$

while for Spanning Forest we find [52]

Table 3. Correlation length critical exponent ν for Percolation ($n=0$) and Spanning Forest ($n=-1$) in $d=5,4,3$. Estimates obtained in the various truncations considered are presented and convergent digits are denoted in bold. Comparison is made with available Monte Carlo simulations or re-summed high order ϵ -expansion estimates.

Dim	n	φ^3	φ^4	φ^5	φ^6	Best	References
5	-1	0.5481	0.6059	0.6071	0.6085	0.59	[52]
	0	0.5358	0.5777	0.5820	0.5810	0.575	[53]
4	-1	0.5551	0.6492	0.7476	0.8087	0.80	[52]
	0	0.5415	0.6083	0.6698	0.7084	0.692	[53]
3	-1	0.5468	0.6238	0.7151	0.8170	1.28	[52]
	0	0.5357	0.5927	0.6537	0.7148	0.897	[53]

Table 4. Critical exponent ω for Percolation ($n=0$) and Spanning Forest ($n=-1$) in $d=5, 4$. Estimates obtained in the various truncations considered are presented and convergent digits (in the LPA') are denoted in bold. Comparison is made with available re-summed high order ϵ -expansion estimates in the $n=0$ case. No estimates have been found in the literature for $n=-1$.

Dim	n	φ^3	φ^4	φ^5	φ^6	Best	References
5	-1	1.157	0.8413	0.603	0.639	0.718	[53]
	0	1.116	0.842	0.627	0.645		
4	-1	2.466	2.500	1.969	1.368	1.2198	[53]
	0	2.346	2.295	1.922	1.587		

$$\nu = \frac{1}{2} + \frac{1}{12}\epsilon + O(\epsilon^2) \quad \eta = -\frac{\epsilon}{15} + O(\epsilon^2). \quad (4.5)$$

Note that for $n \geq 2$ we find $\nu < \nu_{\text{MF}}$, even if one expects these values to be meaningless since in $d \geq 3$ the phase transition is first order. Finally, we remark that the Ising ϵ -expansion can also be correctly recovered if one expands around $d = 4 - \epsilon$ the beta functions (3.34).

4.2. General analysis

The full set of beta functions presented in section 3.2 can be studied only numerically. As already explained, we focus on the non-trivial universality classes in $d > 2$ which, apart Ising, are Percolation and Spanning Forest. The numerical analysis proceeds in steps: for all values $3 \leq p \leq 6$ we first solve numerically the algebraic system $\beta_{i,m} = 0$ with $i = 2, \dots, p$ and $m = 1, \dots, N(p)$ to extract the fixed point coordinates; we compute the stability matrix $M_{ij} = \frac{\partial \beta_i}{\partial \lambda_j}$ symbolically and then we evaluate it at the numerical fixed point; finally we extract the eigenvalues. At each truncation order we find just one negative eigenvalue θ_- from which we can extract the correlation length critical exponent $\nu = -\theta_-^{-1}$; the first positive one θ_+ gives instead the correction to scaling exponent $\omega = \theta_+$; the anomalous dimension is computed from (3.29). At every order of the truncation p , we repeated the procedure for any $3 \leq d \leq 6$; the results for

Table 5. Anomalous dimension η for Percolation ($n=0$) and Spanning Forest ($n=-1$) in $d=5,4$. Estimates obtained in the various truncations considered are presented and convergent digits (in the LPA') are denoted in bold. Comparison is made with available Monte Carlo simulations or re-summed high order ϵ -expansion. As generally happens with LPA' truncations, the estimates for the anomalous dimension are much poorer than those for ν or ω .

Dim	n	φ^3	φ^4	φ^5	φ^6	Best	References
5	-1	-0.055	-0.145	-0.145	-0.1458	-0.08	[52]
	0	-0.041	-0.099	-0.104	-0.102	-0.0565	[53]
4	-1	-0.110	-0.382	-0.590	-0.678	-0.16	[52]
	0	-0.0833	-0.250	-0.363	-0.406	-0.0954	[53]

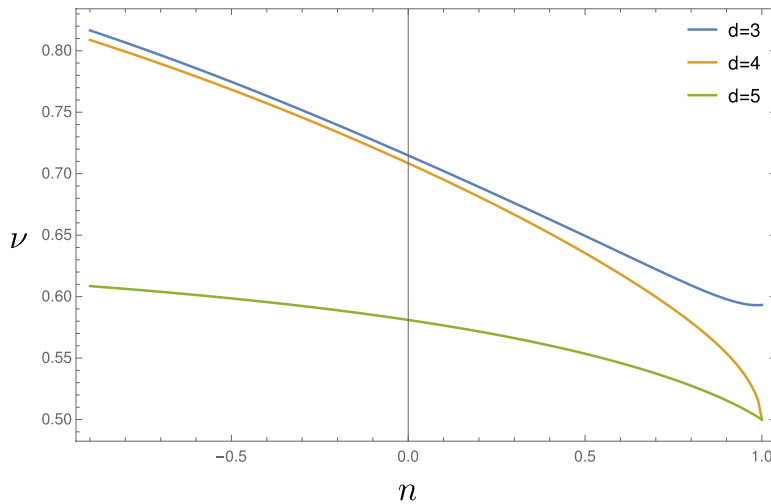


Figure 4. The correlation length critical exponent ν as a function of n for $d=3, 4, 5$ in the φ^6 truncation. Curves interpolate smoothly between Spanning Forest and Ising: in $d=4, 5$ the critical exponent converges to $\nu = \frac{1}{2}$ for $n=1$ since Ising is mean field; in $d=3$ instead the curve approaches $\nu = 0.593$ which is the Ising value at order φ^6 in the LPA'.

the critical exponents in function of the dimension $\nu(d), \omega(d), \eta(d)$ for orders considered $p=3, 4, 5, 6$ are shown in figure 3 for the relevant Percolation and Spanning Forest cases.

It is immediately clear from the plots that as the order of the truncation p increases the curves for the critical exponents converge non-uniformly: for values d closer to the upper critical dimension $d_c = 6$ few orders suffice to obtain a stable estimate. For ν in $d=5$ already $p=4$ returns good estimates; in $d=4$ instead the maximum order available $p=6$ is barely enough; while in $d=3$ convergence is still far from being reached and an improvement of the truncation is needed⁷. In table 3 we report the precise numerical values for the correlation length critical exponent. For ω the convergence is, as expected, slower than for ν and the numerical estimates are reported in table 4, while the results for the anomalous dimension in table 5. The estimates for ν are quite satisfactory and

⁷ As a first guess one can interpolate the curves using the converging parts; the result is consistent with the estimates for ν even in $d=3$. This gives us a reason to expect that larger truncations can deal also with this case.

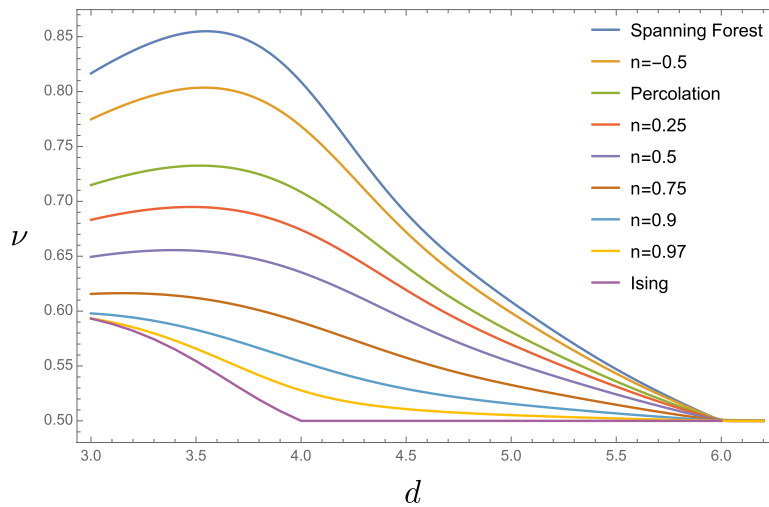


Figure 5. Critical exponent ν as a function of d for various values of n in the φ^6 truncation. As n increases, the critical exponent in the $d > 4$ region flattens toward the $n = 1$ mean field value $\nu = \frac{1}{2}$ (Ising).

testify the success of polynomial truncations in the determination of this exponent. ω requires an improvement of the truncation while η is poor, this being a general trend in LPA' like expansions. In particular, it is difficult to see how the anomalous dimension can change sign in lower dimensions, as expected from $d = 3$ estimates and the exact results in $d = 2$, thus questioning the validity of (3.29) below four dimensions.

In figures 4 and 5 we explore the n -dependence of the correlation length critical exponent. In the first figure we plot $\nu(n)$ for $d = 3, 4, 5$ in the range $-1 \leq n \leq 1$; in the second picture we plot $\nu(d)$ for values of n between $-1 \leq n \leq 1$ to see how the shift of upper critical dimension from $d_c = 6$ to $d_c = 4$ when n approaches one.

As shown in section 2, the number of couplings grows rapidly with the order p of the truncation and even if the general analysis just presented indicate that larger truncations have the strength to fully determine the spectrum down to three dimensions, an explicit derivation of beta functions beyond those reported in section 3.2 demands a significant amount of further work. Finally, while in $d = 5$ a study of scheme dependence is possible already at $p = 6$ since convergence has been achieved, we postpone this study to a future work when also convergence in the other two physically relevant dimensions $d = 3, 4$ is obtained.

5. Conclusion and outlook

The main goal of the present paper, after an initial discussion of the Potts field theory, with particular attention to the construction and enumeration of (non-derivative) invariants, has been the adaptation of functional RG (FRG) methods to the field theory of an n -component scalar with the underlying symmetry of the Potts model: global S_{n+1} .

Our main result has been the development of an algorithm able to compute the beta functions for the couplings of potential interactions, which we then used to perform explicit computations up to order φ^6 , thus obtaining a system of coupled ODE

describing the RG flow of the ten couplings present at this order, for arbitrary d and n . The main technical difficulty in the application of FRG methods to S_{n+1} -symmetry consisted in a systematic enumeration and construction of the invariants and in the development of the necessary trace machinery to reduce the traces present in the expansion of the rhs of the flow equation, unlocking in this way the access to the beta functions.

While it is not possible to obtain a closed equation for the effective potential for arbitrary n , it is instead possible to do so for specific integer values. We derived the improved local potential approximation (LPA') explicitly for the $n=1,2$ cases and explained how to do it for arbitrary positive integer n . A characterising property of the S_{n+1} -symmetry is that the LPA' is a PDE of $(n+1)$ -variables, since n independent invariants can be built out of the field multiplet even without introducing derivatives. In this respect the generalization of a single component \mathbb{Z}_2 -scalar to a multi-component scalar is more involved in the S_{n+1} case than, for example, in the $O(n)$ case.

As a first application, after a simple study of a two coupling truncation comprising mass and φ^3 -coupling, which also allowed us to recover the leading order ϵ -expansion, we obtained estimates for the critical exponents for the Percolation and Spanning Forest universality classes in dimension $d=4,5$ (where a preliminary study of convergence was possible) and made a first analysis in $d=3$. Our numerical estimates for ν and ω turned out to be in quite satisfactory agreement with Monte Carlo simulations and high order ϵ -expansion results, showing that FRG methods are indeed very effective also in the case of S_{n+1} -symmetry.

Apart for these first results and applications, our study serves as groundwork for future enquires of the Potts_{n+1} universality classes. Several questions demand further study or remain unanswered, the main of which are:

1. Study larger truncations to achieve full convergence in $d=3,4$ and study regulator dependence to extract the best estimates for the critical exponents.
2. We recall the problem of the sign change in η from a negative value in $d=6-\epsilon$ to a positive one in $d=2$. To overcome the problem, and to obtain better estimates for η , one must probably consider truncations that go beyond non-derivative interactions (i.e. beyond LPA').
3. One needs to study the two variable PDE encoding the flow in the LPA' for the three states Potts model ($n=2$) to understand for which value of d the phase transitions becomes first order. This value is expected to lie between two and three [21]. A similar study could be made for the three variable PDE pertaining to the $n=3$ case.
4. Much less is known about possible multi-critical phases of models with S_{n+1} -symmetry in any dimension greater than two, and for which more understanding is needed. An analysis of these phases can be done with LPA' scaling solutions following [1, 4–6].
5. Obtain the LPA' for continuous values of n in order to enable a fully functional analysis of the physically interesting limits $n \rightarrow 0$ and $n \rightarrow -1$. A large- n limit will also become available.

As stated in the introduction, this paper is the first of a series devoted to the study of universality classes characterized by discrete global symmetries in arbitrary dimension [54]. We will also be interested in applying the complementary methods of functional perturbative RG [2] and CFT + SDE [3] to Potts field theories in order to obtain their spectrum and OPE coefficients in the ϵ -expansion. Multi-critical models with S_{n+1} -symmetry can also be explored along the lines of the relative one component Lee–Yang family [55] and may represent non-trivial interacting theories $d = 3$. Another line of future developments is to study long range interactions [56] with Potts symmetry, as recently done for Percolation in [57]. Otherwise one can study systems out of equilibrium [58, 59] generalizing the work of [60] or consider the presence of boundaries [61].

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