

Strauss' and Lions' Type Results in $BV(\mathbb{R}^N)$ with an Application to an 1-Laplacian Problem

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Abstract. In this work we state and prove versions of some classical results, in the framework of functionals defined in the space of functions of bounded variation in \mathbb{R}^N . More precisely, we present versions of the Radial Lemma of Strauss, the compactness of the embeddings of the space of radially symmetric functions of $BV(\mathbb{R}^N)$ in some Lebesgue spaces and also a version of the Lions Lemma, proved in his celebrated paper of 1984. As an application, we get existence of a nontrivial bounded variation solution of a quasilinear elliptic problem involving the 1-Laplacian operator in \mathbb{R}^N , which has the lowest energy among all the radial ones. This seems to be one of the very first works dealing with stationary problems involving this operator in the whole space.

Mathematics Subject Classification (2010). Primary 35J62; Secondary 35J93.

Keywords. Bounded variation functions, 1-Laplacian operator, compactness with symmetry.

1. Introduction and some abstract results

When dealing with semilinear elliptic equations in \mathbb{R}^N , the lack of compactness is a problem to be considered. In general, what people are used to do is to impose some symmetry on the problem in order to recover the compactness of the embeddings of the Sobolev space into Lebesgue spaces. In this procedure at least two results are absolutely essential: a kind of Strauss Radial Lemma and a version of the Symmetric Criticality Principle of Palais.

Another very useful tool, mainly when symmetry is broken, is the very known Lions' Lemma, which has been settled by Lions in the celebrated paper [18] and widely used since then.

Giovary M. Figueiredo was partially supported by FAPESP and CNPq, Brazil. Marcos T.O. Pimenta was supported by FAPESP 2017/01756-2 and CNPq 442520/2014-0, Brazil.

As regards quasilinear problems, depending on some features of the differential operator to be considered, it can be necessary to deal with it in the space of functions of bounded variation, $BV(\mathbb{R}^N)$. This is the case when dealing with the mean-curvature operator or with the 1-Laplacian operator, a highly singular version of the usual p -Laplacian operator with $p = 1$. However, the space $BV(\mathbb{R}^N)$, which is going to be precisely defined later on, hasn't a crucial property that the most part of the Sobolev spaces has, the reflexivity. Indeed, the dual of $BV(\mathbb{R}^N)$ is not well known yet. This lack of reflexivity becomes a very difficult task to find critical points of functionals defined in this space and, as a consequence, we can see few works dealing with elliptic problems in \mathbb{R}^N which are modeled in this space.

In this work, to study a quasilinear elliptic problem involving the 1-Laplacian operator in \mathbb{R}^N , in order to deal with the lack of compactness of the embeddings of $BV(\mathbb{R}^N)$ into Lebesgue spaces, we state and prove versions in $BV(\mathbb{R}^N)$ of some classical results, like the Radial Lemma of Strauss and the compactness of embeddings of a subspace of $BV(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$. To be more precise, we prove that $BV_{\text{rad}}(\mathbb{R}^N)$, is compactly embedded into $L^q(\mathbb{R}^N)$, for $1 < q < 1^*$, where $1^* = \frac{N}{N-1}$. The precise statement of such an embedding theorem is the following.

Theorem 1.1. *Let $BV_{\text{rad}}(\mathbb{R}^N) = \{u \in BV(\mathbb{R}^N); u(x) = u(|x|)\}$. Then the embedding below is compact:*

$$BV_{\text{rad}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \quad \text{for } 1 < q < 1^*.$$

In its proof, it is necessary to state and prove a version of the Strauss Radial Lemma (see [22]) to the space $BV(\mathbb{R}^N)$.

As an application of our compactness result, we study the following quasilinear problem:

$$\begin{cases} -\Delta_1 u + \frac{u}{|u|} = f(u) & \text{in } \mathbb{R}^N, \\ u \in BV(\mathbb{R}^N), \end{cases} \quad (1)$$

where the 1-Laplacian operator is formally defined by $\Delta_1 u := \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right)$ and the nonlinearity f satisfies the following set of assumptions:

- (f₁) $f \in C(\mathbb{R})$;
- (f₂) $f(s) = o(1)$ as $s \rightarrow 0$;
- (f₃) There exist constants $c_1, c_2 > 0$ and $p \in (1, 1^*)$ such that

$$|f(s)| \leq c_1 + c_2 |s|^{p-1};$$

- (f₄) There exists $\theta > 1$ such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for } s \neq 0,$$

where $F(s) = \int_0^s f(t)dt$;

- (f₅) f is increasing.

In fact we prove the following result, which states the existence of a nontrivial solution of (1).

Theorem 1.2. *Suppose that f satisfies the conditions (f_1) – (f_5) . Then there exists a nontrivial solution $u \in BV(\mathbb{R}^N)$ of (1). Moreover, this solution has the lowest energy level among all the radial ones.*

Our approach to prove Theorem 1.2 is variational and, in spite of the most part of works dealing with the 1-Laplacian operator, we work in $BV(\mathbb{R}^N)$ itself, rather than extending the energy functional to some Lebesgue space. With this in mind, we have to overcome the lack of the Palais-Smale condition that this choice brings.

In the last years an increasing number of researchers have dedicated their efforts studying problems involving the 1-Laplacian operator. A version of Brézis-Nirenberg problem to 1-Laplacian has been studied in [9] by Degiovanni and Magrone, where they use a nonstandard linking structure in order to get solutions of the problem. In [19], León and Webler study a parabolic problem involving the 1-Laplacian operator and succeed in proving global existence and uniqueness for source and initial data in some adequate space. In [13], the authors seems to be the pioneers in using Nehari types arguments in order to get bounded variation solutions for problems involving the mean-curvature or the 1-Laplacian operators. In [14] it is studied a problem related to (1), but in the presence of vanishing potentials, where it is proved the existence of a nontrivial ground-state solution. In [1] the authors show the existence and concentration of a sequence of solutions of a singularly perturbed version of (1), with a potential satisfying geometrical conditions like in the celebrated paper [20].

Note that (1) is not well defined in every $x \in \mathbb{R}^N$ such that $\nabla u(x) = 0$ or $u(x) = 0$. In fact (1) is just a formal version of the precise Euler-Lagrange equation associated to our problem. More precisely, in Section 3 we are going to prove that a critical point $u \in BV(\mathbb{R}^N)$ of the energy functional associated to the problem, in fact satisfy the following version of (1)

$$\begin{cases} \exists z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N), \|z\|_\infty \leq 1, \operatorname{div} z \in L^N(\mathbb{R}^N), -\int_{\mathbb{R}^N} u \operatorname{div} z dx = \int_{\mathbb{R}^N} |Du|, \\ \exists z_2^* \in L^\infty(\mathbb{R}^N), z_2^* |u| = u \quad \text{a.e. in } \mathbb{R}^N, \\ -\operatorname{div} z + z_2^* = f(u), \quad \text{a.e. in } \mathbb{R}^N. \end{cases}$$

By using a variational approach, we have to deal with an Euler-Lagrange functional which is not smooth, although locally Lipschitz. In fact the sense of solution we consider here has to take into account the concept of generalized gradient developed by Clarke (see [7, 6]). More precisely, the Euler-Lagrange functional of (1) is modeled in a subspace of $BV(\mathbb{R}^N)$ and is given by

$$\Phi(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx - \int_{\mathbb{R}^N} F(u) dx,$$

where Du is the distributional derivative of u , which in turn is a Radon measure. As can be seen in Section 5, we say that $u \in BV(\mathbb{R}^N)$ is a bounded variation solution of (1) if

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \mathcal{F}'(u)(v - u), \quad (2)$$

for all $v \in BV(\mathbb{R}^N)$, where $\mathcal{J}, \mathcal{F} : BV(\mathbb{R}^N) \rightarrow \mathbb{R}$ is given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx.$$

and

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(u) dx.$$

In the end of the proof of Theorem 1.2, in order to assure that the critical point of Φ restricted to $BV_{\text{rad}}(\mathbb{R}^N)$ is a critical point of Φ in all of $BV(\mathbb{R}^N)$, we need a version of the Symmetric Criticality Principle of Palais for non-smooth functionals defined in possibly non-reflexive Banach spaces. This is provided by Squassina in [21] and is used in the end of Section 5.

To finish, we succeed in proving a version of the Lions' Lemma (see [18][Lemma I.1]) to $BV(\mathbb{R}^N)$. Although this result has not been used to prove our main result, we believe it has interest by its own, since it is a classical and largely used tool in the analysis of second order semilinear and quasilinear elliptic problems.

Theorem 1.3 (Lions' Lemma in $BV(\mathbb{R}^N)$). *Suppose there exist $R > 0$, $1 \leq q < 1^*$, and a bounded sequence (u_n) in $BV(\mathbb{R}^N)$ such that*

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^q dx \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then $u_n \rightarrow 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (1, 1^)$.*

The paper is organized as follows. In Section 2 we present a preliminary explanation about the space $BV(\mathbb{R}^N)$. In Section 3 we present the precise form of the Euler–Lagrange equation associated to the functional Φ . In Section 4 we prove the abstract results. In Section 5 we present an application of the abstract results in order to get solutions to a quasilinear elliptic problem involving the 1–Laplacian operator.

The authors would like to warmly thank Prof. Alexandru Kristály for some discussions about the Symmetric Criticality Principle of Palais. This work was written while Giovany M. Figueiredo was as a Visiting Professor at FCT - Unesp in Presidente Prudente - SP. He would like to thank the warm hospitality.

2. Preliminaries

First of all let us introduce the space of functions of bounded variation, $BV(\mathbb{R}^N)$. We say that $u \in BV(\mathbb{R}^N)$, or is a function of bounded variation, if $u \in L^1(\mathbb{R}^N)$, and its distributional derivative Du is a vectorial Radon measure, i.e.,

$$BV(\mathbb{R}^N) = \{u \in L^1(\mathbb{R}^N); \quad Du \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)\}.$$

It can be proved that $u \in BV(\mathbb{R}^N)$ if and only if $u \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |Du| := \sup \left\{ \int_{\mathbb{R}^N} u \operatorname{div} \phi \, dx; \quad \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N), \text{ s.t. } |\phi|_\infty \leq 1 \right\} < +\infty.$$

The space $BV(\mathbb{R}^N)$ is a Banach space when endowed with the norm

$$\|u\| := \int_{\mathbb{R}^N} |Du| + |u|_1,$$

which is continuously embedded into $L^r(\mathbb{R}^N)$ for all $r \in [1, 1^*]$.

As one can see in [3], the space $BV(\mathbb{R}^N)$ has different convergence and density properties than the usual Sobolev spaces. For example, $C_0^\infty(\mathbb{R}^N)$ is not dense in $BV(\mathbb{R}^N)$ with respect to the strong convergence, since the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm of $BV(\mathbb{R}^N)$ is equal to $W^{1,1}(\mathbb{R}^N)$, which is a proper subspace of $BV(\mathbb{R}^N)$. This has motivated people to define a weaker sense of convergence in $BV(\mathbb{R}^N)$, called *intermediate convergence*. We say that $(u_n) \subset BV(\mathbb{R}^N)$ converge to $u \in BV(\mathbb{R}^N)$ in the sense of the intermediate convergence if

$$u_n \rightarrow u, \quad \text{in } L^1(\mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^N} |Du_n| \rightarrow \int_{\mathbb{R}^N} |Du|,$$

as $n \rightarrow \infty$. Fortunately, with respect to the intermediate convergence, $C_0^\infty(\mathbb{R}^N)$ is dense in $BV(\mathbb{R}^N)$. This fact is going to be used later.

For a vectorial Radon measure $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by $\mu = \mu^a + \mu^s$ the usual decomposition stated in the Radon Nikodyn Theorem, where μ^a and μ^s are, respectively, the absolute continuous and the singular parts with respect to the N -dimensional Lebesgue measure \mathcal{L}^N . We denote by $|\mu|$, the absolute value of μ , the scalar Radon measure defined like in [3][pg. 125]. By $\frac{\mu}{|\mu|}(x)$ we denote the usual Lebesgue derivative of μ with respect to $|\mu|$, given by

$$\frac{\mu}{|\mu|}(x) = \lim_{r \rightarrow 0} \frac{\mu(B_r(x))}{|\mu|(B_r(x))}.$$

It easy to see that $\mathcal{J} : BV(\mathbb{R}^N) \rightarrow \mathbb{R}$, given by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx, \quad (3)$$

is a convex functional and Lipschitz continuous in its domain. It is also well know that \mathcal{J} is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$ topology, for $r \in [1, 1^*]$ (see [15] for example). Although non-smooth, the functional \mathcal{J} admits some directional derivatives. More specifically, as is shown in [2], given $u \in BV(\mathbb{R}^N)$, for all $v \in BV(\mathbb{R}^N)$ such that $(Dv)^s$ is absolutely continuous with respect to $(Du)^s$ and $v \equiv 0$, \mathcal{L}^N -a.e. in the set where u vanishes, it follows that

$$\mathcal{J}'(u)v = \int_{\mathbb{R}^N} \frac{(Du)^a(Dv)^a}{|(Du)^a|} dx + \int_{\mathbb{R}^N} \frac{Du}{|Du|}(x) \frac{Dv}{|Dv|}(x) |(Dv)|^s + \int_{\mathbb{R}^N} \text{sgn}(u)v dx, \quad (4)$$

where $\text{sgn}(u(x)) = 0$ if $u(x) = 0$ and $\text{sgn}(u(x)) = u(x)/|u(x)|$ if $u(x) \neq 0$. In particular, note that, for all $u \in BV(\mathbb{R}^N)$,

$$\mathcal{J}'(u)u = \mathcal{J}(u). \quad (5)$$

We have also that $BV(\mathbb{R}^N)$ is a *lattice*, i.e., if $u, v \in BV(\mathbb{R}^N)$, then $\max\{u, v\}$, $\min\{u, v\} \in BV(\mathbb{R}^N)$ and also

$$\mathcal{J}(\max\{u, v\}) + \mathcal{J}(\min\{u, v\}) \leq \mathcal{J}(u) + \mathcal{J}(v), \quad \forall u, v \in BV(\mathbb{R}^N). \quad (6)$$

3. The Euler–Lagrange equation

Since (1) contains expressions that doesn't make sense when $\nabla u = 0$ or $u = 0$, it can be understood just as the formal version of the Euler–Lagrange equation associated to the functional Φ . In this section we present the precise form of an Euler–Lagrange equation satisfied by all bounded variation critical points of Φ . In order to do so we closely follow the arguments of [16].

The first step is to consider the extension of the functionals \mathcal{J}, \mathcal{F} and Φ to $X = L^1(\mathbb{R}^N) \cap L^{1*}(\mathbb{R}^N)$ endowed with the norm $\|w\|_X = |w|_1 + |w|_{1*}$, given respectively by $\overline{\mathcal{J}}, \overline{\mathcal{F}}, \overline{\Phi} : X \rightarrow \mathbb{R} \cup \{+\infty\}$, where

$$\begin{aligned} \overline{\mathcal{J}}(v) &= \begin{cases} \mathcal{J}(v), & \text{if } v \in BV(\mathbb{R}^N), \\ +\infty, & \text{if } v \in X \setminus BV(\mathbb{R}^N), \end{cases} \\ \overline{\mathcal{F}}(u) &= \int_{\mathbb{R}^N} F(u) dx \end{aligned}$$

and $\overline{\Phi} = \overline{\mathcal{J}} - \overline{\mathcal{F}}$. It is easy to see that $\overline{\mathcal{F}}$ belongs to $C^1(X, \mathbb{R})$ and that $\overline{\mathcal{J}}$ is a convex lower semicontinuous functional defined in X . Hence the subdifferential (in the sense of [23]) of $\overline{\mathcal{J}}$, denoted by $\partial \overline{\mathcal{J}}$, is well defined. The following is a crucial result in obtaining an Euler–Lagrange equation satisfied by the critical points of Φ .

Lemma 3.1. *If $u \in BV(\mathbb{R}^N)$ is such that $0 \in \partial \Phi(u)$, then $0 \in \partial \overline{\Phi}(u)$.*

Proof. Suppose that $0 \in \partial \Phi(u)$, i.e., that u satisfies (2). We claim that

$$\overline{\mathcal{J}}(v) - \overline{\mathcal{J}}(u) \geq \overline{\mathcal{F}}'(u)(v - u), \quad \forall v \in X.$$

To see why, let $v \in X$ and note that:

- if $v \in BV(\mathbb{R}^N) \cap X$, then

$$\begin{aligned} \overline{\mathcal{J}}(v) - \overline{\mathcal{J}}(u) &= \mathcal{J}(v) - \mathcal{J}(u) \\ &\geq \mathcal{F}'(u)(v - u) \\ &= \int_{\mathbb{R}^N} f(u)(v - u) dx \\ &= \overline{\mathcal{F}}'(u)(v - u); \end{aligned}$$

- if $v \in X \setminus BV(\mathbb{R}^N)$, since $\overline{\mathcal{J}}(v) = +\infty$ and $\overline{\mathcal{J}}(u) < +\infty$, it follows that

$$\begin{aligned} \overline{\mathcal{J}}(v) - \overline{\mathcal{J}}(u) &= +\infty \\ &\geq \overline{\mathcal{F}}'(u)(v - u). \end{aligned}$$

Therefore the result follows. □

Let us assume that $u \in BV(\mathbb{R}^N)$ is a bounded variation solution of (1). Since $0 \in \partial\Phi(u)$, by the last result it follows that $0 \in \partial\overline{\Phi}(u)$. Since $\overline{\mathcal{F}}$ is convex and $\overline{\mathcal{F}}$ is smooth, it follows that $\overline{\mathcal{F}}'(u) \in \partial\overline{\mathcal{J}}(u)$. In what follows, we set $\overline{\mathcal{J}}^1, \overline{\mathcal{J}}^2 : X \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\overline{\mathcal{J}}^1(v) := \begin{cases} \int_{\mathbb{R}^N} |Dv|, & \text{if } v \in BV(\mathbb{R}^N), \\ +\infty, & \text{if } v \in X \setminus BV(\mathbb{R}^N), \end{cases}$$

and

$$\overline{\mathcal{J}}^2(v) := \int_{\mathbb{R}^N} |v| dx.$$

Note that $\overline{\mathcal{J}}^2 \in C(X, \mathbb{R})$, $\overline{\mathcal{J}}^2 \in C(BV(\mathbb{R}^N), \mathbb{R})$ and

$$\overline{\mathcal{J}}(v) = \overline{\mathcal{J}}^1(v) + \overline{\mathcal{J}}^2(v), \quad \forall v \in X.$$

Since $\overline{\mathcal{J}}^1$ and $\overline{\mathcal{J}}^2$ are convex, and $\overline{\mathcal{J}}^2$ is finite and continuous in every point of $BV(\mathbb{R}^N)$, it follows from [3, Theorem 9.5.4] that

$$\overline{\mathcal{F}}'(u) \in \partial\overline{\mathcal{J}}(u) = \partial\overline{\mathcal{J}}^1(u) + \partial\overline{\mathcal{J}}^2(u).$$

By using the same arguments explored in [5, Theorem 8.15], it follows that $X' \subset L_{\infty,N}(\mathbb{R}^N)$ where

$$L_{\infty,N}(\mathbb{R}^N) = \{g : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : |g|_{\infty,N} < \infty\}$$

where

$$|g|_{\infty,N} = \sup_{|\phi|_1 + |\phi|_{1^*} \leq 1} \left| \int_{\mathbb{R}^N} g\phi dx \right|.$$

It is possible to prove that $|\cdot|_{\infty,N}$ is a norm in $L_{\infty,N}(\mathbb{R}^N)$. Moreover, the inclusion $L_{\infty,N}(\mathbb{R}^N) \hookrightarrow L^N(B_R(0))$ is continuous for all $R > 0$.

From the above commentaries, there are $z_1^*, z_2^* \in L_{\infty,N}(\mathbb{R}^N)$ such that $z_1^* \in \partial\overline{\mathcal{J}}^1(u)$, $z_2^* \in \partial\overline{\mathcal{J}}^2(u)$ and

$$\overline{\mathcal{F}}'(u) = z_1^* + z_2^* \quad \text{in } L_{\infty,N}(\mathbb{R}^N).$$

Following the same arguments in [16, Proposition 4.23, pg. 529], it can be proved that there exists $z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that $|z|_\infty \leq 1$,

$$-\operatorname{div} z = z_1^* \quad \text{in } L_{\infty,N}(\mathbb{R}^N) \tag{7}$$

and

$$-\int_{\mathbb{R}^N} u \operatorname{div} z dx = \int_{\mathbb{R}^N} |Du|, \tag{8}$$

where the divergence in (7) has to be understood in the distributional sense. Moreover, the same result implies that z_2^* is such that

$$z_2^*|u| = u, \quad \text{a.e. in } \mathbb{R}^N. \tag{9}$$

Therefore, it follows from (7), (8) and (9) that u satisfies

$$\begin{cases} \exists z \in L^\infty(\mathbb{R}^N, \mathbb{R}^N), |z|_\infty \leq 1, \operatorname{div} z \in L_{\infty,N}(\mathbb{R}^N), \\ -\int_{\mathbb{R}^N} u \operatorname{div} z \, dx = \int_{\mathbb{R}^N} |Du|, \\ \exists z_2^* \in L_{\infty,N}(\mathbb{R}^N), z_2^*|u| = u \quad \text{a.e. in } \mathbb{R}^N, \\ -\operatorname{div} z + z_2^* = f(u), \quad \text{a.e. in } \mathbb{R}^N. \end{cases} \quad (10)$$

Hence, (10) is the precise version of (1).

4. Proof of the abstract results

In order to prove Theorem 1.1, we first state and prove a counterpart in the space of functions of bounded variation, of a very important result of Strauss (see [22]), with so many applications when dealing with radial functions in Sobolev spaces.

Lemma 4.1 (Radial Lemma in BV). *Let $u \in BV_{\text{rad}}(\mathbb{R}^N)$, then for almost every $x \in \mathbb{R}^N \setminus \{0\}$, it follows that*

$$|u(x)| \leq \frac{1}{|x|^{N-1}} \|u\|.$$

Proof. In this result we use the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $BV(\mathbb{R}^N)$ with respect to the intermediate convergence topology. This result is going to be justified in a remark immediately after this proof. For $u \in BV_{\text{rad}}(\mathbb{R}^N)$, let $(u_n) \subset C_0^\infty(\mathbb{R}^N)$ (which may be chosen radially symmetric) such that

$$u_n \rightarrow u \quad \text{in } L^1(\mathbb{R}^N) \quad (11)$$

and

$$\int_{\mathbb{R}^N} |\nabla u_n| \, dx \rightarrow \int_{\mathbb{R}^N} |Du|, \quad (12)$$

as $n \rightarrow \infty$. Denoting $v(x) = v(|x|) = v(r)$ whenever v is a radial function of \mathbb{R}^N , we have that

$$\frac{d}{d\rho} (\rho^{N-1} |u_n(\rho)|) = (N-1) \rho^{N-2} |u_n(\rho)| + \rho^{N-1} \frac{u_n}{|u_n|} u_n'(\rho) \quad \forall \rho > 0.$$

Integrating both sides over $(r, +\infty)$ we have that

$$\begin{aligned} \int_r^{+\infty} \frac{d}{d\rho} (\rho^{N-1} |u_n(\rho)|) \, d\rho &= \int_r^{+\infty} (N-1) \rho^{N-2} |u_n(\rho)| \, d\rho \\ &\quad + \int_r^{+\infty} \rho^{N-1} \frac{u_n}{|u_n|} u_n'(\rho) \, d\rho \\ &\geq \int_r^{+\infty} \rho^{N-1} \frac{u_n}{|u_n|} u_n'(\rho) \, d\rho. \end{aligned}$$

Since u_n has compact support, $\lim_{\rho \rightarrow +\infty} \rho^{N-1}|u_n(\rho)| = 0$ and then

$$\begin{aligned} r^{N-1}|u_n(r)| &\leq \int_r^{+\infty} |u'_n(\rho)| \rho^{N-1} d\rho \\ &\leq \int_{\mathbb{R}^N} |\nabla u_n| dx. \end{aligned}$$

Then we have that

$$|u_n(r)| \leq \frac{1}{r^{N-1}} \int_{\mathbb{R}^N} |\nabla u_n| dx. \quad (13)$$

Hence by (11), (12) and (13) it follows that

$$|u(r)| \leq \frac{1}{r^{N-1}} \int_{\mathbb{R}^N} |Du|, \quad \text{a.e. in } \mathbb{R}^N. \quad \square$$

Remark 1. The fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $BV(\mathbb{R}^N)$ w.r.t. the intermediate convergence topology can be verified by a two step approximation process. For $u \in BV(\mathbb{R}^N)$, consider φ a smooth function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_1(0)$ and $\varphi = 0$ in $B_2(0)^c$. Then consider $\varphi_n(x) := \varphi(x/n)$, which is a $C_0^\infty(\mathbb{R}^N)$ function. It can be easily verified that $\varphi_n u \rightarrow u$ in the intermediate convergence of $BV(\mathbb{R}^N)$. Now, for each $n \in \mathbb{N}$, since $\varphi_n u \in BV(B_{2n}(0))$, by [3][Theorem 10.1.2], there exists $(v_m^n)_{m \in \mathbb{N}} \subset C^\infty(B_{2n}(0))$ such that $v_m^n \rightarrow \varphi_n u$, as $m \rightarrow \infty$, in the intermediate convergence of $BV(B_{2n}(0))$. Then, a diagonal process proves the claim.

Now we present the proof of the compactness result.

Proof of Theorem 1.1. Let $(u_n) \subset BV_{\text{rad}}(\mathbb{R}^N)$ be a bounded sequence and let $C > 0$ be such that

$$\|u_n\| \leq C, \quad \forall n \in \mathbb{N}.$$

By Lemma 4.1 it follows that, for all $n \in \mathbb{N}$,

$$|u_n(x)| \leq \frac{C}{|x|^{N-1}} \quad \text{a.e. in } \mathbb{R}^N \setminus \{0\}.$$

Since $q > 1$, given $\epsilon > 0$, there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$|u_n(x)|^q \leq \frac{\epsilon}{2C} |u_n(x)| \quad \forall x \in B_R(0)^c.$$

This implies that

$$\int_{B_R(0)^c} |u_n|^q dx \leq \frac{\epsilon}{2C} \int_{B_R(0)^c} |u_n| dx \leq \frac{\epsilon}{2C} \|u_n\| \leq \frac{\epsilon}{2}, \quad (14)$$

for all $n \in \mathbb{N}$. Since (u_n) is bounded in $BV_{\text{rad}}(\mathbb{R}^N)$, there exists $u \in L_{\text{loc}}^q(\mathbb{R}^N)$ such that $u_n \rightarrow u$ in $L_{\text{loc}}^q(\mathbb{R}^N)$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^N (in particular u has radial symmetry). In particular, by the compactness of the embedding $BV(B_R(0)) \hookrightarrow L^q(B_R(0))$, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} |u_n - u|^q dx < \frac{\epsilon}{2}, \quad \forall n \geq n_0. \quad (15)$$

Now let us prove that $u \in BV_{\text{rad}}(\mathbb{R}^N)$. Note that for a given $r > 0$, from the semicontinuity of the norm in $BV(B_r(0))$ w.r.t. the $L^q(B_r(0))$ convergence, it follows that

$$\|u\|_{BV(B_r(0))} \leq \liminf_{n \rightarrow +\infty} \|u_n\|_{BV(B_r(0))} \leq \liminf_{n \rightarrow +\infty} \|u_n\| \leq C, \quad (16)$$

where C does not depend on n or r . Since the last inequality holds for every $r > 0$, then $u \in BV_{\text{rad}}(\mathbb{R}^N)$ and then also $u \in L^q(\mathbb{R}^N)$. Hence, by taking $R > 0$ in (14) also such that $\int_{B_R(0)^c} |u|^q dx < \frac{\epsilon}{2}$, it follows from (15) that, for all $n \geq n_0$,

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n - u|^q dx &= \int_{B_R(0)} |u_n - u|^q dx + \int_{B_R(0)^c} |u_n - u|^q dx \\ &< \frac{\epsilon}{2} + \int_{B_R(0)^c} 2^q (|u_n|^q + |u|^q) dx \\ &< C\epsilon. \end{aligned}$$

Then it is clear that $u_n \rightarrow u$ in $L^q(\mathbb{R}^N)$, as $n \rightarrow \infty$. \square

To end up this section, let us present the proof of the Lions' type result.

Proof of Theorem 1.3. Let $q < s < 1^*$ and $u \in BV(\mathbb{R}^N)$. Since $BV(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $1 \leq r \leq 1^*$, then $u \in L^q(\mathbb{R}^N)$ and $u \in L^{1^*}(\mathbb{R}^N)$.

For $R > 0$, by interpolation inequality with $\theta = \frac{s-q}{1^*-q} \frac{1^*}{s}$, it follows that $0 < \theta < 1$ and

$$\begin{aligned} |u|_{L^s(B_R(y))} &\leq |u|_{L^q(B_R(y))}^{1-\theta} |u|_{L^{1^*}(B_R(y))}^{\theta} \\ &\leq c |u|_{L^q(B_R(y))}^{1-\theta} \|u\|_{BV(B_R(y))}^{\theta}. \end{aligned}$$

Covering \mathbb{R}^N by balls of radius R and center in (y_n) in such a way that each point in \mathbb{R}^N belongs to at maximum $N + 1$ balls, we have that

$$\begin{aligned} &\int_{\mathbb{R}^N} |u|^s dx \\ &\leq \sum_{n=1}^{+\infty} \int_{B_R(y_n)} |u|^s dx \\ &\leq c \sum_{n=1}^{+\infty} |u|_{L^q(B_R(y_n))}^{(1-\theta)s} \|u\|_{BV(B_R(y_n))}^{\theta s} \\ &\leq c \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u|^q dx \right)^{\frac{(1-\theta)s}{q}} \lim_{k \rightarrow +\infty} \sum_{n=1}^k \left(\int_{B_R(y_n)} |Du| + \int_{B_R(y_n)} |u| dx \right)^{\theta s} \\ &= c \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u|^q dx \right)^{\frac{(1-\theta)s}{q}} \lim_{k \rightarrow +\infty} \sum_{n=1}^k \left(\int_{\mathbb{R}^N} \chi_{B_R(y_n)} |Du| + \int_{\mathbb{R}^N} \chi_{B_R(y_n)} |u| dx \right)^{\theta s} \end{aligned}$$

$$\begin{aligned}
&= c \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u|^q dx \right)^{\frac{(1-\theta)s}{q}} \lim_{k \rightarrow +\infty} \left(\int_{\mathbb{R}^N} \sum_{n=1}^k \chi_{B_R(y_n)} |Du| \right. \\
&\quad \left. + \int_{\mathbb{R}^N} \sum_{n=1}^k \chi_{B_R(y_n)} |u| dx \right)^{\theta s} \\
&\leq c \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u|^q dx \right)^{\frac{(1-\theta)s}{q}} (N+1) \|u\|^{\theta s}.
\end{aligned}$$

Then, since (u_n) is bounded in $BV(\mathbb{R}^N)$, by the last inequality and the hypothesis, it follows that

$$u_n \rightarrow 0, \quad \text{in } L^s(\mathbb{R}^N), \quad (17)$$

for all $q < s < 1^*$.

Then, if $q = 1$ we are done. Otherwise, if $1 < q < 1^*$, let us consider $1 < s \leq q$ and take $s_0 \in (q, 1^*)$ in such a way that (17) holds. Note that $u \in L^1(\mathbb{R}^N) \cap L^{s_0}(\mathbb{R}^N)$ and, since $s \in (1, s_0)$, by doing

$$\theta = \frac{s_0 - s}{s(s_0 - 1)}$$

we have that

$$\frac{1}{s} = \frac{\theta}{1} + \frac{1-\theta}{s_0} \quad \text{and} \quad 0 < \theta < 1.$$

Then, again by interpolation inequality, the embedding of $BV(\mathbb{R}^N)$ and (17), it follows that

$$|u_n|_s \leq |u_n|_1 |u_n|_{s_0} \leq \|u_n\| |u_n|_{s_0} \rightarrow 0$$

as $n \rightarrow \infty$, since (u_n) is bounded in $BV(\mathbb{R}^N)$. This completes the proof. \square

5. Application

In this section we present an application of Theorem 1.1 to the following problem

$$\begin{cases} -\Delta_1 u + \frac{u}{|u|} = f(u) & \text{in } \mathbb{R}^N, \\ u \in BV(\mathbb{R}^N), \end{cases} \quad (18)$$

where f satisfies the conditions (f_1) – (f_5) .

Since (18) is variational, let us define the energy functional associated to it, $\Phi : BV(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$\Phi(u) = \mathcal{J}(u) - \mathcal{F}(u), \quad (19)$$

where $\mathcal{F} : BV(\mathbb{R}^N) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} F(u) dx \quad (20)$$

and \mathcal{J} is given by (3).

It is straightforward to see that \mathcal{F} is a smooth functional. Moreover, by (4), $\mathcal{J}'(u)u = \mathcal{J}(u)$ for all $u \in BV(\mathbb{R}^N)$. Then, the directional derivatives $\Phi'(u)u$ exists and

$$\Phi'(u)u = \mathcal{J}(u) - \int_{\mathbb{R}^N} f(u)u \, dx. \quad (21)$$

Before we start to deal with this equation, let us make precise the sense of solution we are considering here. Since Φ can be written as the difference between a Lipschitz and a smooth functional in $BV(\mathbb{R}^N)$, we say that $u_0 \in BV(\mathbb{R}^N)$ is a solution of (18) if $0 \in \partial\Phi(u_0)$, where $\partial\Phi(u_0)$ denotes the generalized gradient of Φ in u_0 , as defined in [6]. It follows that this is equivalent to $\mathcal{F}'(u_0) \in \partial\mathcal{J}(u_0)$ and, since \mathcal{J} is convex, this can be written as

$$\mathcal{J}(v) - \mathcal{J}(u_0) \geq \mathcal{F}'(u_0)(v - u_0), \quad \forall v \in BV(\mathbb{R}^N). \quad (22)$$

Hence all $u_0 \in BV(\mathbb{R}^N)$ such that (22) holds is going to be called a bounded variation solution of (18).

Proof of Theorem 1.2. First of all let us prove that the restriction of Φ to the Banach space $BV_{\text{rad}}(\mathbb{R}^N)$ satisfies the first geometry of the Mountain Pass Theorem. Note that by (f_2) and (f_3) it follows that for all $\eta > 0$, there exists $A_\eta > 0$ such that

$$|F(s)| \leq \eta|s| + A_\eta|s|^p, \quad \forall s \in \mathbb{R}. \quad (23)$$

Note that, by (23) and the embeddings of $BV(\mathbb{R}^N)$, it follows that

$$\begin{aligned} \Phi(u) &= \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| \, dx - \int_{\mathbb{R}^N} F(u) \, dx \\ &\geq \|u\| - \eta\|u\|_1 - A_\eta\|u\|_p^p \\ &\geq \|u\| - \eta\|u\| - c_3\|u\|^p \\ &= \|u\| (1 - \eta - c_3\|u\|^{p-1}) \\ &\geq \alpha, \end{aligned}$$

for all $u \in BV_{\text{rad}}(\mathbb{R}^N)$, such that $\|u\| = \rho$, where $0 < \eta < 1$ is fixed, $0 < \rho < \left(\frac{1-\eta}{c_3}\right)^{\frac{1}{p-1}}$ and $\alpha = \rho(1 - \eta - c_3\rho^{p-1})$.

Now let us prove that Φ satisfies the second condition of the Mountain Pass Theorem. First note that condition (f_4) implies that there exists constants $d_1, d_2 > 0$ such that

$$F(s) \geq d_1|s|^\theta - d_2, \quad \forall s \in \mathbb{R}. \quad (24)$$

Let $u \in BV_{\text{rad}}(\mathbb{R}^N)$, with compact support, $u \neq 0$ and let $t > 0$. Then

$$\Phi(tu) \leq t\|u\| - d_1t^\theta|u|_\theta^\theta + d_2|\text{supp}(u)| \rightarrow -\infty,$$

as $t \rightarrow +\infty$, since $\theta > 1$ and then we can choose $e \in BV_{\text{rad}}(\mathbb{R}^N)$ such that $\Phi(e) < 0$.

Then, Mountain Pass Theorem [23][Theorem 3.2] implies that, given a sequence $\epsilon_n \rightarrow 0$, there exists $(u_n) \subset BV_{\text{rad}}(\mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} \Phi(u_n) = c \quad (25)$$

and

$$\mathcal{J}(v) - \mathcal{J}(u_n) \geq \int_{\mathbb{R}^N} f(u_n)(v - u_n)dx - \epsilon_n \|v - u_n\|, \quad \forall v \in BV_{\text{rad}}(\mathbb{R}^N), \quad (26)$$

where c is given by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \Phi(\gamma(t))$$

and $\Gamma = \{\gamma \in C^0([0,1], BV_{\text{rad}}(\mathbb{R}^N)); \gamma(0) = 0 \text{ and } \gamma(1) = e\}$. In fact, suppose by contradiction that it doesn't exist a $(PS)_c$ sequence to the functional Φ . Then Φ satisfies the Palais-Smale condition at the level c , since otherwise, it would exist a (PS) sequence at the level c which would not be strongly convergent in $BV(\mathbb{R}^N)$, contradicting the non-existence of such a sequence.

Let us prove that the sequence (u_n) is bounded in $BV_{\text{rad}}(\mathbb{R}^N)$. In (26), let us take as test function $v = 2u_n$ and note that

$$\|u_n\| \geq \int_{\mathbb{R}^N} f(u_n)u_n dx - \epsilon_n \|u_n\|,$$

which implies that

$$(1 + \epsilon_n)\|u_n\| \geq \int_{\mathbb{R}^N} f(u_n)u_n dx. \quad (27)$$

Then, by (f_4) and (27), note that

$$\begin{aligned} c + o_n(1) &\geq \Phi(u_n) \\ &= \|u_n\| + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(u_n)u_n - F(u_n) \right) dx - \int_{\mathbb{R}^N} \frac{1}{\theta} f(u_n)u_n dx \\ &\geq \|u_n\| \left(1 - \frac{1}{\theta} - \frac{\epsilon_n}{\theta} \right) \\ &\geq C\|u_n\|, \end{aligned}$$

for some $C > 0$ uniform in $n \in \mathbb{N}$. Then it follows that (u_n) is bounded.

By the boundedness of $(u_n) \subset BV_{\text{rad}}(\mathbb{R}^N)$ and Theorem 1.1, it follows that there exists $u \in BV_{\text{rad}}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ in $L^r(\mathbb{R}^N)$ for all $1 < r < 1^*$. Note that the limit function in $L^r(\mathbb{R}^N)$, in fact belongs to $BV_{\text{rad}}(\mathbb{R}^N)$ (see [15][Theorem 1.9]).

Claim. $\int_{\mathbb{R}^N} f(u_n)u_n dx = \int_{\mathbb{R}^N} f(u)u dx + o_n(1).$

In fact, by (f_2) and (f_3) , for given $\eta > 0$, there exists $A_\eta > 0$ such that

$$f(s)s \leq \eta|s| + A_\eta|s|^p, \quad s \in \mathbb{R}. \quad (28)$$

Since $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$, then there exists $R > 0$ such that

$$\int_{B_R(0)^c} |u_n|^p dx < \eta. \quad (29)$$

Since (u_n) is bounded in $L^1(\mathbb{R}^N)$, there exists $C > 0$ such that $|u_n|_1 \leq C$, for all $n \in \mathbb{N}$. Then, from this, (28), (29)

$$\int_{B_R(0)^c} f(u_n)u_n dx \leq (C + A_\eta)\eta. \quad (30)$$

Since $u_n \rightarrow u$ in $L^s(B_R(0))$ for all $s \in [1, 1^*)$, Lebesgue Dominated Convergence Theorem imply that

$$\int_{B_R(0)} f(u_n)u_n dx = \int_{B_R(0)} f(u)u dx + o_n(1). \quad (31)$$

Then, (30), (31) and the integrability of $f(u(\cdot))u(\cdot)$ (which follows by the continuous embeddings of $BV_{\text{rad}}(\mathbb{R}^N)$ and (f_3)), imply in the Claim.

Then, by the last claim and the lower semicontinuity of \mathcal{J} with respect to the $L^r(\mathbb{R}^N)$ convergence, it follows calculating the lim sup both sides of (26) that

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_{\mathbb{R}^N} f(u)(v - u), \quad \forall v \in BV_{\text{rad}}(\mathbb{R}^N). \quad (32)$$

From the last equation, by taking $v = u + tu$ and doing $t \rightarrow 0$ it follows that $\mathcal{J}(u) = \int_{\mathbb{R}^N} f(u)u dx$. Then

$$\mathcal{J}(u_n) = \int_{\mathbb{R}^N} f(u_n)u_n dx + o_n(1) \rightarrow \int_{\mathbb{R}^N} f(u)u dx = \mathcal{J}(u),$$

as $n \rightarrow \infty$. Hence, since $\Phi(u_n) = c + o_n(1)$, we have that

$$\Phi(u) = c$$

and then $u \neq 0$.

Now, in order to prove that u satisfies (32) for all $v \in BV(\mathbb{R}^N)$, we need a sort of Symmetric Criticality Principle of Palais (called here as SCPP), which is well known to hold for smooth functionals. Since we are dealing with a functional which is given as the difference between a locally Lipschitz and a smooth functional, the classical version of SCPP cannot be used. This in fact is a great field of research and there are some papers dealing with the extension of this principle to functionals which are not smooth (see [17] and [4] for example). Something that makes our problem even worse is the fact that $BV(\mathbb{R}^N)$ is not a reflexive space. Fortunately, in [21], Squassina succeed in proving a version of SCPP in a situation which comprises exactly our situation. Hence, by [21][Theorem 4], it follows that u satisfies (32) for all $v \in BV(\mathbb{R}^N)$ and then is a nontrivial bounded variation solution of (18).

Now, what is left to justify is just that the solution u in fact is a ground-state solution among the radial ones, i.e., that u has the lowest energy level among all nontrivial bounded variation radial solutions. In order to prove it, we have to recall [13], where is proved that we can define the Nehari set associated to Φ , given by

$$\mathcal{N} = \left\{ u \in BV_{\text{rad}}(\mathbb{R}^N) \setminus \{0\}; \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx = \int_{\mathbb{R}^N} f(u)u dx \right\}.$$

It can be proven as in [13] that \mathcal{N} is a set which contains all nontrivial bounded variation radial solutions of (18). Then, if we manage to prove that the solution u

is such that $\Phi(u) = \inf_{\mathcal{N}} \Phi$, then u would have the lowest energy level among the radial nontrivial solutions of (18).

By using the same kind of arguments that Rabinowitz in [20] (which consists in study the maps $t \mapsto \Phi(tv)$ and verify that it has a unique maximum point $t_v > 0$, that is such that $t_v v \in \mathcal{N}$), in the light of $(f_1) - (f_5)$, one can see that \mathcal{N} is radially homeomorphic to the unit sphere in $BV_{\text{rad}}(\mathbb{R}^N)$ and also that the minimax level c satisfies

$$c = \inf_{v \in BV(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} \Phi(tv) = \inf_{v \in \mathcal{N}} \Phi(v).$$

Since u is such that $\Phi(u) = c$, it follows that u is a solution which has the lowest energy among all the radial ones. \square

Acknowledgment

The authors would like to express their gratitude to the anonymous referee by careful reading the manuscript and by his/her suggestions.

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Received: July, 5, 2017.

Revised: November, 28, 2017.

Accepted: March, 5, 2018.