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# Green–Naghdi dynamics of surface wind waves in finite depth

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## Abstract

The Miles' quasi laminar theory of waves generation by wind in finite depth  $h$  is presented. In this context, the fully nonlinear Green–Naghdi model equation is derived for the first time. This model equation is obtained by the non perturbative Green–Naghdi approach, coupling a nonlinear evolution of water waves with the atmospheric dynamics which works as in the classic Miles' theory. A depth-dependent and wind-dependent wave growth  $\gamma$  is drawn from the dispersion relation of the coupled Green–Naghdi model with the atmospheric dynamics. Different values of the dimensionless water depth parameter  $\delta = gh/U_1$ , with  $g$  the gravity and  $U_1$  a characteristic wind velocity, produce two families of growth rate  $\gamma$  in function of the dimensionless theoretical wave-age  $c_0$ : a family of  $\gamma$  with  $h$  constant and  $U_1$  variable and another family of  $\gamma$  with  $U_1$  constant and  $h$  variable. The allowed minimum and maximum values of  $\gamma$  in this model are exhibited.

Keywords: surface waves, wind-waves, Miles' mechanism, nonlinear Green–Naghdi model, waves in finite depth

## 1. Introduction

The problem of wind-generation of surface waves has as starting point the Euler (or the Navier–Stokes) equations governing both the airflow and the surface wave dynamics. Winds generate surface wind waves and this, in its turn, modifies the airflow due to the loss of energy and momentum. In this way the atmosphere depends on the state of waves. The physical

mechanism behind is the *anti-dissipation*. Energy and momentum flow continuously from the air to the surface wave. Consequently the wave amplitude grows exponentially in time more or less quickly according to the value of the growth rate which depends on the wind speed and the water depth.

It is a wonderful issue and until now several approximations and assumptions have been done to understand and describe this problem. The pioneer theoretical works to describe surface wind-waves growth began with the works of Jeffreys (1925, 1926) and Miles (1957, 1997).

In the usual Miles' theory, the water is assumed deep, the viscosity is disregarded, the equations of motions are linearized and the model has two spatial dimensions. The air domain is also considered inviscid, the equations are linearized around a prescribed mean wind velocity and air flow turbulence is only used to set up the mean wind profile.

In finite depth, the pioneer experiments and numerical studies were conducted by Thijsse (1949), Ijima and Tang (2011) and in particular, the experiments in Lake George, described by Young and Verhagen (1996a). They provide one of the first systematic attempts to understand the physics of wave-wind generation in finite depth water. The results of the field experiments in fetch limited growth have been presented in Young and Verhagen (1996a, 1996b).

In Montalvo *et al* (2013a) the linear Miles' mechanism of wind-wave generation was extended to finite depth. In that work, the experimental results explained only via empirical formula of Young and Verhagen (1996a, 1996b) have been derived from theoretical expressions. The wave growth rate has been determined in Montalvo *et al* (2013) as a function of the wave age and the wind velocities, which among the linear Miles' and Jeffreys' mechanisms would prevail. The first attempt to include the simultaneous competing effects of *weakly nonlinearity, dispersion and anti-dissipation* due to the wind action in finite depth was done in Manna *et al* (2014).

In the present paper, our aim is to derive a *fully nonlinear* model equation of surface wind-wave generation. Hence we are going to describe the air/water interaction from a quasi laminar point of view, in which the water obeys the fully nonlinear Green–Naghdi model and the air flow is kept, as in Miles' theory, linear and obeying the linear Euler equation of motion.

This paper is organized as follows. In section 2 the nonlinear problem of surface waves in water of finite depth is presented. In section 3 the Green–Naghdi hypothesis is introduced and a Green–Naghdi model coupled with the air dynamics is derived. Section 4 concerns the linear air domain and the associated Miles' mechanisms of surface wind waves generation. At the end of this section the Green–Naghdi model equations under the wind action via the Miles' mechanisms is exhibited for the first time. appropriate dimensionless variables and scalings in order to compute the wave growth rate are introduced in section 5. Finally section 6 draws perspectives and conclusions.

## 2. The water domain

In the water domain we consider the Euler equations for finite depth. The horizontal and vertical velocities of the fluid are  $u(x, z, t)$ , and  $w(x, z, t)$ . The continuity equation and the equations of motion read

$$u_x + w_z = 0, \quad (1)$$

$$\rho_w(u_t + uu_x + wu_z) = -P_x, \quad (2)$$

$$\rho_w(w_t + uw_x + ww_z) = -P_z - g\rho_w, \quad (3)$$

where  $P(x, z, t)$  is the pressure,  $g$  the gravitational acceleration,  $\rho_w$  is the water density and subscripts in  $u$ ,  $w$  and  $P$  denote partial derivatives. The boundary conditions at  $z = -h$  and at  $z = \eta(x, t)$  are

$$w = 0; \quad z = -h, \quad (4)$$

$$\eta_t + u\eta_x - w = 0; \quad z = \eta, \quad (5)$$

$$P = P_a; \quad z = \eta, \quad (6)$$

where  $P_a(x, z, t)$  is the air pressure. Thus, equation (6) is the continuity of the pressure across the air/water interface. As these are vital assumptions for the growth mechanism, let us give a more pleasant expression to these equations. So, let us introduce a reduced pressure defined by

$$\mathbf{P}(x, z, t) = P(x, z, t) + \rho_w gz - P_0, \quad (7)$$

where  $P_0$  is the atmospheric pressure. In terms of (7) equations (1)–(3) read

$$u_x + w_z = 0, \quad (8)$$

$$\rho_w(u_t + uu_x + wu_z) = -\mathbf{P}_x, \quad (9)$$

$$\rho_w(w_t + uw_x + ww_z) = -\mathbf{P}_z, \quad (10)$$

and (4)–(6)

$$w = 0; \quad z = -h, \quad (11)$$

$$\eta_t + u\eta_x - w = 0; \quad z = \eta, \quad (12)$$

$$\mathbf{P} - \rho_w g\eta + P_0 = P_a; \quad z = \eta. \quad (13)$$

### 3. Coupling the fully nonlinear Green–Naghdi model with the atmospheric dynamics

Shallow water model equations are usually derived by performing an asymptotic analysis directly from equations (8)–(13). Our approach is somehow different since, instead of studying the entire problem via a perturbation theory, we are going to consider the nonlinear evolution of a given Ansatz for the velocity field. We assume indeed that the horizontal velocity  $u(x, z, t)$  is independent of  $z$ , that is

$$u = u(x, t). \quad (14)$$

This *ab initio* given velocity profile is known as the *columnar flow hypothesis*. It was introduced long ago by Su (1969), Serre (1953) and Green and Naghdi (1974, 1976). From (8), (14) and (12) we have

$$w(x, z, t) = -(z + h)u_x(x, t), \quad (15)$$

so (9) and (10) read

$$\rho_w(u_t + uu_x) = -\mathbf{P}_x, \quad (16)$$

$$-\rho_w(z+h)(u_{xt} + uu_{xx} - u_x^2) = -\mathbf{P}_z. \quad (17)$$

Integrating (17) and using (13) we obtain the pressure  $\mathbf{P}(x, z, t)$

$$\begin{aligned} \mathbf{P}(x, z, t) &= \frac{1}{2}\rho_w[(z+h)^2 - (\eta+h)^2](u_{xt} + uu_{xx} - u_x^2) \\ &+ P_a(x, \eta, t) + \rho_w g\eta - P_0. \end{aligned} \quad (18)$$

Substituting (18) in (16) and integrating in  $z$  for  $-h \leq z \leq \eta$  we have

$$\begin{aligned} u_t + uu_x + g\eta_x &= -\frac{1}{\rho_w}[P_a(x, \eta, t)]_x \\ &+ \frac{1}{3(\eta+h)}\{(\eta+h)^3(u_{xt} + uu_{xx} - u_x^2)\}_x, \end{aligned} \quad (19)$$

and (12) read

$$\eta_t + [u(\eta+h)]_x = 0. \quad (20)$$

If  $P_a = P_0$  equations (19) and (20) are reduced to the usual Green–Naghdi equations (Green and Naghdi 1974, 1976). But this is not our case and we must determine  $P_a(x, z, t)$  which depends on the atmospheric dynamics. So the system (19) and (20) couples the fully nonlinear Green–Naghdi model equations with the atmospheric dynamics.

#### 4. The Miles' mechanism in the air domain

Let us consider the linearized (inviscid) governing equation of a steady air flow, with a prescribed mean horizontal velocity  $U(z)$  depending on the vertical coordinate  $z$ . We are going to study perturbations to the mean flow  $U(z)$ :  $u_a(x, z, t)$ ,  $w_a(x, z, t)$  and  $P_a(x, z, t)$  (subscript  $a$  stands for *air*). So with  $\mathbf{P}_a(x, z, t) = P_a(x, z, t) + \rho_a gz - P_0$ ,  $\rho_a$  the air density, and  $U' = dU/dz$  we have the following equations

$$u_{a,x} + w_{a,z} = 0, \quad (21)$$

$$\rho_a[u_{a,t} + U(z)u_{a,x} + U'(z)w_a] = -\mathbf{P}_{a,x}, \quad (22)$$

$$\rho_a[w_{a,t} + U(z)w_{a,x}] = -\mathbf{P}_{a,z}, \quad (23)$$

which must be completed by some appropriate boundary conditions. The first one is the kinematic boundary condition for air, evaluated at the aerodynamic sea surface roughness  $z_0$  placed just above the interface. It reads

$$\eta_t + U(z_0)\eta_x = w_a(z_0). \quad (24)$$

Through this paper,  $z_0$  will be regarded as a constant, independent from the sea state. This is a widely used approximation, first proposed by Charnock (1955). For the datasets used later on, the wind speed ranges are such that the roughness may be seen as a constant (Fairall *et al* 1996).

We choose  $U(z)$  to be the following wind profile

$$U(z) = U_1 \ln(z/z_0), \quad U_1 = \frac{u_*}{\kappa}, \quad \kappa \approx 0.41, \quad (25)$$

where  $u_*$  is the friction velocity and  $\kappa$  the Von Kármán constant. This logarithmic wind profile is commonly used to describe the vertical distribution of the horizontal mean wind speed within the lowest portion of the air-side of the marine boundary layer

(Garratt *et al* 1996, Tennekes 1972). With (25) the equation (24) reduces to

$$\eta_t = w_a(z_0). \tag{26}$$

This equation describes the influence of the surface perturbation dynamics ( $\eta_t$ ) on the atmosphere (through the perturbed wind speed  $w_a(z_0)$ ). Consequently *the air feels the water*. So, the atmosphere dynamics (beyond, the initial logarithmic profile  $U(z)$ ) is determined by the surface wave dynamics.

Now we assume  $\mathbf{P}_a = \mathcal{P}_a(z)\exp(i\theta)$ ,  $u_a = \mathcal{U}_a(z)\exp(i\theta)$ ,  $w_a = \mathcal{W}_a(z)\exp(i\theta)$  with  $\theta = x - ct$ ,  $c$  being the phase speed and we add the following boundary conditions on  $\mathcal{W}_a$  and  $\mathcal{P}_a$ ,

$$\lim_{z \rightarrow +\infty} (\mathcal{W}_a' + k\mathcal{W}_a) = 0, \tag{27}$$

$$\lim_{z \rightarrow z_0} \mathcal{W}_a = W_0, \tag{28}$$

$$\lim_{z \rightarrow +\infty} \mathcal{P}_a = 0, \tag{29}$$

that is, the disturbance plus its  $z$ -derivative vanish at infinity, and the vertical component of the wind speed is enforced by the wave movement at the sea surface. Then, using equations (21)–(23) and (29) we obtain

$$w_a(x, z, t) = \mathcal{W}_a \exp(i\theta), \tag{30}$$

$$u_a(x, z, t) = \frac{i}{k} \mathcal{W}_{a,z} \exp(i\theta), \tag{31}$$

$$\mathbf{P}_a(x, z, t) = ik\rho_a \exp(i\theta) \int_z^\infty [U - c] \mathcal{W}_a dz'. \tag{32}$$

Removing the pressure from the Euler equations, we find the well-known Rayleigh equation  $\forall z \setminus z_0 < z < +\infty$  (Rayleigh 1880, Conte and Miles 1959)

$$(U - c)(\mathcal{W}_a'' - k^2\mathcal{W}_a) - U''\mathcal{W}_a = 0, \tag{33}$$

which is singular at  $z_c = z_0 e^{c\kappa/u_*} > z_0 > 0$ , where  $U(z_c) = c$ . We calculate  $P_a(x, \eta, t)$

$$P_a(x, \eta, t) = P_0 - \rho_a g \eta + ck^2 \rho_a \frac{\eta}{W_0} \int_{z_0}^\infty [U - c] \mathcal{W}_a dz, \tag{34}$$

where the lower integration bound is taken at the roughness height  $z_0$  instead of  $z = \eta$  since we are studying the linear problem and we have used equation (26) to eliminate the term  $ik\rho_a \exp(i\theta)$ . Now in terms of the integrals  $I_1$  and  $I_2$  defined as follow

$$I_1 = \int_{z_0}^\infty U \mathcal{W}_a dz, \quad I_2 = \int_{z_0}^\infty \mathcal{W}_a dz. \tag{35}$$

Equation (34) reads

$$P_a(x, \eta, t) = P_0 - \left[ g - \frac{ck^2}{W_0} (I_1 - cI_2) \right] \rho_a \eta. \tag{36}$$

In equations (30)–(31) and (36) neither  $\mathcal{W}_a$  nor  $c$  are known. In order to find  $c$  we use the linear system associated with (19) and (20) and equation (36)

$$u_t + g\eta_x(1 - s) + sk^2c \frac{\eta_x}{W_0} (I_1 - cI_2) + \frac{h^2}{3} u_{xxt} = 0, \tag{37}$$

$$\eta_t + hu_x = 0. \quad (38)$$

Now, writing  $u = u_0 e^{i\theta}$  and  $\eta = \eta_0 e^{i\theta}$  and substituting them in (37), (38) yields to a linear system of equations for  $u_0$  and  $\eta_0$ . The determinant of this system gives the following equation

$$c^2 \left\{ 1 + \frac{k^2 h^2}{3} + s \frac{k^2 h}{W_0} I_2 \right\} - s \frac{ck^2 h - 1}{W_0} - gh(1 - s) = 0. \quad (39)$$

The parameter  $s = \rho_a/\rho_w$  is small ( $\sim 10^{-3}$ ) and allow us to expand  $c$

$$c = \mathbb{A} + \mathbb{B}s + O(s^2). \quad (40)$$

Substituting (40) in (39), yields at order  $s^0$

$$\mathbb{A} = c_0 = \sqrt{\frac{gh}{1 + \frac{k^2 h^2}{3}}}. \quad (41)$$

and at order  $s^1$

$$\mathbb{B} = -\frac{c_0}{2} + \frac{c_0 k^2}{2gW_0} (c_0 I_1 - c_0^2 I_2). \quad (42)$$

The relation (39) reads then

$$c = c_0 \left( 1 - \frac{s}{2} \right) + s \frac{c_0 k^2}{2gW_0} (c_0 I_1 - c_0^2 I_2) + O(s^2). \quad (43)$$

If  $s = 0$  we obtain the classical dispersion relation of the Green–Naghdi equations (Green and Naghdi 1976)

$$\omega_0 = kc_0 = k \sqrt{\frac{gh}{1 + \frac{k^2 h^2}{3}}}. \quad (44)$$

Now substituting (43) in (33) and neglecting the products of  $s$  with perturbative quantities (and their derivatives) the Rayleigh equation can be solved. This allows us to compute  $I_1$  and  $I_2$ .

Due to the complex form of  $w_a(x, z, t)$  (30), the term  $I_1 - c_0 I_2$  is complex (see Beji and Nadaoka 2004), so we can write

$$c = c_0 + s \left\{ -\frac{c_0}{2} + \frac{c_0^2 k^2}{2gW_0} [\mathbf{Re}(I_1 - c_0 I_2)] \right\} + s \frac{c_0^2 k^2}{2gW_0} i [\mathbf{Im}(I_1 - c_0 I_2)]. \quad (45)$$

The linear expression of  $P_a(x, \eta, t)/\rho_w$  can be written at order  $s$  as

$$P_a/\rho_w = P_0/\rho_w - \left[ g - \frac{c_0 k^2}{W_0} \mathbf{Re}(I_1 - c_0 I_2) \right] s \eta + \frac{c_0 k}{W_0} \mathbf{Im}(I_1 - c_0 I_2) s \eta_x. \quad (46)$$

Equations (46) and (19) yield the Green–Naghdi model equations under the linear wind-action. They read

$$\begin{aligned} u_t + uu_x + g\eta_x = & \left[ g - \frac{c_0 k^2}{W_0} \mathbf{Re}(I_1 - c_0 I_2) \right] s \eta_x - \frac{sc_0 k}{W_0} \mathbf{Im}(I_1 - c_0 I_2) \eta_{xx} \\ & + \frac{1}{3(\eta + h)} [(\eta + h)^3 (u_{xt} + uu_{xx} - u_x^2)]_x, \end{aligned} \quad (47)$$

$$\eta_t + [u(\eta + h)]_x = 0. \quad (48)$$

The first term at the right-hand side of (47) changes the linear phase velocity of the system. The second one is an *anti-dissipative* term. This term transports (continuously) energy from the wind to the wave. The last terms at the right-hand side of (48) are the well known nonlinear and dispersive terms of the usual Green–Naghdi equations (Green and Naghdi 1974, 1976).

## 5. The finite depth parameters

In this section we introduce suitable non-dimensional variables. The problem is non-dimensionalized using the undisturbed depth of the water  $h$ , a typical wavenumber  $k$  and the acceleration due to gravity  $g$ . It follows that the appropriate non-dimensionalization for the horizontal velocities (in the water and in the air) is the typical wind velocity  $U_1$  (defined in (25)), the appropriate non-dimensionalization for the vertical wind velocity perturbation is  $W_0$  and the appropriate time scale is  $U_1/g$ . So we have

$$\begin{aligned} x &= h\hat{x}, & z &= h\hat{z}, & \eta &= h\hat{\eta}, & t &= \frac{U_1}{g}\hat{t}, & U &= U_1\hat{U}, \\ u &= U_1\hat{u}, & c_0 &= U_1\hat{c}_0, & \mathcal{W}_a &= W_0\hat{\mathcal{W}}_a, \end{aligned} \quad (49)$$

where *hats* mean dimensionless variables. With these definitions the Green–Naghdi equations under the wind-action are transformed in a dimensionless system which reads (dropping the hats)

$$\begin{aligned} \delta u_t + uu_x + \delta \eta_x &= [\delta - \epsilon^2 c_0 \mathbf{Re}(I_1 - c_0 I_2)] s \eta_x - \epsilon c_0 \mathbf{Im}(I_1 - c_0 I_2) s \eta_{xx} \\ &+ \frac{1}{3(\eta + 1)} [(\eta + 1)^3 (\delta u_{xt} + uu_{xx} - u_x^2)]_x, \end{aligned} \quad (50)$$

$$\delta \eta_t + [u(\eta + 1)]_x = 0, \quad (51)$$

where the dimensionless parameters  $\epsilon$ ,  $\delta$  and  $c_0$ , are defined as

$$\epsilon = kh, \quad \delta = \frac{gh}{U_1^2}, \quad c_0 = \frac{\delta^{1/2}}{(1 + \epsilon^2/3)^{1/2}}. \quad (52)$$

Obviously  $\epsilon$  is the parameter measuring dispersion. The parameter  $\delta$  is the ratio of the square of the dispersionless phase velocity of the system  $\sqrt{gh}$  and the square of the wind velocity  $U_1$ .  $c_0$  is the ratio of the phase velocity of the usual Green–Naghdi equations and  $U_1$ . A given value of  $\sqrt{\delta}$  characterizes the relative value between the extreme linear phase velocity and the wind.

The allowed values of  $\epsilon$  and  $c_0$  are

$$0 < \epsilon < 1, \quad \sqrt{\frac{3\delta}{4}} < c_0 < \sqrt{\delta}. \quad (53)$$

Parameters  $\epsilon$  and  $\delta$  are two independent dimensionless parameters.

Imagine a constant wind in direction and magnitude blowing perpendicular to an ideal straight lake's coastline. In the region where the wind meets the water, short surface waves are generated. As the waves progress, their phase velocities increase such that at intermediate distances the wavelengths are largest. Hence at largest distance from the shoreline, the phase velocities are constant and  $\sim \sqrt{\delta}$ . A suitable parameter able to describe this pattern is  $c_0$ . The phase velocity grows running from a minimum value  $\sim \sqrt{\frac{3\delta}{4}}$  (for maximum dispersion with



$\epsilon \sim 1$ ) to a maximum value  $\sim \sqrt{\delta}$  (for minimum dispersion  $\epsilon \sim 0$ ). Consequently,  $c_0$  is a kind of *theoretical wave-age*.

The term  $(\delta - \epsilon^2 c_0 \mathbf{Re}(I_1 - c_0 I_2)) s \eta_x$  can be disregarded because it contributes only to the linear phase velocity without introducing any changes on the growth rate of the waves. With this approximation, the dimensionless phase velocity of the system (50), (51) is given by

$$c = c_0 + i \frac{\epsilon^2 c_0^2 s}{2\delta} \mathbf{Im}(I_1 - c_0 I_2). \quad (54)$$

Now  $c_0$  together with the linear progressive dispersionless wave solution  $\eta \sim \exp(\epsilon x - (\epsilon/\delta)t)$  yield

$$\eta \sim \exp\left[i\epsilon\left(x - \frac{c_0}{\delta}\right)\right] t \times \exp\left[\frac{\epsilon^3 c_0^2 s}{2\delta^2} \mathbf{Im}(I_1 - c_0 I_2)\right] t. \quad (55)$$

The real exponential determines the growth rate of the the plane wave solution  $\gamma$ ,

$$\gamma(\epsilon, \delta, c_0) = \frac{\epsilon^3 c_0^2 s}{2\delta^2} \mathbf{Im}(I_1 - c_0 I_2). \quad (56)$$

The existence of a finite depth  $h$  transforms the uniquely defined function of wave growth rate in deep water in *families of functions* indexed by  $\delta = gh/U_1^2$ , i.e., a function for each value of  $\delta$ . A family of  $\gamma$ -functions is obtained with  $h$  constant and  $U_1$  variable and another family of  $\gamma$ -functions can be obtained if  $U_1$  is constant and  $h$  is variable.

There exist two extreme limits for  $\gamma$  ( $\gamma_{\max}$  and  $\gamma_{\min}$ ) obtained from two extreme limits for the wave age  $c_0$  ( $c_0^1$  and  $c_0^2$ )

$$c_0^1 = \sqrt{\frac{3\delta}{4}}, \quad \rightarrow \gamma_{\max} = \frac{3s}{8\delta} \mathbf{Im}\left(I_1 - \sqrt{\frac{3\delta}{4}} I_2\right), \quad (57)$$

$$c_0^2 = \sqrt{\delta}, \quad \rightarrow \gamma_{\min} = 0. \quad (58)$$

Expression (57) and (58) are the maximum and the minimum growth rate in this model.

## 6. Perspectives and conclusions

We have introduced a quasi laminar Miles' mechanism of wind waves generation in finite depth. We have linearized the equations of motion governing the air dynamics but the water dynamics was considered fully nonlinear and obeying to the Green–Naghdi model equation. The model is in addition weakly dispersive and the action of the wind appears by an anti-dissipative term. Anti-dissipation causes growth of the free water surface and we have computed the wave growth of the plane wave solutions  $\gamma$  in terms of the dimensionless parameters  $\epsilon$ ,  $\delta$  and  $c_0$ . The extreme value  $\gamma_{\max}$  is obtained for the minimum value of the wave age  $c_0^1 = \sqrt{3\delta/4}$  according to the fact that in young seas the surface waves are short with large aspect ratios, so the wind anti-dissipative action is maximal in this case. The extreme value  $\gamma_{\min} = 0$  is obtained for the maximum value of the wave age  $c_0^2 = \sqrt{\delta}$  according to the fact that in old seas the surface waves are large with small aspect ratios, so the wind anti-dissipative action is minimal.

It worth remarking that although the Green–Naghdi equation gives an excellent approximation of one layer fluid (Lannes 2013), it does not improve significantly the basic ill-posedness of classical two layer shallow water model (Liska *et al* 1995). Indeed, according to Liska *et al* (1995) (figure 1) for small  $s = \rho_a/\rho_w$  there exists a narrow region of

ill-posedness which corresponds to the region around the singularity of the equation (33) where the energy transfer from the air to water operates through resonance.

Some important analytical and numerical works are still to be done in upcoming studies. Namely, a numerical study of the Rayleigh equation in order to compute (explicitly)  $I_1$  and  $I_2$ , the blow-up in finite time associated with the soliton solution of the anti-dissipative Green–Naghdi model (47) and (48) and last but not least, the comparison of our analytical and numerical results with experimental results.

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## References

- Beji S and Nadaoka K 2004 *J. Fluid Mech.* **500** 65  
 Charnock H 1955 *Q. J. R. Meteorol. Soc.* **81** 639  
 Conte S and Miles J 1959 *J. Soc. Ind. Appl. Math.* **7** 361  
 Fairall C W, Grachev A A, Bedard A and Nishiyama R T 1996 *NOAA Technical Memorandum ERL ETL-268*  
 Garratt J, Hess G, Physick W and Bougeault P 1996 *Bound.-Layer Meteorol.* **78** 9  
 Green A E and Naghdi P M 1974 *Proc. R. Soc. A* **337** 451  
 Green A E and Naghdi P M 1976 *J. Fluid Mech.* **78** 237  
 Ijima T and Tang F L W 2011 *Coastal Engineering Proc. (North America)*  
 Jeffreys H 1925 *Proc. R. Soc. A* **107** 189  
 Jeffreys H 1926 *Proc. R. Soc. A* **110** 241  
 Lannes D 2013 *The Water Waves Problem: Mathematical Analysis and Asymptotics (Mathematical Surveys and Monographs)* (Providence, RI: American Mathematical Society)  
 Liska R, Margolin L and Wendroff B 1995 *Comput. Math. Appl.* **29** 25–37  
 Manna M A, Montalvo P and Kraenkel R A 2014 *Phys. Rev. E* **90** 013006  
 Miles J 1957 *J. Fluid Mech.* **3** 185  
 Miles J 1997 *Appl. Mech. Rev.* **50** R5  
 Montalvo P, Dorignac J, Manna M, Kharif C and Branger H 2013a *Coast. Eng.* **77** 49  
 Montalvo P, Kraenkel R, Manna M A and Kharif C 2013b *Nat. Hazards Earth Syst. Sci.* **13** 2805  
 Rayleigh L 1880 *Proc. London Math. Soc.* **XI** 57  
 Serre F 1953 *Houille Blanche* **3** 374  
 Su C S and Gardner C S 1969 *J. Math. Phys.* **10** 536  
 Tennekes H 1972 *J. Atmos. Sci.* **30** 234  
 Thijsse J T 1949 *Gen. Assem. Assoc. d'Océanographie Phys., Procés-Verbaux* **4** 80  
 Young I R and Verhagen L A 1996a *Coastal Eng.* **29** 47  
 Young I R and Verhagen L A 1996b *Coastal Eng.* **29** 79