Mediterr. J. Math. (2018) 15:137
https://doi.org/10.1007/s00009-018-1186-9
1660-5446/18/030001-23
published online May 31, 2018

Mediterranean Journal of Mathematics

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Multiplicity Results for the Fractional Laplacian in Expanding Domains

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Abstract. In this paper, we establish a multiplicity result of nontrivial weak solutions for the problem $(-\Delta)^{\alpha}u + u = h(u) \quad \text{in } \Omega_\lambda,\ u = 0$ on $\partial\Omega_\lambda$, where $\Omega_\lambda = \lambda\Omega$, Ω is a smooth and bounded domain in $\mathbb{R}^N, N > 2\alpha$, λ is a positive parameter, $\alpha \in (0,1)$, $(-\Delta)^{\alpha}$ is the fractional Laplacian and the nonlinear term $h(u)$ has subcritical growth. We use minimax methods, the Ljusternick–Schnirelmann and Morse theories to get multiplicity results depending on the topology of Ω.

Mathematics Subject Classification. 35A15, 35S05, 58E05, 74G35.

Keywords. Fractional Laplacian, multiplicity of solutions, Ljusternick–Schnirelmann category, Morse theory.

1. Introduction

This paper is concerned with the following problem:

$$\begin{cases} (-\Delta)^{\alpha}u + u = h(u) & \text{in } \Omega_\lambda, \\ u = 0 & \text{on } \partial\Omega_\lambda, \end{cases} \tag{1.1}$$

where $\Omega_\lambda = \lambda\Omega$, Ω is a smooth and bounded domain in $\mathbb{R}^N, N > 2\alpha$, λ is a positive parameter, $\alpha \in (0,1)$, $(-\Delta)^{\alpha}$ is the fractional Laplace operator, whose definition will be briefly recalled in the next section, and h satisfies suitable assumptions.

We are motivated in studying an equation involving the fractional Laplacian due to the great attention which has been given in these last years to problems involving fractional operators, both in $\mathbb{R}^N$ and in bounded domains. Indeed these problems appear in many areas such as physics, economy, finance, optimization, obstacle problems, fractional diffusion and probabilistic. In particular, from a probability point of view, the fractional Laplacian is the

Giovany M. Figueiredo was partially supported by CNPq, Capes, FAPDF, Brazil. Marcos T.O. Pimenta was supported by CNPq, Capes, Fapesp, Brazil. Gaetano Siciliano was partially supported by CNPq, Capes, Fapesp, Brazil.

infinitesimal generator of a Lévy process, see e.g. [11]. We also recall that a fractional Schrödinger equation has been derived by Laskin in the framework of the Fractional Quantum Mechanics. More information and applications are contained in some references such as [7,19,26,27,30].

On the other hand, in a beautiful series of papers, Benci, Cerami and Passaseo (see [8–10]) investigate the existence and multiplicity of positive solutions for equations of type $-\Delta u + \lambda u = u^{p-1}$ (that with a simple change of variable can be transformed in a problem involving an expanding domain like (1.1) for $\alpha = 1$) or $-\varepsilon\Delta u + u = f(u)$ in a bounded domain Ω with Dirichlet boundary conditions. In particular, they develop a tool which allows to estimate the number of positive solutions depending on the "shape" of the domain (or of suitable "nearby" domains), whenever the parameters λ, ε or p tend to a suitable limit value. They use variational methods, and introduce suitable maps which permit to see "a photography" of Ω in a certain sublevel set of the energy functional related to the equation. Then the Ljusternick–Schnirelmann and Morse theory, based on the properties of the *category* and some Morse relations, are used to obtain the existence of multiple solutions. Later on, these general ideas have been successfully applied also in other contexts, such as the "zero mass" case in [29], Klein–Gordon and Schrödinger–Poisson type equations in [23,24,28], p-Laplacian equations in [1,2,15–18], quasilinear equations in [3,5], fractional Schrödinger equation in $\mathbb{R}^N$ with a potential in [22], problems involving magnetic fields in expanding domains in [4,6], among many others.

The aim of this paper is to show existence and multiplicity results of solutions for the fractional scalar field equation (1.1) in the expanding domain Ω_λ. We obtain the same type of results of the papers cited above: roughly speaking, for λ large enough the number of nonnegative solutions is bounded below by topological invariants related to Ω_λ. In the proof of our results, we use some arguments that can be found in [1,4,5]. However, due to the presence of the fractional Laplacian, some more refined estimates are need, such as in Propositions 4.1 and 4.3, for instance. To the best of our knowledge, our paper is the first one where the Morse theory is applied to the fractional Laplacian to obtain multiplicity of solutions depending on the domain topology. For other applications of the Morse theory to fractional operators see, e.g., [25].

More precisely, let us assume that $h : \mathbb{R} \to \mathbb{R}$ is a C^1-function verifying the following conditions:

- (H0) $h(s) = 0$ for $s \leq 0$;
- (H1) $\lim_{|s|\to\infty} h(s)/|s|^{q-1} = 0$ for some $q \in (2, 2^*_\alpha)$ where $2^*_\alpha = 2N/(N-2\alpha)$;
- (H2) there exists $\theta > 2$ such that $0 < \theta H(s) \leq sh(s)$ for all $s > 0$, where $H(s) = \int_0^s h(t)\,\mathrm{d}t$;
- (H3) the function $s \mapsto h(s)/s$ is increasing for $s > 0$.

The typical function satisfying the above conditions is $h(s) = s^\mu$ for $s \geq 0$, with $1 < \mu < q-1$, and $h(s) = 0$ for $s < 0$.

Note that by the regularity of h and (H0) it holds $h(s) = o(|s|)$ near the origin, and $h'(0) = 0$.

Our main results are the following.

Theorem 1.1. *Suppose that (H0)–(H3) hold. Then there exists* $\lambda^* > 0$ *such that for* $\lambda \geq \lambda^*$, *problem* (1.1) *has at least* $cat\,\Omega_\lambda$ *nonnegative weak solutions.*

For $Y \subset X$, we are denoting with $\mathrm{cat}_X Y$ the Ljusternick–Schnirelmann category of X in Y, i.e., the least number of closed and contractible sets in X which cover Y. When $X = Y$ we just write $\mathrm{cat}\, X$.

As usual, we get one more solution if the domain Ω_λ is not contractible, i.e.,

Theorem 1.2. *Beside the assumptions of the previous theorem, assume that* $cat\,\Omega_\lambda \geq 2$. *Then there exists* $\lambda^* > 0$ *such that for* $\lambda \geq \lambda^*$, *problem* (1.1) *has at least* $cat\,\Omega_\lambda + 1$ *nonnegative weak solutions.*

If we replace (H1) by a slightly stronger condition to deal with the second variation of the energy functional associated to problem (1.1), we can get a better result using the Morse theory. To this aim, let

(H1') $\lim_{|s| \to \infty} h'(s)/|s|^{q-2} = 0$ for some $q \in (2, 2^*_\alpha)$.

Then, we have

Theorem 1.3. *Suppose that (H0)–(H1')–(H2)–(H3) hold. Then there exists* $\lambda^* > 0$ *such that for* $\lambda \geq \lambda^*$, *the Eq.* (1.1) *has at least* $2\mathcal{P}_1(\Omega_\lambda) - 1$ *nonnegative weak solutions, if counted with their multiplicity.*

Here $\mathcal{P}_1(\Omega_\lambda)$ denotes the Poincaré polynomial of Ω_λ evaluated in $t = 1$. This definition will be recalled later during the proof.

To prove our results, we use variational methods. Indeed a functional on a Hilbert space can be defined in such a way that its critical points are exactly the solutions of (1.1). In this framework, the assumptions on h are quite natural to deal with Nehari manifolds, Mountain Pass arguments and Palais–Smale condition. We recall that if I is a C^1 functional on a Hilbert manifold $\mathcal{M}$ and $c \in \mathbb{R}$, a sequence $\{v_n\} \subset \mathcal{M}$ is said to be a Palais–Smale sequence for I at level c (briefly, a $(PS)_c$ sequence) if $I(v_n) \to c$ and $I'(u_n) \to 0$ in the tangent bundle. Furthermore, I is said to satisfy the Palais–Smale condition at level c if every $(PS)_c$ sequence has a convergent subsequence.

The functional related to our problem will turn out to be bounded from below on the "manifold solution" and verify the Palais–Smale condition at every level c, so the "photography method" of Benci and Cerami can be implemented and the classical Ljusternick–Schnirelmann and Morse theory can be used to estimate the number of critical points of the functional, that is, the number of solutions of (1.1).

1.1. Notations

Let us introduce here few notations that will be used throughout the paper.

- $B_R(x)$ denotes the open ball in $\mathbb{R}^N$ of radius R centered in x; if $x = 0$ we write B_R. In all the paper, we assume without loss of generality that $0 \in \Omega$.
- For $U \subset \mathbb{R}^N$, we denote with $\mathcal{C}_U$ the half cylinder $U \times (0, +\infty) \subset \mathbb{R}^{N+1}$. In particular, $\mathcal{C}_{\mathbb{R}^N} = \mathbb{R}^N \times (0, +\infty)$. Whenever an element of $\mathcal{C}_U$ is

written as (x, y), it has always to be intended as $x \in U, y \in (0, +\infty)$. If $U \neq \mathbb{R}^N$, the lateral boundary of the cylinder is $\partial_L \mathcal{C}_U = \partial U \times [0, +\infty)$.

Other notations will be introduced along the paper as soon as we need. Finally, we will use $C_1, C_2, \ldots$ to denote suitable positive constants, whose exact value may change from line to line.

The plan of the paper is the following. In Sect. 2, we recall some facts on the fractional Laplacian and write the variational framework in which we will work. Section 3 is devoted to study the limit problem associated to our equation; in particular compactness results are proved and, *en passant*, also the existence of a ground state solution for (1.1). In Sect. 4, we introduce the barycenter map and its properties. Moreover, a careful analysis of the ground states level in terms of λ is carried out. Finally, in Sect. 5, we give the proof of Theorems 1.1 and 1.2, and finally in Sect. 6, after recalling some facts and introducing some notations in classical Morse theory, we prove Theorem 1.3.

2. Preliminary Results and the Variational Framework

In this section, we start by introducing the functional framework necessary to apply variational methods and recover some known results about the different forms of definition of the fractional power of the Laplacian with Dirichlet boundary condition.

Let us consider the half cylinder with base Ω_λ, i.e., $\mathcal{C}_{\Omega_\lambda}$ and let

$$H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) = \Big\{ v \in H^1(\mathcal{C}_{\Omega_\lambda});\ v = 0 \text{ on } \partial_L \mathcal{C} \text{ and } \|v\|_\alpha < \infty \Big\},$$

where

$$\|v\|_\alpha = \left(k_\alpha^{-1} \int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha} |\nabla v|^2 \mathrm{d}x\mathrm{d}y + \int_{\Omega_\lambda} |tr_{\Omega_\lambda} v(x)|^2 \mathrm{d}x \right)^{1/2},$$

$k_\alpha = 2^{1-2\alpha}\Gamma(1-\alpha)/\Gamma(\alpha)$, $\alpha \in (0,1)$ and tr_{Ω_λ} is the trace operator given by $tr_{\Omega_\lambda} v = v(\cdot, 0)$ for $v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$. It is not difficult to see that $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a Hilbert space when endowed with the norm $\|\cdot\|_\alpha$, which comes from the following inner product

$$\langle v, w \rangle_\alpha = \int_{\mathcal{C}_{\Omega_\lambda}} k_\alpha^{-1} y^{1-2\alpha} \nabla v \nabla w \mathrm{d}x\mathrm{d}y + \int_{\Omega_\lambda} v(x,0) w(x,0) \mathrm{d}x.$$

Consider the following space:

$$\mathcal{V}_0^\alpha(\Omega_\lambda) = \Big\{ tr_{\Omega_\lambda} v;\ v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \Big\}.$$

By [14, Proposition 2.1], there exists a trace operator from $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ into the fractional Sobolev space $H_0^\alpha(\Omega_\lambda)$. Then $\mathcal{V}_0^\alpha(\Omega_\lambda)$ is a subspace of the fractional Sobolev space $H^\alpha(\Omega_\lambda)$ endowed with the norm

$$\|u\|_{\mathcal{V}_0^\alpha(\Omega_\lambda)} = \left(\|u\|^2_{L^2(\Omega_\lambda)} + \int_{\Omega_\lambda} \int_{\Omega_\lambda} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}} \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

Moreover, by the Trace theorem and embeddings of the fractional Sobolev spaces (see [20, Theorem 6.7] for instance) it follows that

$$\|tr_{\Omega_\lambda} v\|_{L^p(\Omega_\lambda)} \le C\|v\|_\alpha, \quad \forall v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}),$$

where $p \in (1, 2^*_\alpha)$.

By [14, Proposition 2.1] it holds that

$$\mathcal{V}_0^\alpha(\Omega_\lambda) = \left\{ u \in L^2(\Omega_\lambda);\ u = \sum_{k=1}^\infty b_k\varphi_k \ \text{ such that } \sum_{k=1}^\infty b_k^2 \mu_k^\alpha < \infty \right\},$$

where hereafter (μ_k, φ_k) are the eigenpairs of $(-\Delta, H_0^1(\Omega_\lambda))$, μ_k repeated as much as its multiplicity.

Given $u \in C_0^\infty(\Omega_\lambda)$, with $u = \sum_{k=1}^\infty b_k\varphi_k$, we define the operator

$$(-\Delta)^\alpha u = \sum_{k=1}^\infty \mu_k^\alpha b_k \varphi_k \tag{2.1}$$

which extends by density on $\mathcal{V}_0^\alpha(\Omega_\lambda)$.

Instead of working with this definition, we can get a local realization of $(-\Delta)^\alpha$ by adding one more dimension. Indeed, as proved in [14, Section 2.1], for each $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ there exists a unique $\tilde{u} \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$, called the *$\alpha$-harmonic extension of* u such that

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla\tilde{u}) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ \tilde{u} = 0 & \text{on } \partial_L\mathcal{C}_{\Omega_\lambda} \\ \tilde{u}(\cdot, 0) = u & \text{on } \Omega_\lambda. \end{cases}$$

Moreover, if $u = \sum_{k=1}^\infty b_k\varphi_k$ then

$$\tilde{u}(x,y) = \sum_{k=1}^\infty b_k\varphi_k(x)\psi(\mu_k^{1/2} y), \quad \forall (x,y) \in \mathcal{C}_{\Omega_\lambda}, \tag{2.2}$$

where ψ solves the Bessel equation

$$\begin{cases} \psi''(s) + \dfrac{1-2\alpha}{s}\psi'(s) = \psi\,,\ s > 0 \\ -\lim\limits_{s\to 0^+} s^{1-2\alpha}\psi'(s) = k_\alpha \\ \psi(0) = 1. \end{cases} \tag{2.3}$$

Now, for a fixed $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ define the functional $\left.\dfrac{1}{k_\alpha}\dfrac{\partial\tilde{u}}{\partial y^\alpha}\right|_{\Omega_\lambda\times\{0\}} \in \mathcal{V}_0^\alpha(\Omega_\lambda)^*$ by

$$\left\langle \frac{1}{k_\alpha}\frac{\partial\tilde{u}}{\partial y^\alpha}(\cdot,0), g \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} := \frac{1}{k_\alpha}\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}\nabla\tilde{u}\nabla\tilde{g}\,\mathrm{d}x\mathrm{d}y, \quad g \in \mathcal{V}_0^\alpha(\Omega_\lambda).$$

Integration by parts in the right-hand side of the last equality explains the notation chosen to the functional, since

$$\left\langle \frac{1}{k_\alpha}\frac{\partial\tilde{u}}{\partial y^\alpha}(\cdot,0), g \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} = \left\langle \frac{1}{k_\alpha}\frac{\partial\tilde{u}}{\partial y^\alpha}(\cdot,0), g \right\rangle_{L^2(\Omega_\lambda)}$$

for all $g \in \mathcal{V}_0^\alpha(\Omega_\lambda)$, where

$$\frac{\partial \tilde{u}}{\partial y^\alpha}(x,0) = -\lim_{y\to 0^+} y^{1-2\alpha}\frac{\partial \tilde{u}}{\partial y}(x,y) \qquad \forall x \in \Omega_\lambda.$$

Then we can define an operator $A_\alpha : \mathcal{V}_0^\alpha(\Omega_\lambda) \to \mathcal{V}_0^\alpha(\Omega_\lambda)^*$ such that

$$A_\alpha u := \frac{1}{k_\alpha}\frac{\partial \tilde{u}}{\partial y^\alpha}\bigg|_{\Omega_\lambda\times\{0\}}.$$

Let us prove that the operators A_α and $(-\Delta)^\alpha$ defined in (2.1) are in fact the same, i.e., that for all $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$,

$$A_\alpha u = \sum_{k=1}^{\infty} \mu_k^\alpha b_k \varphi_k, \qquad \text{where} \qquad u = \sum_{k=1}^{\infty} b_k\varphi_k.$$

It is enough to show that for all $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$,

$$\left\langle \frac{1}{k_\alpha}\frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot,0), \varphi_k \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))} = \langle (-\Delta)^\alpha u, \varphi_k\rangle_{L^2(\Omega_\lambda)}, \quad \text{for all } k \in \mathbb{N}.$$

For $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ and $k \in \mathbb{N}$, by (2.2),

$$\tilde{u}(x,y) = \sum_{k=1}^{\infty} b_k\varphi_k(x)\psi(\mu_k^{1/2}y) \quad \text{and} \quad \widetilde{\varphi_k}(x,y) = \varphi_k(x)\psi(\mu_k^{1/2}y).$$

Now, integration by parts implies that, for $y > 0$,

$$\int_{\Omega_\lambda} y^{1-2\alpha}\nabla\tilde{u}(x,y)\nabla\widetilde{\varphi_k}(x,y)\mathrm{d}x = y^{1-2\alpha}b_k\left(\mu_k\psi(\mu_k^{1/2}y)^2 + \psi_k'(\mu_k^{1/2}y)^2\right).$$

Then, by (2.3)

$$\begin{aligned}
&\left\langle \frac{1}{k_\alpha}\frac{\partial \tilde{u}}{\partial y^\alpha}(\cdot,0), \varphi_k \right\rangle_{(\mathcal{V}_0^\alpha(\Omega_\lambda)^*, \mathcal{V}_0^\alpha(\Omega_\lambda))}\\
&= \frac{1}{k_\alpha}\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}\nabla\tilde{u}\nabla\widetilde{\varphi_k}\,\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{k_\alpha}\int_0^{+\infty} y^{1-2\alpha}b_k\left(\mu_k\psi(\mu_k^{1/2}y)^2 + \psi_k'(\mu_k^{1/2}y)^2\right)\mathrm{d}y\\
&= \frac{1}{k_\alpha}\lim_{\eta\to 0^+} y^{1-2\alpha}\mu_k^{1/2}b_k\psi'(\mu_k^{1/2}y)\psi(\mu_k^{1/2}y)\bigg|_{y=\eta}\\
&= b_k\mu_k^\alpha\\
&= \langle (-\Delta)^\alpha u, \varphi_k\rangle_{L^2(\Omega_\lambda)}.
\end{aligned}$$

Hence, in (1.1) we are going to understand $(-\Delta)^\alpha$ as A_α.

Let us pass to the definition of weak solution for problems involving the fractional Laplacian. We say that a function u is a solution of the linear problem

$$\begin{cases} (-\Delta)^\alpha u = f(x) & \text{in } \Omega_\lambda\\ u = 0 & \text{on } \partial\Omega_\lambda, \end{cases}$$

where $f \in \mathcal{V}_0^\alpha(\Omega_\lambda)^*$, if $u = tr_{\Omega_\lambda} v$, where $v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a solution of

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\Omega_\lambda} \\ \dfrac{1}{k_\alpha}\dfrac{\partial v}{\partial y^\alpha}(x,0) = f(x) & x \in \Omega_\lambda. \end{cases}$$

Analogously, we say that $u \in \mathcal{V}_0^\alpha(\Omega_\lambda)$ is a weak solution of (1.1) if $u = tr_{\Omega_\lambda} v$, where $v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ is a weak solution of

$$\begin{cases} -\mathrm{div}(y^{1-2\alpha}\nabla v) = 0 & \text{in } \mathcal{C}_{\Omega_\lambda} \\ v = 0 & \text{on } \partial_L \mathcal{C}_{\Omega_\lambda} \\ \dfrac{1}{k_\alpha}\dfrac{\partial v}{\partial y^\alpha} + v(x,0) = h(v(x,0)) & x \in \Omega_\lambda, \end{cases}$$

that is,

$$\int_{\mathcal{C}_{\Omega_\lambda}} k_\alpha^{-1} y^{1-2\alpha} \nabla v \nabla \psi \mathrm{d}x\mathrm{d}y + \int_{\Omega_\lambda} v(x,0)\psi(x,0)\mathrm{d}x$$
$$= \int_{\Omega_\lambda} h(v(x,0))\psi(x,0)\mathrm{d}x, \quad \forall \psi \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}).$$

As it is easy to see, this is equivalent to say that v is a critical point of the C^1 functional

$$I_\lambda(v) = \frac{k_\alpha^{-1}}{2}\int_{\mathcal{C}_\lambda} y^{1-2\alpha}|\nabla v|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\Omega_\lambda} |v(x,0)|^2 \mathrm{d}x - \int_{\Omega_\lambda} H(v(x,0))\mathrm{d}x$$

in $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.

It is not difficult to see that, in virtue of the assumptions on the nonlinearity h, the functional I_λ possesses a Mountain Pass Geometry: the mountain pass level will be denoted with $c(\Omega_\lambda) > 0$. We also define the Nehari manifold associated to I_λ by

$$\mathcal{M}_\lambda = \Big\{ v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \backslash \{0\} : J_\lambda(v) = 0 \Big\}, \tag{2.4}$$

where

$$J_\lambda(v) := I'_\lambda(v)[v] = k_\alpha^{-1}\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}|\nabla v|^2 \mathrm{d}x\mathrm{d}y + \int_{\Omega_\lambda} |v(x,0)|^2 \mathrm{d}x$$
$$- \int_{\Omega_\lambda} h(v(x,0))v(x,0)\mathrm{d}x.$$

We will need the following properties of $\mathcal{M}_\lambda$ stated in the Lemma below. They are standard, as well, and just based on the hypothesis made on the nonlinearity. Observe first that for $v \in \mathcal{M}_\lambda$, it is

$$meas\{x \in \Omega_\lambda : v(x,0) > 0\} > 0.$$

Otherwise by the definition of $\mathcal{M}_\lambda$ and using (H0) we arrive at

$$k_\alpha^{-1}\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}|\nabla v|^2 \mathrm{d}x\mathrm{d}y + \int_{\Omega_\lambda} |v(x,0)|^2 \mathrm{d}x = 0$$

which gives $v(\cdot,0) = 0$ on Ω_λ and then $\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}|\nabla v|^2 \mathrm{d}x\mathrm{d}y = 0$, that is $v = 0$ which is a contradiction. In virtue of this one can define the map

$$\sigma_v : t \in [0, +\infty) \mapsto \frac{1}{t}\int_{\Omega_\lambda} h(tv(x,0))v(x,0)\mathrm{d}x$$

and then repeat the arguments of [9, Lemma 2.2] to get the next result.

Lemma 2.1. *Let $\lambda > 0$. The following propositions hold true:*

1. *for every $v \in \mathcal{M}_\lambda$ it is $J'_\lambda(v)[v] < 0$;*
2. *$\mathcal{M}_\lambda$ is a differentiable manifold radially diffeomorphic to*

$$\mathrm{S} = \{v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : \|v\|_\alpha = 1\}\backslash\{v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : v(x,0) \le 0 \textit{ a.e. in } \Omega_\lambda\}$$

   and is bounded away from 0;
3. *I_λ is bounded from below on $\mathcal{M}_\lambda$ and*

$$0 < c(\Omega_\lambda) = \inf_{\mathcal{M}_\lambda} I_\lambda = \inf_{u\neq 0}\sup_{t>0} I_\lambda(tu). \tag{2.5}$$

In particular, every nonzero function $v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})\backslash\{v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : v \le 0 \text{ a.e.}\}$ can be "projected" on $\mathcal{M}_\lambda$; in other words, we have an homeomorphism which just multiplies a function by a positive constant (depending on the function)

$$v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})\backslash\{v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : v(x,0) \le 0 \text{ a.e.}\} \longmapsto t_\lambda v \in \mathcal{M}_\lambda. \tag{2.6}$$

It is clear that $\mathcal{M}_\lambda$ is a natural constraint for I_λ in the sense that

Corollary 2.2. *If v is a critical point of I_λ on $\mathcal{M}_\lambda$, then v is a nontrivial critical point of I_λ on $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.*

Moreover, standard arguments show that the Palais–Smale sequences for I_λ restricted to $\mathcal{M}_\lambda$ are Palais–Smale sequences for the free functional I_λ, and I_λ satisfies the Palais–Smale condition on $\mathcal{M}_\lambda$ if and only if it satisfies the same condition on $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$.

Remark 1. In the next sections, we will use some auxiliary functionals: they differ from I_λ just for the domain on which these functionals are defined. In a similar way as in (2.4), we will define the Nehari manifolds related to these functionals and it is clear that analogous properties to that stated for $\mathcal{M}_\lambda$ hold, since they are essentially based on the structure of the functional, on the hypothesis made on the nonlinearity, and on the definition of the Nehari manifold. For this reason, the above cited properties will be used without any other comment through the paper.

3. Compactness Results and Existence of a Ground State Solution for I_λ

Now let us consider the half cylinder with base $\mathbb{R}^N$, $\mathcal{C}_{\mathbb{R}^N}$, and define

$$H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}) = \{v \in H^1(\mathcal{C}_{\mathbb{R}^N}) : \ \|v\|_{\mathcal{C}_{\mathbb{R}^N}} < \infty\},$$

where

$$\|v\|_{\mathcal{C}_{\mathbb{R}^N}} = \left(k_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla v|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |v(x,0)|^2 \mathrm{d}x \right)^{1/2}.$$

It is easy to see that $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a Hilbert space when endowed with the norm $\|\cdot\|_{\mathcal{C}_{\mathbb{R}^N}}$, which comes from the following inner product:

$$\langle v, w\rangle_{\mathcal{C}_{\mathbb{R}^N}} = k_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} \nabla v \nabla w \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} v(x,0) w(x,0) \mathrm{d}x.$$

An important result we are going to use in this work is related with the existence of a nonnegative ground state solution of the *limit problem*

$$(P_\infty) \qquad (-\Delta)^\alpha u + u = h(u) \ \text{ in } \ \mathbb{R}^N,$$

i.e., the least energy solution for the functional

$$I_\infty(v) = \frac{k_\alpha^{-1}}{2} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla v|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^N} |v(x,0)|^2 \mathrm{d}x - \int_{\mathbb{R}^N} H(v(x,0)) \mathrm{d}x.$$

It is standard to see that I_∞ has a Mountain Pass Geometry in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$, whose mountain pass level is denoted by $c(\mathbb{R}^N) > 0$. Moreover, we can define the Nehari manifold associated to I_∞ by

$$\mathcal{M}_\infty = \left\{ v \in H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}) \backslash \{0\} : I'_\infty(v)[v] = 0 \right\}$$

and standard computations give

$$0 < c(\mathbb{R}^N) = \inf_{\mathcal{M}_\infty} I_\infty.$$

The theorem below states the existence of a ground state solution for (P_∞), hence $c(\mathbb{R}^N)$ is achieved on a function of mountain pass type. The result is known in the literature (it can be obtained with similar arguments used in [1, Theorem 3.1]) but for completeness, and since it will be very useful for us, we prefer to give the proof.

Lemma 3.1. *Let $\{v_n\} \subset \mathcal{M}_\infty$ be a sequence satisfying $I_\infty(v_n) \to c(\mathbb{R}^N)$. Then, either*

(a) *$\{v_n\}$ has a strongly convergent subsequence in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$*

or

(b) *there exists a sequence $\{x_n\} \subset \mathbb{R}^N$ such that, up to a subsequence, $|x_n| \to +\infty$ and $\overline{v}_n(x,y) := v_n(x - x_n, y)$ strongly converges in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.*

In particular, there exists a nonnegative minimizer, hereafter denoted by $\mathfrak{w}_\infty$, for $c(\mathbb{R}^N)$.

Proof. By the Ekeland Variational Principle we can assume without loss of generality that $\{v_n\}$ is a $(PS)_{c(\mathbb{R}^N)}$ sequence for I_∞ on $\mathcal{M}_\infty$ which is nonnegative and then, by very known arguments, it follows that it is a $(PS)_{c(\mathbb{R}^N)}$ sequence for I_∞ on $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. In a standard way, one can prove that $\{v_n\}$ is bounded in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ and then, up to a subsequence, $v_n \rightharpoonup v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.

First case: $v \neq 0$. It is a simple matter to prove in this case that $I'_\infty(v) = 0$. It follows from the Fatou Lemma, (H2) and the weak lower semicontinuity of the norm that

$$\begin{aligned}
c(\mathbb{R}^N) &\leq I_\infty(v) \\
&= I_\infty(v) - \frac{1}{\theta} I'_\infty(v)[v] \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|v\|^2_{\mathcal{C}_{\mathbb{R}^N}} + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v(x,0))v(x,0) - H(v(x,0))\right) \mathrm{d}x \\
&\leq \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\theta}\right) \|v_n\|^2_{\mathcal{C}_{\mathbb{R}^N}} + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v_n(x,0))v_n(x,0) - H(v_n(x,0))\right) \mathrm{d}x\right] \\
&= c(\mathbb{R}^N),
\end{aligned}$$

which implies that $I_\infty(v) = c(\mathbb{R}^N)$. Now let us prove that $v_n \to v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ and for this it is enough to show that $\|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} \to \|v\|_{\mathcal{C}_{\mathbb{R}^N}}$. By the weak lower semicontinuity of the norm it follows that

$$\|v\|_{\mathcal{C}_{\mathbb{R}^N}} \leq \liminf_{n\to\infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}}. \tag{3.1}$$

Assuming by contradiction that

$$\limsup_{n\to\infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} > \|v\|_{\mathcal{C}_{\mathbb{R}^N}},$$

the Fatou Lemma implies that

$$\begin{aligned}
c(\mathbb{R}^N) &= \limsup_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\theta}\right) \|v_n\|^2_{\mathcal{C}_{\mathbb{R}^N}} \\
&\qquad + \limsup_{n\to\infty} \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v_n(x,0))v_n(x,0) - H(v_n(x,0))\right) \mathrm{d}x \\
&> \left(\frac{1}{2} - \frac{1}{\theta}\right) \|v\|^2_{\mathcal{C}_{\mathbb{R}^N}} + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} h(v(x,0))v(x,0) - H(v(x,0))\right) \mathrm{d}x \\
&= c(\mathbb{R}^N),
\end{aligned}$$

which is a contradiction. Then it follows that

$$\limsup_{n\to\infty} \|v_n\|_{\mathcal{C}_{\mathbb{R}^N}} \leq \|v\|_{\mathcal{C}_{\mathbb{R}^N}}$$

and this together with (3.1) implies that $v_n \to v$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$.

Second case $v = 0$. Then $\{v_n\}$ is not strongly convergent; indeed, if this were not the case, we would have a contradiction with the fact that $I_\infty(v_n) \to c(\mathbb{R}^N) > 0$. Hence, there are $R, \gamma > 0$ and $\{x_n\} \subset \mathbb{R}^N$ such that, up to a subsequence

$$\int_{B_R(x_n)} |v_n(x,0)|^2 \mathrm{d}x \geq \gamma > 0.$$

Otherwise by [21, Lemma 2.2], we get $v_n(\cdot, 0) \to 0$ in $L^q(\mathbb{R}^N)$ for $2 < q < 2^*_\alpha$. This fact together with conditions (H0)-(H3), implies that

$$I_\infty(v_n) = \int_{\mathbb{R}^N} \left(\frac{1}{2} h(v_n(x,0))v_n(x,0) - H(v_n(x,0))\right) \mathrm{d}x + o_n(1) = o_n(1),$$

which contradicts again $I_\infty(v_n) \to c(\mathbb{R}^N) > 0$. Moreover, since $v = 0$, it follows that $|x_n| \to +\infty$, otherwise the Sobolev embedding would give $v \neq 0$. Since $\mathbb{R}^N$ is invariant by translation, defining $\overline{v}_n(x,y) := v_n(x - x_n, y)$ we still have a $(PS)_{c(\mathbb{R}^N)}$ sequence for I_∞, which is contained on $\mathcal{M}_\infty$ and is bounded in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. Then $\overline{v}_n \rightharpoonup \overline{v} \neq 0$ and hence, by the first case, $\overline{v}_n \to \overline{v}$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$, $I_\infty(\overline{v}) = c(\mathbb{R}^N)$ and $\overline{v}$ is a ground state for I_∞. □

For what concerns our functional, we have

Lemma 3.2. *For every $\lambda > 0$, the functional I_λ satisfies the Palais–Smale condition on $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$, and hence on $\mathcal{M}_\lambda$.*

Proof. Let $\{v_n\} \subset H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ be a sequence such that

$$I_\lambda(v_n) \to c \quad \text{and} \quad I'_\lambda(v_n) \to 0.$$

Thus, by (H2), we get

$$C_1 + o_n(1)\|v_n\|_\alpha \geq I_\lambda(v_n) - \frac{1}{\theta} I'_\lambda(v_n)[v_n] \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|v_n\|^2_\alpha,$$

which gives that $\{v_n\}$ is bounded in $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$. Then we may assume that, up to a subsequence, $v_n \rightharpoonup v$ in $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ and hence $tr_{\Omega_\lambda} v_n \to tr_{\Omega_\lambda} v$ in $L^s(\Omega_\lambda)$, with $2 \leq s < 2^*_\alpha$. Thus, since the nonlinearity h has subcritical growth, by standard calculations, we see that I_λ satisfies the Palais–Smale condition. □

Then, taking into account that I_λ is bounded from below on $\mathcal{M}_\lambda$ we have

Theorem 3.3. *For every $\lambda > 0$, $c(\Omega_\lambda)$ is achieved on a ground state solution denoted with $\mathfrak{w}_{\Omega_\lambda}$.*

4. The Barycenter Map and Behavior of the Mountain Pass Levels

In this section, we study the behavior of some minimax levels with respect to the parameter λ. To do so, some preliminaries are in order.

Recall we are assuming that $0 \in \Omega_\lambda$. Following [9], for $v \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ with compact support and such that $tr_{\Omega_\lambda} v^+ \not\equiv 0$, we define the *barycenter* or *center of mass* of v in the following way: first consider the "trivial" extension of $v^+(\cdot, 0) = tr_{\Omega_\lambda} v^+$ to the whole $\mathbb{R}^N$ (denoted by the same symbol) and then set

$$\beta(v) := \beta(v^+(\cdot,0)) = \frac{\displaystyle\int_{\mathbb{R}^N} x|v^+(x,0)|^2 \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |v^+(x,0))|^2 \mathrm{d}x} \in \mathbb{R}^N.$$

For $R > r > 0$ let us denote by $A_{R,r}(\tilde{x})$ the open annulus in $\mathbb{R}^N$ centered in $\tilde{x}$

$$A_{R,r}(\tilde{x}) = B_R(\tilde{x}) \backslash \overline{B}_r(\tilde{x}).$$

Define the functional on $H^1_{0,L}(\mathcal{C}_{A_{\lambda R,\lambda r}(\tilde{x})}, y^{1-\alpha})$

$$\widehat{I}_{\lambda,\tilde{x}}(v) = \frac{1}{2}\int_{\mathcal{C}_{A_{\lambda R,\lambda r}(\tilde{x})}} y^{1-2\alpha}|\nabla v|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{A_{\lambda R,\lambda r}(\tilde{x})} |v(x,0)|^2 \mathrm{d}x \\ - \int_{A_{\lambda R,\lambda r}(\tilde{x})} H(v(x,0))\mathrm{d}x, \tag{4.1}$$

and set

$$\widehat{\mathcal{M}}_{\lambda,\tilde{x}} = \left\{ v \in H^1_{0,L}(\mathcal{C}_{A_{\lambda R,\lambda r}(\tilde{x})}, y^{1-2\alpha})\backslash\{0\};\ \widehat{I}'_{\lambda,\tilde{x}}(v)[v] = 0 \right\} \tag{4.2}$$

$$a(R,r,\lambda,\tilde{x}) = \inf\left\{ \widehat{I}_{\lambda,\tilde{x}}(v) : v \in \widehat{\mathcal{M}}_{\lambda,\tilde{x}} \text{ and } \beta(v) = \tilde{x} \right\}. \tag{4.3}$$

As is customary, when $\tilde{x} = 0$ we simply write $\widehat{I}_\lambda$, $\widehat{\mathcal{M}}_\lambda$ and $a(R,r,\lambda)$. We observe that the value $a(R,r,\lambda,\tilde{x})$ does not depend on the "center" $\tilde{x}$.

Since $\widehat{I}_{\lambda,\tilde{x}}$ is bounded from below on $\widehat{\mathcal{M}}_{\lambda,\tilde{x}}$ and satisfies the Palais–Smale condition, the infimum $a(R,r,\lambda,\tilde{x})$ is attained.

The next result will be useful in future estimates with the barycenter map.

Proposition 4.1. *The number $a(R,r,\lambda)$ satisfies*

$$\liminf_{\lambda\to\infty} a(R,r,\lambda) > c(\mathbb{R}^N).$$

Proof. From the definition of $a(R,r,\lambda)$ and $c(\mathbb{R}^N)$, we get

$$a(R,r,\lambda) > c(\mathbb{R}^N).$$

Suppose by contradiction that there exist $\lambda_n \to \infty$ such that $a(R,r,\lambda_n) \to c(\mathbb{R}^N)$. Since $a(R,r,\lambda_n)$ is achieved there exists $v_n \in \widehat{\mathcal{M}}_{\lambda_n}$ such that

$$\beta(v_n) = 0 \quad \text{and} \quad \widehat{I}_\lambda(v_n) = a(R,r,\lambda_n) \to c(\mathbb{R}^N).$$

Since $h \geq 0$, by (H0) and Lemma 5.1 it is $v_n \geq 0$ for all $n \in \mathbb{N}$. Moreover, since $v_n = 0$ on $\partial_L \mathcal{C}_{A_{\lambda_n R,\lambda_n r}}$, by considering the trivial extension on $\mathcal{C}_{\mathbb{R}^N}\backslash\mathcal{C}_{A_{\lambda_n R,\lambda_n r}}$ (which we denote with the same symbol), we obtain a function in $H^1_{0,L}(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$. Consequently,

$$v_n \rightharpoonup 0 \text{ in } H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha}),\ I_\infty(v_n) = a(R,r,\lambda_n) \to c(\mathbb{R}^N) \text{ and } v_n \in \mathcal{M}_\infty.$$

Recalling that $c(\mathbb{R}^N) > 0$, we have that $\{v_n\}$ is not strongly convergent. From Lemma 3.1, we get (recall $z = (x,y)$)

$$v_n(z) = w_n(z+z_n) + \mathfrak{w}_\infty(z+z_n)$$

where $\{w_n\} \subset H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a sequence converging strongly to 0, $\{z_n\} = \{(x_n,0)\} \subset \mathbb{R}^{N+1}$ is such that $|x_n| \to \infty$ and $\mathfrak{w}_\infty \in H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ is a nonnegative function verifying

$$I_\infty(\mathfrak{w}_\infty) = c(\mathbb{R}^N) \quad \text{and} \quad I'_\infty(\mathfrak{w}_\infty) = 0.$$

Due to the fact that I_∞ is rotationally invariant on functions of type $w(\cdot,0)$, we can assume that

$$z_n = (x^1_n, 0, 0, \ldots, 0) \text{ and } x^1_n < 0.$$

Now we set

$$M=\int_{\mathbb{R}^N}|\mathfrak{w}_\infty(x,0)|^2\mathrm{d}x>0.$$

Since $\|w_n\|_\alpha\to 0$, it follows that

$$\int_{B_{r\lambda_n/2}(x_n)}|w_n(x+x_n,0)+\mathfrak{w}_\infty(x+x_n,0)|^2\mathrm{d}x\to M,$$

from which we obtain

$$\int_{\Theta_n}|v_n(x,0)|^2\mathrm{d}x\to M,\quad \text{where}\quad \Theta_n=B_{r\lambda_n/2}(x_n)\cap A_{\lambda_nR,\lambda_nr}$$

and hence

$$\int_{\Upsilon_n}|v_n(x,0)|^2\mathrm{d}x\to 0,\quad \text{where}\quad \Upsilon_n=A_{\lambda_nR,\lambda_nr\backslash B_{\lambda_nr/2}(x_n)}. \tag{4.4}$$

From $\beta(v_n)=0$, we get

$$0=\int_{A_{\lambda_nR,\lambda_nr}}x^1|v_n(x,0)|^2\mathrm{d}x=\int_{\Theta_n}x^1|v_n(x,0)|^2\mathrm{d}x+\int_{\Upsilon_n}x^1|v_n(x,0)|^2\mathrm{d}x.$$

Thus,

$$-\frac{r\lambda_n}{2}(M+o_n(1))+R\lambda_n\int_{\Upsilon_n}|v_n(x,0)|^2\mathrm{d}x\geq 0$$

with $o_n(1)\to 0$. Then,

$$\int_{\Upsilon_n}|v_n(x,0)|^2\mathrm{d}x\geq\frac{rM}{2R}-o_n(1)$$

which contradicts (4.4), finishing the proof. □

The other auxiliary functional we need is $I_{B_\xi}:H^1_{0,L}(\mathcal{C}_{B_\xi},y^{1-2\alpha})\to\mathbb{R}$, where $\xi>0$, given by

$$I_{B_\xi}(v)=\frac{k_\alpha^{-1}}{2}\int_{\mathcal{C}_{B_\xi}}y^{1-2\alpha}|\nabla v|^2\mathrm{d}x\mathrm{d}y+\frac{1}{2}\int_{B_\xi}|v(x,0)|^2\mathrm{d}x-\int_{B_\xi}H(v(x,0))\mathrm{d}x. \tag{4.5}$$

This functional has a Mountain Pass Geometry and we denote with $c(B_\xi)$ the mountain pass level. If

$$\mathcal{M}_{B_\xi}=\left\{v\in H^1_{0,L}(\mathcal{C}_{B_\xi},y^{1-2\alpha})\backslash\{0\}:I'_{B_\xi}(v)[v]=0\right\}$$

denotes the Nehari manifold associated to I_{B_ξ}, then, as usual,

$$c(B_\xi)=\inf_{v\in\mathcal{M}_{B_\xi}}I_{B_\xi}(v). \tag{4.6}$$

Arguing as in Theorem 3.3 and using Schwartz symmetrization techniques, we get

Proposition 4.2. *The functional I_{B_ξ} defined in (4.5) satisfies the (PS) condition on $\mathcal{M}_{B_\xi}$. In particular, there exists a ground state solution $\mathfrak{w}_{B_\xi}\in\mathcal{M}_{B_\xi}$ and $\mathfrak{w}_{B_\xi}(\cdot,0)$ is radially symmetric with respect to the origin.*

The next result will be fundamental.

Proposition 4.3. *The numbers* $c(\Omega_\lambda)$ *and* $c(B_\xi)$*, defined, respectively, in* (2.5) *and* (4.6)*, verify the limits*

$$\lim_{\lambda\to\infty} c(\Omega_\lambda) = c(\mathbb{R}^N) \quad \text{and} \quad \lim_{\xi\to\infty} c(B_\xi) = c(\mathbb{R}^N).$$

Proof. Here we will just prove the first limit, since the second one follows from the same kind of arguments.

Let us fix a $\bar{\lambda} > 0$ and $R > 0$ such that $B_R \subset \Omega_\lambda$. By density, let $w_k \in C_c^\infty(\mathbb{R}_+^{N+1})$ be such that $w_k \to \mathfrak{w}_\infty$ in $H^1(\mathcal{C}_{\mathbb{R}^N}, y^{1-2\alpha})$ and let $\Phi_R \in C_c^\infty(\mathbb{R}_+^{N+1}), \Phi_R \geq 0$ such that $\Phi_R = 1$ on $\mathbb{D}_{R/2}$ and $\Phi_R = 0$ on $\mathbb{R}_+^{N+1} \backslash \mathbb{D}_R$. Here

$$\mathbb{D}_R = \{z = (x, y) \in \mathcal{C}_{\mathbb{R}^N} : |z| \leq R\}.$$

In particular, $\operatorname{supp}(\Phi_R w_k) \subset B_R$. Let $t_{R,k} > 0$ such that $\eta_{R,k} := t_{R,k}\Phi_R w_k \in \mathcal{M}_\lambda$. Then

$$c(\Omega_\lambda) \leq I_\lambda(t_{R,k}\Phi_R w_k) = I_\infty(t_{R,k}\Phi_R w_k) \ \text{ for all } \ \lambda \geq \bar{\lambda}. \tag{4.7}$$

Claim 1 : $\lim_{R\to\infty} t_{R,k} = t_k > 0$.

Indeed, since $t_{R,k}\Phi_R w_k \in \mathcal{M}_\lambda$, we get

$$\begin{aligned}
\|\Phi_R w_k\|^2_{\mathcal{C}_{\mathbb{R}^N}} &= k_\alpha^{-1} \int_{\mathcal{C}_{\mathbb{R}^N}} y^{1-2\alpha} |\nabla \eta_{R,k}|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} |\eta_{R,k}(x,0)|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} h(\eta_{R,k}(x,0)) t_{R,k}^{-1} \Phi_R(x,0) w_k(x,0) \mathrm{d}x \\
&\geq \int_{|x|\leq a} h(t_{R,k} m_k) t_{R,k}^{-1} m_k \mathrm{d}x,
\end{aligned}$$

where $m_k = \min_{|x|\leq a} \Phi_R(x,0) w_k(x,0) > 0$ and the ball of radius a is contained in $B_{R/2}$. It follows that $\{t_{R,k}\}_R$ has to be bounded by (H3) and we can assume $\lim_{R\to\infty} t_{R,k} = t_k \geq 0$. Moreover, if there exists $R_n \to \infty$ with $t_{R_n,k} \to 0$, recalling that $h(s) = o(|s|)$ near zero and (H1), for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}
\|\Phi_{R_n} w_k\|^2_{\mathcal{C}_{\mathbb{R}^N}} &= \int_{\mathbb{R}^N} h(\eta_{R_n,k}) t_{R_n,k}^{-1} \Phi_{R_n}(x,0) w_k(x,0) \mathrm{d}x \\
&\leq \varepsilon \int_{\mathbb{R}^N} |\Phi_{R_n},(x,0) w_k(x,0)|^2 \mathrm{d}x + C_\varepsilon t_{R_n,k}^{q-2} \int_{\mathbb{R}^N} |w_k(x,0)|^q \mathrm{d}x.
\end{aligned}$$

and the contradiction follows by passing to the limit as $R_n \to +\infty$ and using the arbitrariness of ε.

Then passing to the limit in R in (4.7)

$$c(\Omega_\lambda) \leq I_\infty(t_k w_k). \tag{4.8}$$

Claim 2: $\lim_{k\to+\infty} t_k = t_0 > 0$.

First observe, by passing to the limit in $R \to \infty$ in

$$\|t_{R,k}\Phi_R w_k\|^2_{\mathcal{C}_{\mathbb{R}^N}} = \int_{\mathbb{R}^N} h(t_{R,k}\Phi_R w_k) t_{R,k}\Phi_R w_k \mathrm{d}x$$

that $t_k w_k \in \mathcal{M}_\infty$. Then arguing as in Claim 1, the claim holds.

Passing to the limit in k in (4.8), we get

$$c(\Omega_\lambda) \le I_\infty(t_0\mathfrak{w}_\infty) \le I_\infty(\mathfrak{w}_\infty) = c(\mathbb{R}^N)$$

and hence

$$\limsup_{\lambda\to+\infty} c(\Omega_\lambda) \le c(\mathbb{R}^N). \tag{4.9}$$

On the other hand, by the definition of $c(\Omega_\lambda)$ and $c(\mathbb{R}^N)$, we get $c(\Omega_\lambda) \ge c(\mathbb{R}^N)$ for all $\lambda > 0$, which implies

$$\liminf_{\lambda\to\infty} c(\Omega_\lambda) \ge c(\mathbb{R}^N). \tag{4.10}$$

The conclusion follows by (4.9) and (4.10). $\square$

Before to proceed, we need to introduce other notations. Given $a \in (-\infty, +\infty]$, we set

- $I_\lambda^a := \left\{u \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : I_\lambda(u) \le a\right\}$, the a-sublevel of I_λ;
- $\mathcal{M}_\lambda^a := \mathcal{M}_\lambda \cap I_\lambda^a$.

Moreover, from now on we fix a real number $r > 0$ such that $B_r \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^N : d(x, \Omega) \le r\}$$

and

$$\Omega_r^- = \{x \in \Omega : d(x, \partial\Omega) \ge r\}$$

are homotopically equivalent to $\overline{\Omega}$; then $B_{\lambda r} \subset \Omega_\lambda$, so that $\mathcal{M}_\lambda^{c(B_{\lambda r})} \ne \emptyset$.

The next proposition will be of primary importance in order to apply the "barycenter method". We use the notation $\Omega_{\lambda,r}^+ = \lambda\Omega_r^+$.

Proposition 4.4. *There exists $\lambda^* > 0$ such that for all $\lambda \ge \lambda^*$,*

$$v \in \mathcal{M}_\lambda^{c(B_{\lambda r})} \implies \beta(v) \in \Omega_{\lambda,r}^+.$$

Proof. Suppose that there exist $\lambda_n \to \infty$, $v_n \in \mathcal{M}_{\lambda_n}^{c(B_{\lambda_n r})}$, that we may assume positive, such that

$$x_n := \beta(v_n) \notin \Omega_{\lambda_n,r}^+.$$

Fixing $R > \operatorname{diam}(\Omega)$, we have that

$$A_{\lambda_n R, \lambda_n r}(x_n) \supset \Omega_{\lambda_n}$$

and so, recalling (4.1)–(4.3),

$$a(R, r, \lambda_n) = a(R, r, \lambda_n, x_n) \le I_{\lambda_n}(v_n) \le c(B_{\lambda_n r}). \tag{4.11}$$

Sending $n \to \infty$ in (4.11) and using Proposition 4.3, it follows that

$$\limsup_{n\to\infty} a(R, r, \lambda_n) \le c(\mathbb{R}^N)$$

which contradicts Proposition 4.1. $\square$

For $\lambda > 0$, we define the injective operator $\Psi_{\lambda,r} : \Omega^-_{\lambda,r} \to H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ given, for every $\tilde{x} \in \Omega^-_{\lambda,r}$ by

$$[\Psi_{\lambda,r}(\tilde{x})](x,y) = \begin{cases} t_\lambda \mathfrak{w}_{B_{\lambda r}}(\tilde{x}-x,y) & \text{for } (x,y) \in \mathcal{C}_{B_{\lambda r}(\tilde{x})} \\ 0 & \text{for } (x,y) \in \mathcal{C}_{\Omega_\lambda \setminus B_{\lambda r}(\tilde{x})}, \end{cases}$$

where $\mathfrak{w}_{B_{\lambda r}}$ is the ground state solution given in Proposition 4.2 and $t_\lambda > 0$ is such that $\Psi_{\lambda,r}(\tilde{x}) \in \mathcal{M}_\lambda$, see (2.6). Note that for every $\tilde{x} \in \Omega^-_{\lambda,r}$, it holds

$$\beta(\Psi_{\lambda,r}(\tilde{x})) = \beta([\Psi_{\lambda,r}(\tilde{x})](\cdot,0)) = \tilde{x}$$

and since

$$I_\lambda(\Psi_{\lambda,r}(\tilde{x})) = I_{B_{\lambda r}}(t_\lambda \mathfrak{w}_{B_{\lambda r}}(\tilde{x}-\cdot,\cdot)) \leq I_{B_{\lambda r}}(\mathfrak{w}_{B_{\lambda r}}(\tilde{x}-\cdot,\cdot)) = c(B_{\lambda r}),$$

we infer also

$$\Psi_{\lambda,r}(\tilde{x}) \in \mathcal{M}_\lambda^{c(B_{\lambda r})}.$$

Then, we have

Lemma 4.5. *For $\lambda \geq \lambda^*$ given in Proposition 4.4, the composite map*

$$\Omega^-_{\lambda,r} \xrightarrow{\Psi_{\lambda,r}} \mathcal{M}_\lambda^{c(B_{\lambda r})} \xrightarrow{\beta} \Omega^+_{\lambda,r}$$

is well defined and coincides with the inclusion map of $\Omega^-_{\lambda,r}$ into $\Omega^+_{\lambda,r}$

The next result is a consequence of the above setting, but for the sake of completeness we give the proof. It is understood, from now on, that for λ^* we mean the one given in Proposition 4.4.

Proposition 4.6. *For every $\lambda \geq \lambda^*$, we have*

$$cat\, \mathcal{M}_\lambda^{c(B_{\lambda r})} \geq cat\, \Omega_\lambda.$$

Proof. Assume that $cat\, \mathcal{M}_\lambda^{c(B_{\lambda r})} = n$. This means that n is the smallest positive integer such that

$$\mathcal{M}_\lambda^{c(B_{\lambda r})} = \bigcup_{j=1}^n A_j,$$

where $A_j, j = 1,\dots,n$ are closed and contractible in $\mathcal{M}_\lambda^{c(B_{\lambda r})}$; that is, there exist $h_j \in C([0,1] \times A_j, \mathcal{M}_\lambda^{c(B_{\lambda r})})$ and fixed elements $w_j \in \mathcal{M}_\lambda^{c(B_{\lambda r})}$ such that

$$h_j(0,u) = u \ \text{ for all } u \in A_j \qquad \text{and} \qquad h_j(1,u) = w_j \text{ for all } u \in A_j.$$

Consider the closed sets $D_j = \Psi^{-1}_{\lambda,r}(A_j)$ and note that

$$\Omega^-_{\lambda,r} = \bigcup_{j=1}^n D_j.$$

Using the deformation $g_j : [0,1] \times D_j \to \Omega^+_{\lambda,r}$ given by

$$g_j(t,x) = \beta\Big((h_j(t,\Psi_{\lambda,r}(x))^+(\cdot,0)\Big),$$

we have for $j = 1, \ldots, n$ and $x \in D_j$

$$g_j(0, x) = \beta\Big((h_j(0, \Psi_r(x)))^+(\cdot, 0)\Big) = \beta\Big(\Psi_{\lambda,r}(x)(\cdot, 0)\Big) = x$$

and

$$g_j(1, x) = \beta\Big((h_j(1, \Psi_r(z)))^+(\cdot, 0)\Big) = \beta\Big(w_j(\cdot, 0)^+\Big) \in \Omega^+_{\lambda,r}.$$

This means that $D_j, j = 1, \ldots, n$ is contractible in $\Omega^+_{\lambda,r}$, hence $cat_{\Omega^+_{\lambda,r}} \Omega^-_{\lambda,r} \leq n$. The conclusion follows since $\Omega^+_{\lambda,r}$ and $\Omega^-_{\lambda,r}$ are homotopically equivalent to $\overline{\Omega}_\lambda$. □

5. Proof of Theorems 1.1 and 1.2

We remark here once for all the solutions we find in this section and in the next one are nonnegative in virtue of the following.

Lemma 5.1. *Let $\Gamma \subset \mathbb{R}^N$ be a smooth domain and $v \in H^1_{0,L}(\mathcal{C}_\Gamma, y^{1-2\alpha})$ such that*

$$\begin{cases} -\ div(y^{1-2\alpha}\nabla v) = 0 & in\ \mathcal{C}_\Gamma \\ v = 0 & on\ \partial_L \mathcal{C}_\Gamma \\ \dfrac{1}{k_\alpha}\dfrac{\partial v}{\partial y^\alpha}(x, 0) + v(x, 0) = f(x) & on\ \Gamma. \end{cases} \tag{5.1}$$

in the weak sense. If $f \geq 0$, then $v \geq 0$ in $\mathcal{C}_\Gamma$.

Proof. Since v satisfies (5.1), it follows that for all $\psi \in H^1_{0,L}(\mathcal{C}_\Gamma, y^{1-2\alpha})$, we have

$$k_\alpha^{-1} \int_{\mathcal{C}_\Gamma} y^{1-2\alpha} \nabla v \nabla \psi \mathrm{d}x\mathrm{d}y + \int_\Gamma v(x, 0)\psi(x, 0)\mathrm{d}x = \int_\Gamma f(x)\psi(x, 0)\mathrm{d}x.$$

If we take v^- (where $v = v^+ + v^-$) as a test function in the last expression, we get

$$k_\alpha^{-1} \int_{\mathcal{C}_\Gamma} y^{1-2\alpha} |\nabla v^-|^2 \mathrm{d}x\mathrm{d}y + \int_\Gamma |v^-(x, 0)|^2 \mathrm{d}x = \int_\Gamma f(x)v^-(x, 0)\mathrm{d}x \leq 0.$$

But this implies that $v^- \equiv 0$ and then $v \geq 0$. □

Let us fix $\lambda \geq \lambda^*$ given in Proposition 4.4. Since I_λ satisfies the Palais–Smale condition on $\mathcal{M}_\lambda$, applying the Ljusternik–Schnirelmann theory and Proposition 4.6, we get that I_λ on $\mathcal{M}_\lambda$ has at least $cat\, \Omega_\lambda$ critical points whose energy is less than $c(B_{\lambda r})$.

To get another solution, and then proving Theorem 1.2, we use the same ideas of [10]. Since Ω_λ is not contractible, the compact set $A := \overline{\Phi_{\lambda,r}(\Omega^-_{\lambda,r})}$ cannot be contractible in $\mathcal{M}_\lambda^{c(B_{\lambda r})}$. Moreover, as we have seen, functions on the Nehari manifold have to be positive on a set of nonzero measure.

In the following, for $u \in H^1_{0,L}(\Omega_\lambda, y^{1-2\alpha})\backslash\{0\}$ we denote with $t_\lambda(u) > 0$ the unique positive number such that $t_\lambda(u)u \in \mathcal{M}_\lambda$.

Take $u^* \in H^1_{0,L}(\Omega_\lambda, y^{1-2\alpha})$ such that $u^* \geq 0$, and $I_\lambda(t_\lambda(u^*)u^*) > c(B_{\lambda r})$. Consider the cone

$$\mathcal{K} := \Big\{tu^* + (1-t)u : t \in [0,1], u \in A\Big\}$$

(which is compact and contractible) and, since functions in $\mathcal{K}$ have to be positive on a set of nonzero measure, $0 \notin \mathcal{K}$. Then it makes sense to project the cone on the Nehari manifold

$$t_\lambda(\mathcal{K}) := \Big\{t_\lambda(w)w : w \in \mathcal{K}\Big\} \subset \mathcal{M}_\lambda$$

and consider the number

$$c := \max_{t_\lambda(\mathcal{K})} I_\lambda > c(B_{\lambda r}).$$

Since $A \subset t_\lambda(\mathcal{K}) \subset \mathcal{M}_\lambda$ and $t_\lambda(\mathcal{K})$ is contractible in $\mathcal{M}_\lambda^c$, we infer that also A is contractible in $\mathcal{M}_\lambda^c$. In conclusion, A is contractible in $\mathcal{M}_\lambda^c$, not contractible in $\mathcal{M}_\lambda^{c(B_{\lambda r})}$, and $c > c(B_{\lambda r})$; this is only possible, since I_λ satisfies the Palais–Smale condition, if there is a critical level between $c(B_{\lambda r})$ and c, that is, another solution to our problem.

6. Proof of Theorem 1.3

Before proving the theorem, we recall some basic facts of Morse theory and fix some notations. For a pair of topological spaces (X,Y), $Y \subset X$, let $H_*(X,Y)$ be its singular homology with coefficients in some field $\mathbb{F}$ (from now on omitted) and

$$\mathcal{P}_t(X,Y) = \sum_k \dim H_k(X,Y)t^k$$

the Poincaré polynomial of the pair. If $Y = \emptyset$, it will be always omitted in the objects which involve the pair. Recall that if H is an Hilbert space, $I : H \to \mathbb{R}$ a C^2 functional and u an isolated critical point with $I(u) = c$, the *polynomial Morse index* of u is

$$\mathcal{I}_t(u) = \sum_k \dim C_k(I,u)t^k$$

where $C_k(I,u) = H_k(I^c \cap U, (I^c \backslash \{u\}) \cap U)$ are the critical groups. Here $I^c = \{u \in H : I(u) \leq c\}$ and U is a neighborhood of the critical point u. The multiplicity of u is the number $\mathcal{I}_1(u)$.

It is known that for a non-degenerate critical point u (that is, the self-adjoint operator associated to $I''(u)$ is an isomorphism) it is $\mathcal{I}_t(u) = t^{\mathfrak{m}(u)}$, where $\mathfrak{m}(u)$ is the *(numerical) Morse index of* u: the maximal dimension of the subspaces where $I''(u)[\cdot,\cdot]$ is negative definite.

Coming back to our functional, we know that I_λ satisfies the Palais–Smale condition (see Lemma 3.2). Moreover, I_λ is of class C^2 and for $v, v_1, v_2 \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ it is

$$I''_\lambda(v)[v_1,v_2] = k_\alpha^{-1}\int_{\mathcal{C}_{\Omega_\lambda}} y^{1-2\alpha}\nabla v_1\nabla v_2\,\mathrm{d}x\mathrm{d}y \\ + \int_{\Omega_\lambda} v(x,0)w(x,0)\mathrm{d}x - \int_{\Omega_\lambda} h'(v(x,0))v_1(x,0)v_2(x,0)\mathrm{d}x.$$

So $I''_\lambda(v)$ is represented by the operator

$$\mathrm{L}_\lambda(v) := \mathrm{R}_\lambda(v) - \mathrm{K}_\lambda(v) : H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) \to \Big(H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})\Big)', \quad (6.1)$$

where $\mathrm{R}_\lambda(v)$ is the Riesz isomorphism and $\mathrm{K}_\lambda(v)$ is compact. Indeed let $v_n \rightharpoonup 0$ in $H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$ and $w \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha})$; using that $h'(0) = 0$ and (H1'), for a given $\xi > 0$ there exists some constant $C_\xi > 0$ such that

$$\int_{\Omega_\lambda} \Big|h'(v(x,0))v_n(x,0)w(x,0)\Big|\mathrm{d}x \le \xi \int_{\Omega_\lambda} |v_n(x,0)w(x,0)|\mathrm{d}x \\ + C_\xi \int_{\Omega_\lambda} |v(x,0)|^{q-2}|v_n(x,0)w(x,0)|\mathrm{d}x.$$

Using that $v_n \rightharpoonup 0$ and the arbitrariness of ξ, we get

$$\|\mathrm{K}_\lambda(v)[v_n]\| = \sup_{\|w\|_\alpha=1}\Big|\int_{\Omega_\lambda} h'(v(x,0))v_n(x,0)w(x,0)\mathrm{d}x\Big| \to 0.$$

In particular $\mathrm{L}_\lambda(v)$ is a Fredholm operator with index zero. Moreover, for $a \in (-\infty, +\infty]$, we set

- $\mathrm{Crit}_\lambda := \Big\{u \in H^1_{0,L}(\mathcal{C}_{\Omega_\lambda}, y^{1-2\alpha}) : I'_\lambda(u) = 0\Big\}$, the set of critical points of I_λ;
- $(\mathrm{Crit}_\lambda)^a := \mathrm{Crit}_\lambda \cap I^a_\lambda$;
- $(\mathrm{Crit}_\lambda)_a := \Big\{u \in \mathrm{Crit}_\lambda : I_\lambda(u) > a\Big\}$.

In the remaining part of this section, we will follow [6,9]. We will not give the proofs of the next Lemma 6.1 and Corollary 6.2 since they follows by general arguments.

Let $\lambda^* > 0$ as given in Proposition 4.4 and $\lambda \ge \lambda^*$ be *fixed* from now on. In view of Corollary 2.2, to prove Theorem 1.3 it is sufficient to show that I_λ restricted to $\mathcal{M}_\lambda$ has at least $2\mathcal{P}_1(\Omega_\lambda) - 1$ critical points.

First note that we can assume that $c(B_{\lambda r})$ is a regular value for I_λ. Otherwise, we can choose a $\rho \in (0, r)$ so that the new sets

$$\Omega^+_\rho = \{x \in \mathbb{R}^N; d(x,\Omega) \le \rho\} \quad \text{and} \quad \Omega^-_\rho = \{x \in \Omega; d(x,\partial\Omega) \ge \rho\}$$

are still homotopically equivalent to Ω, $c(B_{\lambda\rho}) > c(B_{\lambda r})$ and $c(B_{\lambda\rho})$ is a regular value; and we rename $c(B_{\lambda\rho})$ as $c(B_{\lambda r})$. Of course, we can also assume that Crit_λ is discrete. Since I_λ is bounded from below on $\mathcal{M}_\lambda$, let us say by a $\delta_\lambda > 0$, we have

$$(\mathrm{Crit}_\lambda)^{c(B_{\lambda r})} = \Big\{v \in \mathrm{Crit}_\lambda : 0 < \delta_\lambda < I_\lambda(v) \le c(B_{\lambda r})\Big\}$$

and $(\mathrm{Crit}_\lambda)^{c(B_{\lambda r})}$ and $(\mathrm{Crit}_\lambda)_{c(B_{\lambda r})}$ are (critical) the isolated sets covering Crit_λ.

By Lemma 4.5 and the fact that $(\Psi_{\lambda,r})_*$ induces monomorphism between the homology groups $H_*(\Omega^-_{\lambda,r})$ and $H_*(\mathcal{M}_\lambda^{c(B_{\lambda r})})$, it is standard to see that

$$\mathcal{P}_t(\mathcal{M}_\lambda^{c(B_{\lambda r})}) = \mathcal{P}_t(\Omega^-_{\lambda,r}) + \mathcal{Q}_t, \quad \mathcal{Q} \in \mathbb{P}, \tag{6.2}$$

where we are denoting with $\mathbb{P}$ the set of polynomial with nonnegative integer coefficients. Recall that $c(\Omega_\lambda) = \min_{\mathcal{M}_\lambda} I_\lambda$. As in [9, Lemma 5.2] (the proof just uses a topological lemma and a general deformation argument) one proves the following.

Lemma 6.1. *Let $d \in (0, c(\Omega_\lambda))$ and $l \in (d, +\infty]$ a regular level for I_λ. Then*

$$\mathcal{P}_t(I^l_\lambda, I^d_\lambda) = t\mathcal{P}_t(\mathcal{M}^l_\lambda).$$

From this lemma, (6.2) and the fact that $\pi_1(\mathcal{M}_\lambda) \approx \{0\}$, it follows that

$$\mathcal{P}_t(I_\lambda^{c(B_{\lambda r})}, I^d_\lambda) = t\Big(\mathcal{P}_t(\Omega^-_{\lambda,r}) + \mathcal{Q}_t\Big) \tag{6.3}$$

and

$$\mathcal{P}_t(H^1_{0,L}(y^{1-2\alpha}), I^d_\lambda) = t\mathcal{P}_t(\mathcal{M}_\lambda) = t. \tag{6.4}$$

Finally, we need the next result, whose proof is a matter of algebraic topology (see [6, Lemma 2.4] or [9, Lemma 5.6]).

Corollary 6.2. *We have*

$$\mathcal{P}_t(H^1_{0,L}(y^{1-2\alpha}), I_\lambda^{c(B_{\lambda r})}) = t^2\Big(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t - 1\Big), \quad \mathcal{Q} \in \mathbb{P}. \tag{6.5}$$

Then the Morse theory, (6.3), (6.4) and (6.5) give

$$\begin{aligned}\sum_{v \in (\mathrm{Crtit}_\lambda)^{c(B_{\lambda r})}} \mathcal{I}_t(v) &= \mathcal{P}_t(I_\lambda^{c(B_{\lambda r})}, I^d_\lambda) + (1+t)\mathcal{Q}'_t \\ &= t\Big(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t\Big) + (1+t)\mathcal{Q}'_t\end{aligned}$$

and

$$\begin{aligned}\sum_{v \in (\mathrm{Crtit}_\lambda)_{c(B_{\lambda r})}} \mathcal{I}_t(v) &= \mathcal{P}_t(H^1_{0,L}(y^{1-2\alpha}), I_\lambda^{c(B_{\lambda r})}) + (1+t)\mathcal{Q}''_t \\ &= t^2\Big(\mathcal{P}_t(\Omega_\lambda) + \mathcal{Q}_t - 1\Big) + (1+t)\mathcal{Q}''_t\end{aligned}$$

for some $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'' \in \mathbb{P}$. As a consequence, we obtain

$$\sum_{v \in \mathrm{Crtit}_\lambda} \mathcal{I}_t(v) = t\mathcal{P}_t(\Omega_\lambda) + t^2\Big(\mathcal{P}_t(\Omega_\lambda) - 1\Big) + (1+t)\mathcal{Q}_t \tag{6.6}$$

for a suitable $\mathcal{Q} \in \mathbb{P}$.

For a non-degenerate critical point v (that is, $\mathrm{L}_\lambda(v)$ given in (6.1) is an isomorphism) it is $\mathcal{I}_t(v) = t^{\mathfrak{m}(v)}$.

Then, if the solutions are non-degenerate, (6.6) easily gives the existence of at least $2\mathcal{P}_1(\Omega_\lambda) - 1$ solutions, completing the proof of Theorem 1.3.

Acknowledgements

The first author would like to thank UNESP - Presidente Prudente, specially to Professor Marcos Pimenta for his attention and friendship. This work was done while he was visiting that institution. The authors would like to thanks prof. Claudianor Alves for his suggestion to improve the proof of Proposition 4.3 and the anonymous referee for useful suggestions.

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Received: June 8, 2017.
Revised: April 9, 2018.
Accepted: May 23, 2018.