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Mascon gravitation model using a shaped polyhedral source

T. G. G. Chanut,⋆ S. Aljbaae⋆ and V. Carruba⋆

Univ. Estadual Paulista – UNESP, Grupo de Dinâmica Orbital & Planetologia, Guaratinguetá, CEP 12516-410, SP, Brazil

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ABSTRACT

In the last two decades, new computational tools have been developed in order to aid space missions to orbit around irregular small bodies. One of the techniques consists in rebuilding their shape in tetrahedral polyhedron. This method is well suited to determine the shape and estimate certain physical features of asteroids. However, a large computational effort is necessary depending on the quantity of triangular faces chosen. Another method is based on a representation of the central body in terms of mascons (discrete spherical masses). The main advantage of the method is its simplicity which makes the calculation faster. Nevertheless, the errors are non-negligible when the attraction expressions are calculated near the surface of the body. In this work, we carry out a study to develop a new code that determines the centre of mass of each tetrahedron of a shaped polyhedral source and evaluates the gravitational potential function and its first- and second-order derivatives. We performed a series of tests and compared the results with the classical polyhedron method. We found good agreement between our determination of the attraction expressions close to the surface, and the same determination by the classical polyhedron method. However, this agreement does not occur inside the body. Our model appears to be more accurate in representing the potential very close to the body's surface when we divide the tetrahedron in three parts. Finally, we have found that in terms of CPU time requirements, the execution of our code is much faster compared with the polyhedron method.

Key words: gravitation – methods: numerical – celestial mechanics – minor planets, asteroids: general.

1 INTRODUCTION

Since the 90's decade, only four probes were launched having as the main goal to study asteroids. In 1996, the American probe *Near-Shoemaker* sent images of the asteroid 253 Mathilde and in 2002 it approached and landed on the asteroid 433 Eros. In 2005 the Japanese Spacecraft *Hayabusa* reached the asteroid 25143 Itokawa, and began a period of vicinity operation about the body. ESA's *Rosetta* spacecraft flew past 21 Lutetia, at a distance of 3162 km on 2010 July 10. The *Dawn* spacecraft, launched by NASA, visited one of the largest asteroids of the main belt, 4 Vesta in 2012 and is currently visiting 1 Ceres. Future space missions as the *Ossiris-rex* mission that will be launched in 2016 September in direction to the asteroid (101955) Bennu will require new models and tools to predict and control the navigation and dynamical evolution of an orbiter around a very irregular body in its complex gravity field. Several methods have been employed to calculate the gravitational potential around these irregular shape bodies. The mascons approach, very simple from a conceptual point of view, has been devised in order to calculate the gravitational attraction of bodies with a very irregular shape. Here a body of given shape is approximated by a set of point masses (mascons) placed in a suitable way in order to reproduce the object mass distribution (Geissler et al. 1997). This method is easier to develop but has several deficiencies due to the fact that we replace the true body's continuous mass distribution with a field derived of spheres with a density approximately twice the nominal density and with about 48 per cent of the body being vacant. The Polyhedron method is another approach of precisely representing the shape consisting of abundant planar faces meeting along straight edges or at isolated point called vertices (Werner 1994). The polyhedral approach that describes the total volume of a constant density polyhedron can evaluate with a certain precision the gravitational field around a specific asteroid. The same approach using the harmonic expansion with a constant density polyhedron has as well been developed by Werner (1997). Werner & Scheeres (1997), combining these methods, have shown that the errors are larger when the attraction expressions are calculated by the mascons near the surface of (4769) Castalia if compared them with the spherical harmonics or the polyhedral approach. Rossi, Marzari & Farinella (1999) have also tested a number of faces/mascons

⋆ E-mail: thierry@feg.unesp.br (TGGC); safwan.aljbaae@obspm.fr (SA); vcarruba@feg.unesp.br (VC)

(i.e. 1521 faces and 5835 mascons) to describe an ellipsoid in the orbit propagation tests. They found that the mascons code was somewhat faster but the polyhedral code, especially for orbits getting close to the primary's surface, had a better accuracy. Nevertheless, singularities appear in the numerical evaluation of the polyhedral model. Tsoulis & Petrović (2001) refined the approach by presenting the derivation of certain singularity terms, which emerge at special locations of the computation point with respect to the attracting polyhedral source. However, it is necessary a large computational effort depending on the quantity of triangular faces chosen. In this paper, we carry out a new model of the representation of the mascons, develop a new code that determines the centre of mass of each tetrahedron of a shaped polyhedral source, and evaluate the gravitational potential function and its first- and second-order derivatives. We present the gravitational of an irregular body using two different models of mass concentrations in Section 2. In Section 3, we detail the numerical simulations and the sequence of the script. We also perform a series of tests for the asteroids (216) Kleopatra, (433) Eros, (4769) Castalia and (4179) Toutatis and the results of the comparison with the classical polyhedron method are presented in Section 4. Then, we discuss and conclude in Section 5.

2 GRAVITATIONAL POTENTIAL WITH THE AID OF POLYHEDRAL MODEL

2.1 Modelling an irregular body with polyhedron

The first paper that addressed the potential of three-dimensional bodies by the polyhedron method was developed by Werner (1994). Through data collection by the radio telescope of Arecibo, Puerto Rico, Ostro et al. (2000) have created polyhedron's designs for many asteroids considering a constant density (NASA Planetary Data System, 2004). The polyhedron is partitioned into a collection of simple tetrahedra. Each one, with one of the vertices at the origin and the opposite face represented by a trinomial with predefined orientation, is shown in Fig. 1. By analytical calculation, the volume of the tetrahedron is given by the sixth part of the scalar triple product of the vectors represented by three concurrent edges of this solid,

$$V = \frac{1}{6}(\boldsymbol{u} \times \boldsymbol{v} \cdot \boldsymbol{w}), \quad (1)$$

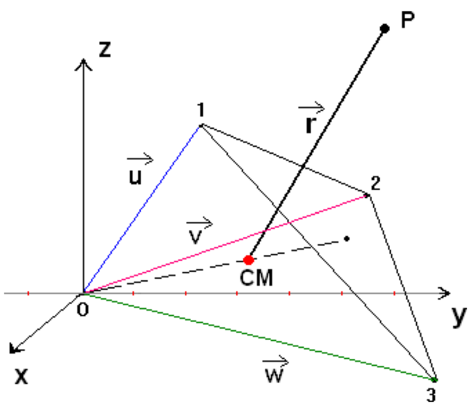

Figure 1. Representation of a tetrahedron with vertex 0 at the origin and the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ representing the three concurrent edges coming out of this vertex. The vector $\boldsymbol{r} = \boldsymbol{r}_{\mathrm{P}} - \boldsymbol{r}_{\mathrm{CM}}$ represent the distance between the centre of mass (CM) of the tetrahedron and the outer point P.

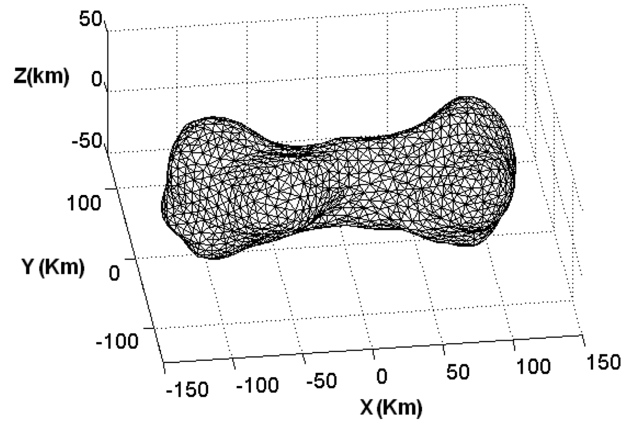

Figure 2. Polyhedron model 3D of asteroid (216) Kleopatra. The shape was built with 4092 faces.

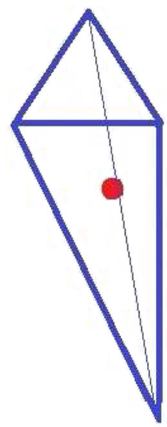

Figure 3. Mascon 1 model for a tetrahedron with the centroid in red (Venditti 2013).

where

$$\boldsymbol{u} \times \boldsymbol{v} \cdot \boldsymbol{w} = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix}. \quad (2)$$

We choose the data set of EAR-A-5-DDR-RADARSHAPE-MODELS-V2.0 from NASA Planetary Data System (2004) to build the polyhedral model with 2048 vertices and 4092 faces of asteroid (216) Kleopatra as illustrated in Fig. 2.

2.2 Two different models of mass concentrations using tetrahedra

(i) Mascon 1

Each triangular face is connected to the centre of the asteroid to form a tetrahedron. The centroid of each tetrahedron is determined, and the mass is proportional to the volume (e.g. Fig. 3).

(ii) Mascon 3

Each triangular face is connected to the centre of the asteroid to form a tetrahedron, which is divided into three parts in order to obtain three layers of volumes within each tetrahedron. The centroid of each figure is determined, and the mass is proportional to the volume of each figure enhanced in blue (e.g. Fig. 4).

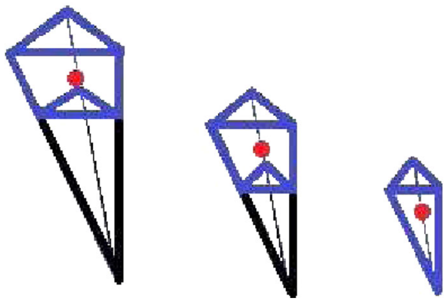

Figure 4. Mascon 3 model for the main tetrahedron divided into three parts in blue, with the centroid in red (Venditti 2013).

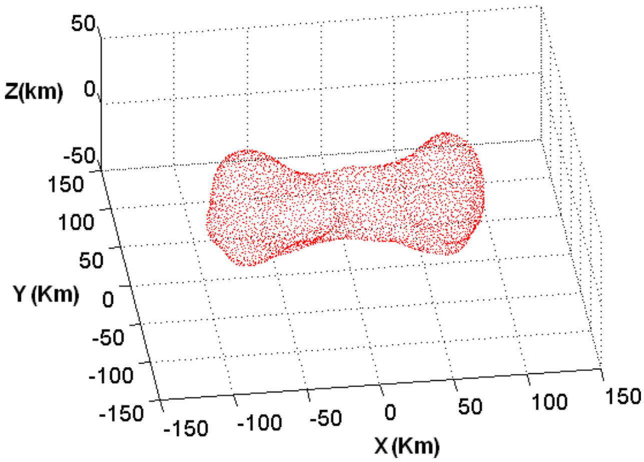

Figure 5. Centroids of the polyhedral shape of (216) Kleopatra with the Mascon 1 model.

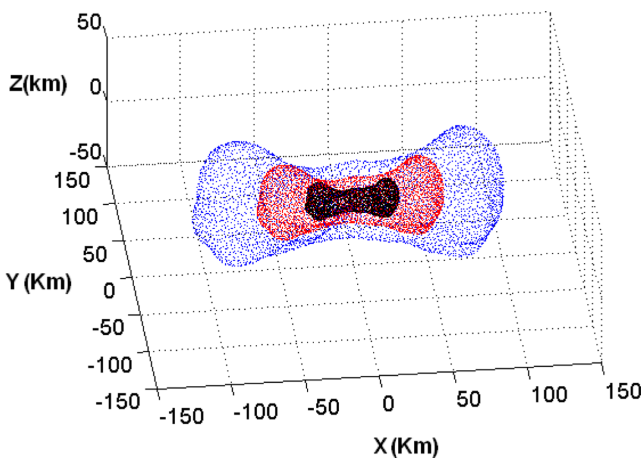

Figure 6. Centroids of the three volumes of each tetrahedron of the polyhedral shape of (216) Kleopatra with the Mascon 3 model.

Two test models were fitted to the data using the faces and the vertices of asteroid (216) Kleopatra and are shown in Figs 5 and 6. The points in the figures refer to the position according to the centroids of each model.

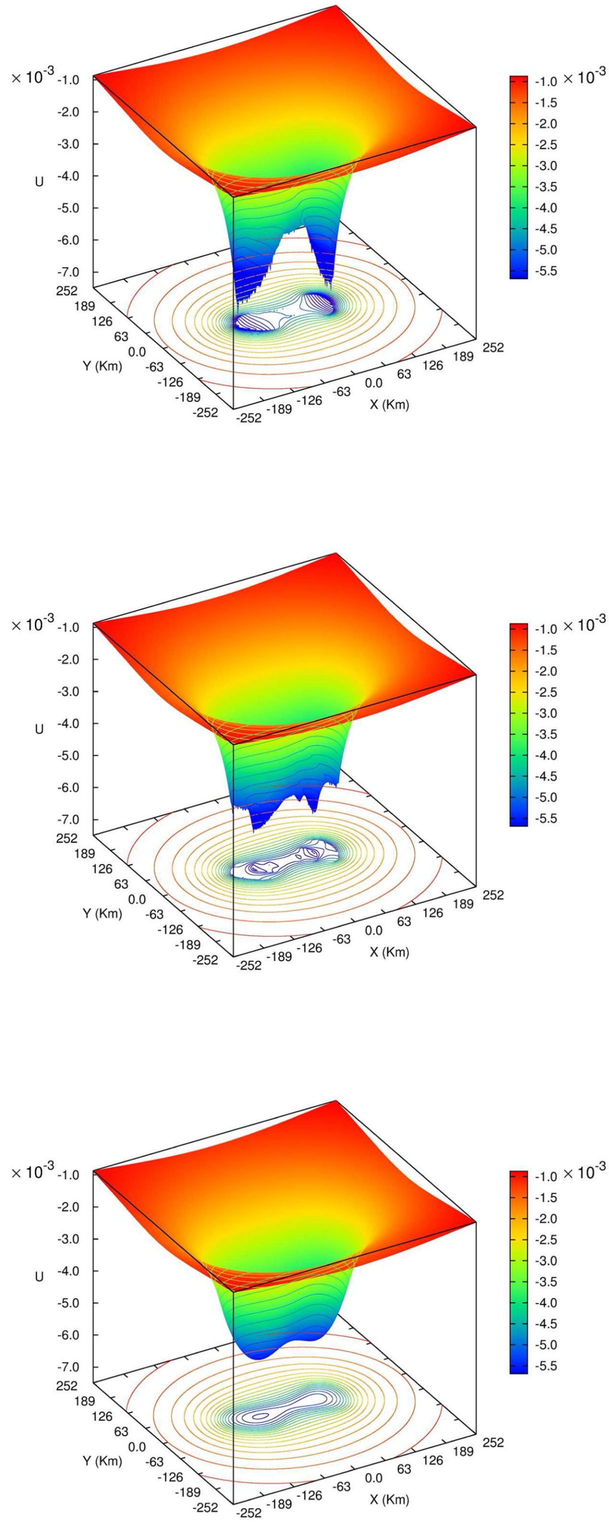

Figure 7. Gravitational potential of asteroid (216) Kleopatra computed by the methods: Mascon 1 (top), Mascon 3 (middle) and Tsoulis (bottom). The colour code gives the intensity of the potential in $km^2\ s^{-2}$.

2.3 Gravitational potential calculations

From Fig. 1, we can calculate the gravitational potential suffered by the external point P in relation to the tetrahedron

$$U_{\mathrm{T}} = \frac{\mu}{r}, \tag{3}$$

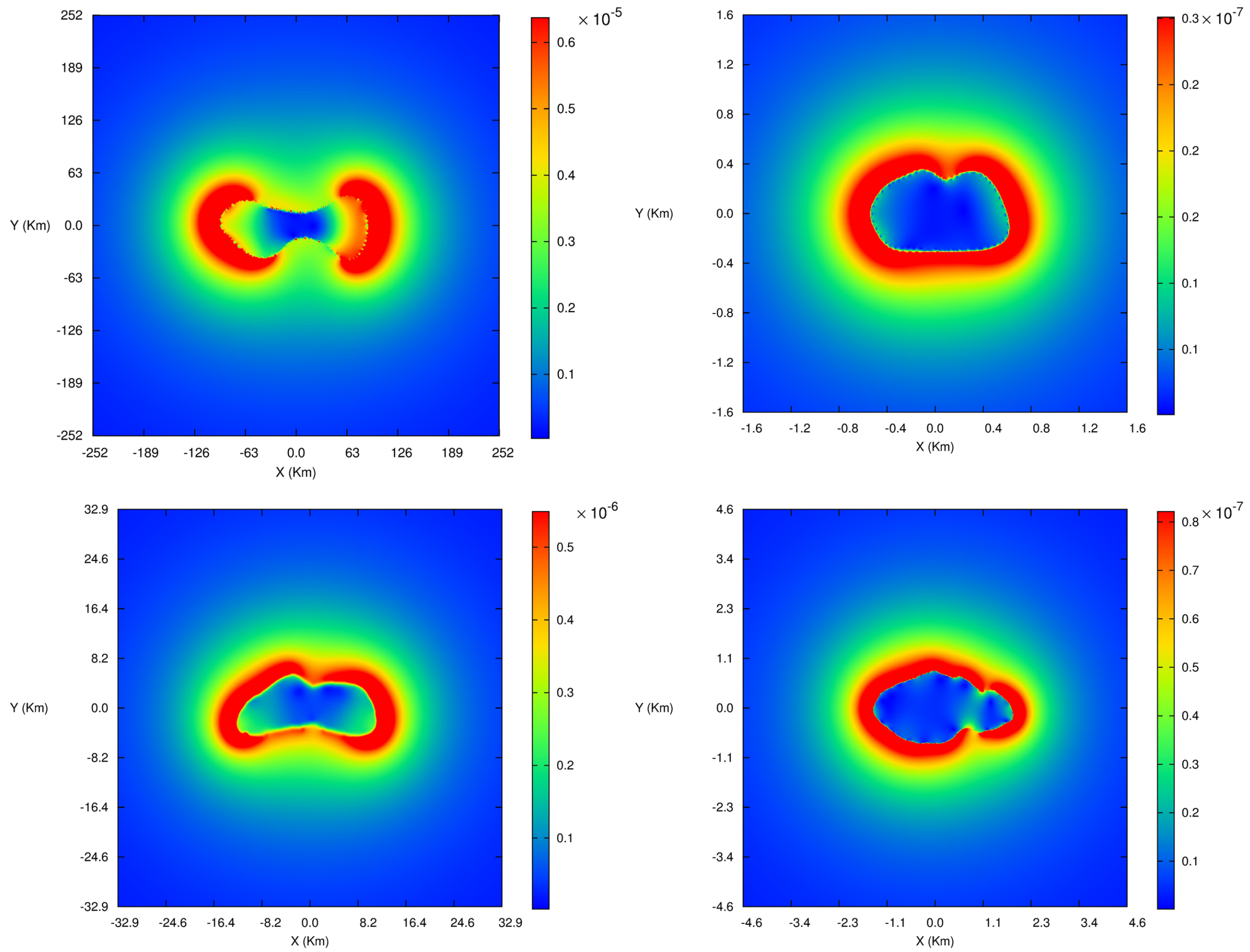

Figure 8. Intensity of the gravitational potential gradient computed using the model Mascon 1 close to asteroids (216) Kleopatra, (4769) Castalia, (433) Eros and (4179) Toutatis from the top to the bottom respectively. The colour code gives the intensity of the gravitational force in km s^{-2}.

where $r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$ is the distance between the centre of mass of the tetrahedron and the external point P with $\xi = x_{\rm P} - x_{\rm CM}$, $\eta = y_{\rm P} - y_{\rm CM}$ and so forth. Here, $\mu = GM_{\rm T}$ is the gravitational parameter of the tetrahedron with $G = 6.672\,59 \times 10^{-20}$ km^3 kg^{-1} s^{-2}. Thus, the potential, the first- and the second-order derivatives of the shaped polyhedral source are given by

$$U = \sum_{i=1}^{n} \frac{\mu_i}{r_i} \tag{4}$$

$$U_r = \sum_{i=1}^{n} \frac{\partial U}{\partial r_i} = \sum_{i=1}^{n} -\frac{\mu_i}{r_i^2}. \tag{5}$$

From equation (5), in terms of the coordinates ξ, η and ζ, we have

$$U_\xi = \sum_{i=1}^{n} \frac{\partial U}{\partial \xi_i} = \sum_{i=1}^{n} \left(\frac{\partial U}{\partial r_i}\right)\left(\frac{\partial r_i}{\partial \xi_i}\right) = \sum_{i=1}^{n} -\frac{\mu_i \xi_i}{r_i^3} \tag{6}$$

$$U_\eta = \sum_{i=1}^{n} \frac{\partial U}{\partial \eta_i} = \sum_{i=1}^{n} -\frac{\mu_i \eta_i}{r_i^3} \tag{7}$$

$$U_\zeta = \sum_{i=1}^{n} \frac{\partial U}{\partial \zeta_i} = \sum_{i=1}^{n} -\frac{\mu_i \zeta_i}{r_i^3}. \tag{8}$$

It follows from the equations (6), (7) and (8) that

$$U_{\xi\xi} = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial \xi_i^2} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial \xi_i}\right)\left(\frac{\partial U}{\partial \xi_i}\right) = \sum_{i=1}^{n} \frac{3\mu_i \xi_i^2}{r_i^5} \tag{9}$$

$$U_{\eta\eta} = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial \eta_i^2} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial \eta_i}\right)\left(\frac{\partial U}{\partial \eta_i}\right) = \sum_{i=1}^{n} \frac{3\mu_i \eta_i^2}{r_i^5} \tag{10}$$

$$U_{\zeta\zeta} = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial \zeta_i^2} = \sum_{i=1}^{n} \left(\frac{\partial}{\partial \zeta_i}\right)\left(\frac{\partial U}{\partial \zeta_i}\right) = \sum_{i=1}^{n} \frac{3\mu_i \zeta_i^2}{r_i^5}. \tag{11}$$

The sum represent the total quantity of tetrahedra used to build the shape where n is the number of faces and i the index of each face with their respective parameters.

3 NUMERICAL SIMULATIONS

The main goal of this work is to develop a new code in FORTRAN to model the external gravitational field of a small celestial body. As discussed above, our model consists in applying the Mascon gravity framework using a shaped polyhedral source, instead of replacing the body by a topologically different one composed of

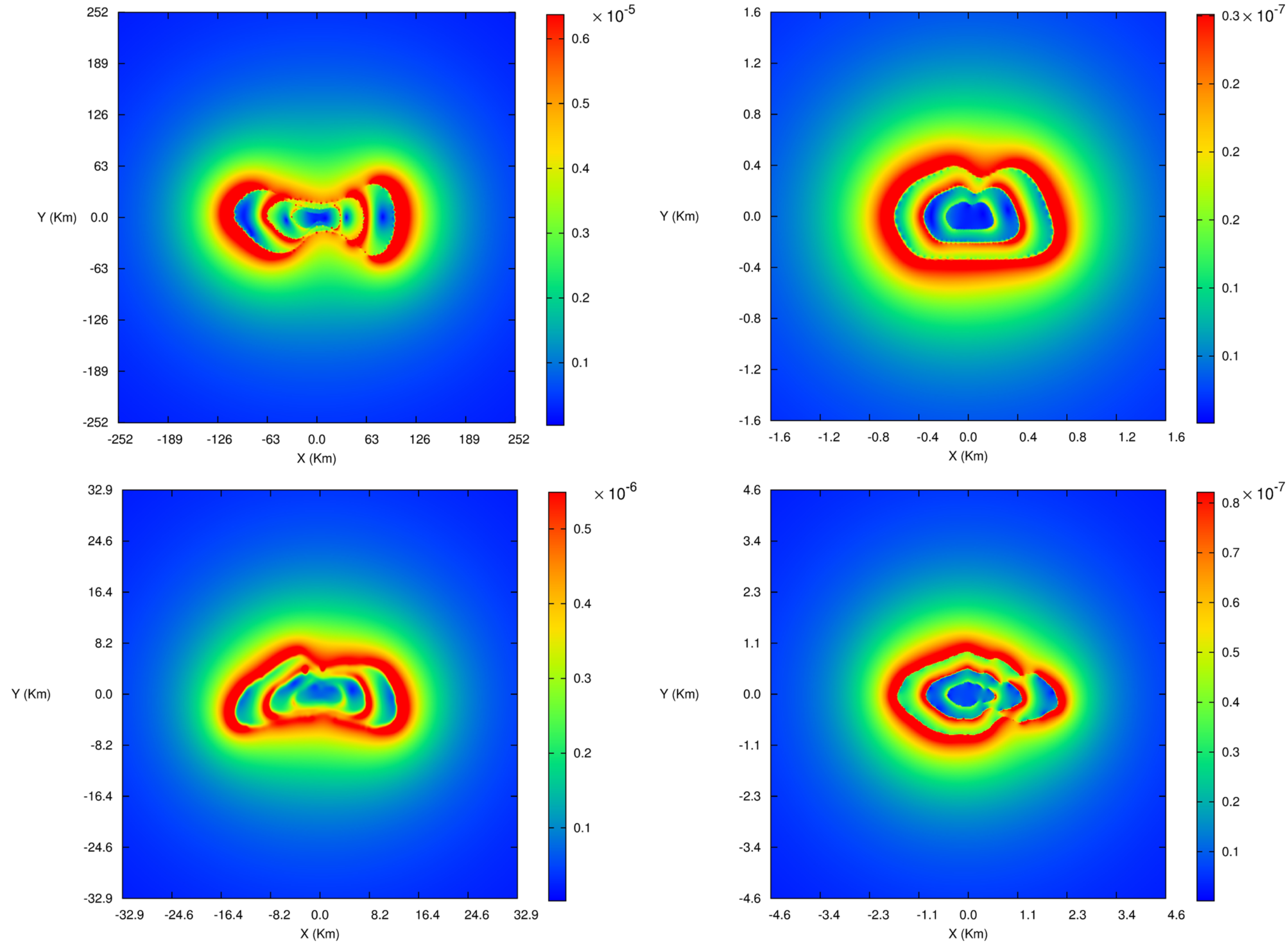

Figure 9. Intensity of the gravitational potential gradient computed using the model Mascon 3 close to the same asteroids shown in Fig. 8.

spherical spheres. In this way, we develop a new approach to evaluate the exterior gravitational potential and acceleration components of a homogeneous polyhedron with constant density. To build the polyhedral model of each studied asteroid in this paper, we use the surface observed by the *Near Earth Asteroid Rendezvous* mission (*NEAR*[1]) in the case of (433) Eros with 10 152 faces, or Small Body Radar Shape Models[2] in the case of (216) Kleopatra with 4092 faces, (4769) Castalia with 4092 faces and (4179) Toutatis with 39 996 faces. Our approach starts by applying the algorithm of Mirtich (1996) to calculate the polyhedral mass properties for each asteroid, considering a uniform bulk density of 4.27 g cm^{-3} for Kleopatra (Carry 2012), 2.65 g cm^{-3} for Eros (Chanut, Winter & Tshuchida 2014), 2.1 g cm^{-3} for Castalia (Scheeres et al. 1996) and 2.5 g cm^{-3} for Toutatis (Scheeres et al. 1998). We also calculate the barycentre and the momentum of inertia of the principal axes of each asteroid. A particular sequence of rotations is required to align the principal axes of inertia with the axes of coordinates. We use initially the algorithm of Tsoulis (2012) to calculate the gravitational field around the asteroids cited above. Done that, we find the coordinates of the barycentre of each volume with the possibility to divide each tetrahedron in three parts to obtain three layers of volume. Furthermore, using equations (4)–(11), we calculate the gravitational potential and its first and second derivatives referring to the two models presented in Section 2. This calculation is done at each point of a grid uniformly spaced in the equatorial plane with a width twice the length of the asteroid subdivided in 500 shares. Thus each grid consists of about 10^{6} points. Finally, we use the GNUPLOT software to generate all figures.

4 RESULTS OF THE COMPARISON TESTS

This section presents the comparative results of our set of asteroids between our models (Mascon 1 and 3) with the classical polyhedron model method (Tsoulis & Petrović 2001).

We apply our model to compute the potential and gravitational force of asteroids Kleopatra, Castalia, Eros and Toutatis at arbitrary points placed in the equatorial plane of each asteroid. The gravitational potential of Kleopatra is displayed in Fig. 7 while the intensity of the gravitational potential gradient close to each asteroid, also called gravitational force, is shown, respectively, in Figs 8, 9 and 10. It is important to notice that the all gravitational fields founded follow very well the shape of the concerning asteroids with the algorithm modelled by Tsoulis (2012) as shown in the bottom of Fig. 7 and in Fig. 10.

In a general way, we find that the difference in the potential and its gradient occurs between the three models in the interior and close to the surface of the body with some divergences. The difference in the intensity of the gravitational force is higher at the edges where the distance with the body's centre of mass is greater.

[1] http://sbn.psi.edu/pds/resource/nearmod.html
[2] http://sbn.psi.edu/pds/resource/rshape.html

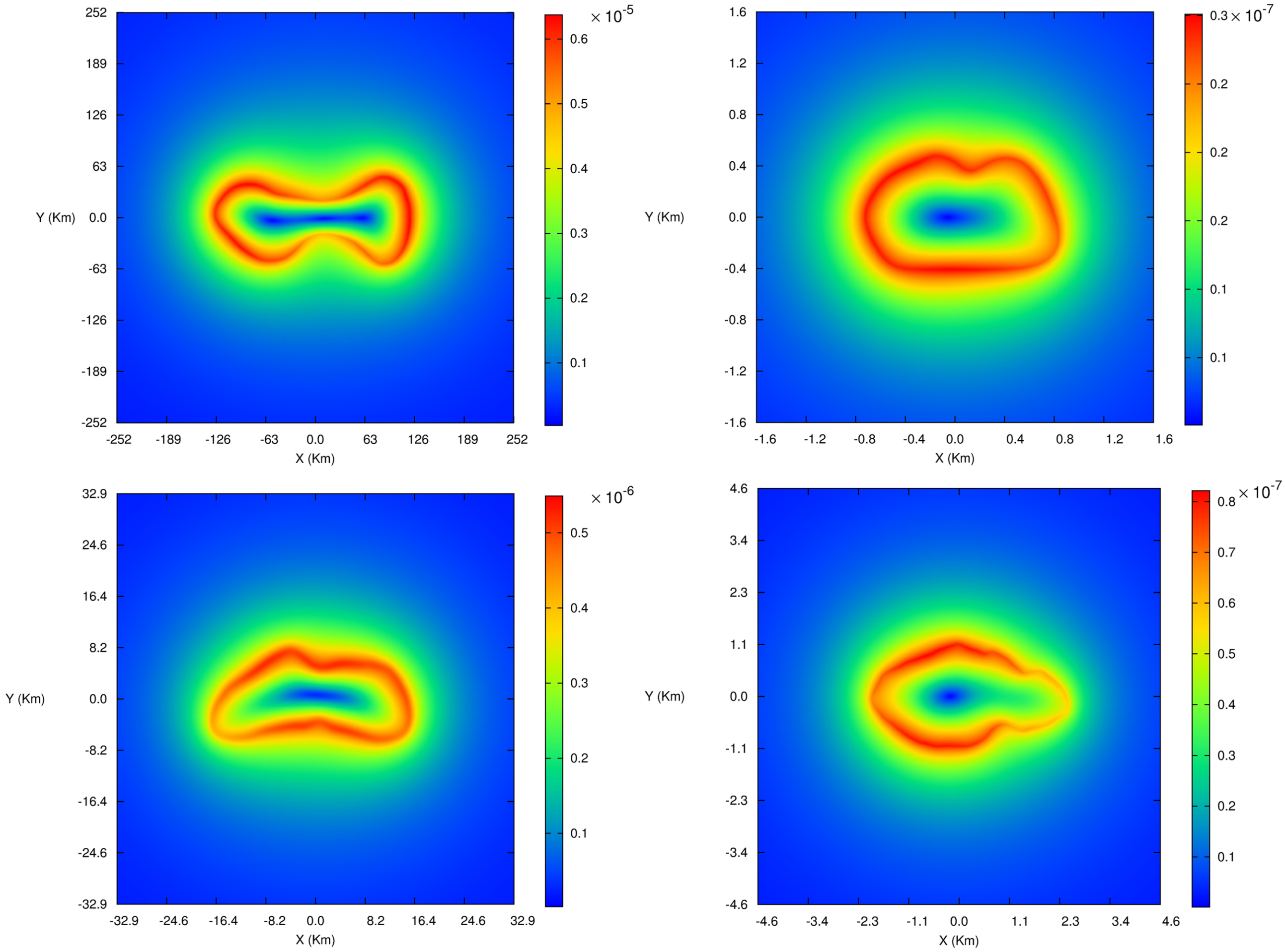

Figure 10. Intensity of the gravitational potential gradient computed using the classical polyhedron method of Tsoulis close to the same asteroids shown in Fig. 8.

This is particularly notable for very irregular and highly elongated bodies like Kleopatra as shows the Fig. 8. In the case of Castalia, the difference is more evenly distributed close to the surface. However, this uniformity vanishes in the case of asteroids Eros and Toutatis, respectively, displayed in the bottom panels of the figures cited above. Another point is that Fig. 8 represents the Mascon 1 model and Fig. 9 the Mascon 3 model for each asteroid. If we compare the two models with results of the Tsoulis approach method, we note that the difference close to the surface is higher with the Mascon 1 model while the difference inside the body seems to be lesser with the Mascon 3 model. In that way, it is interesting and challenging to compare our numerical results qualitatively with the traditional ones calculated by Tsoulis. In Fig. 11, we present, respectively, the relative errors between the potential estimated by the Mascon 1 model or Mascon 3, and the same potential estimated by Tsoulis for the four asteroids. For each studied body, the corresponding figures show good agreement between the estimations of the two Mascon models. However, this agreement does not occur inside the body particularly with the Mascon 1 model which confirms what we saw in Figs 8 and 9. We verify also that our model appears to be more accurate in representing the potential very close to the body's surface when we divide the tetrahedron in three parts (Mascon 3 model). On the other hand, if we compare the two graphics of the top of the Fig. 11, we also show that the error is lower for less irregular bodies. With a higher number of faces, as in the case of the bottom of the figure, the model appear to be somewhat more accurate.

Finally, In Table 1 we present the CPU time needed on computers Pentium 3.10 GHz and Pentium 2.27 GHz. It is worth mentioning that our code considerably reduce the computation processing time with respect to the polyhedron classical method, even if we divide the tetrahedron in three parts (three layers of volume). We also note that the CPU processing time depends on the number of faces but is not exactly proportional. Further the CPU processing time is from 20 up to 50 per cent larger between Mascon 3 and Mascon 1 models, while the Tsoulis method is 30 times slower than the Mascon 1 model with a computer Pentium 3.10 GHz. Using a slower computer this difference is even higher and reaches 40 times the processing speed between the Tsoulis method and the Mascon 1 model.

5 CONCLUSIONS

The main goal of this work was to develop a new FORTRAN code to model the external gravitational field of a small celestial body. Our model consists in applying the Mascon gravity framework using a shaped polyhedral source, instead of replacing the body by a topologically different one composed of spherical spheres. In this way, we have developed a new approach to evaluate the

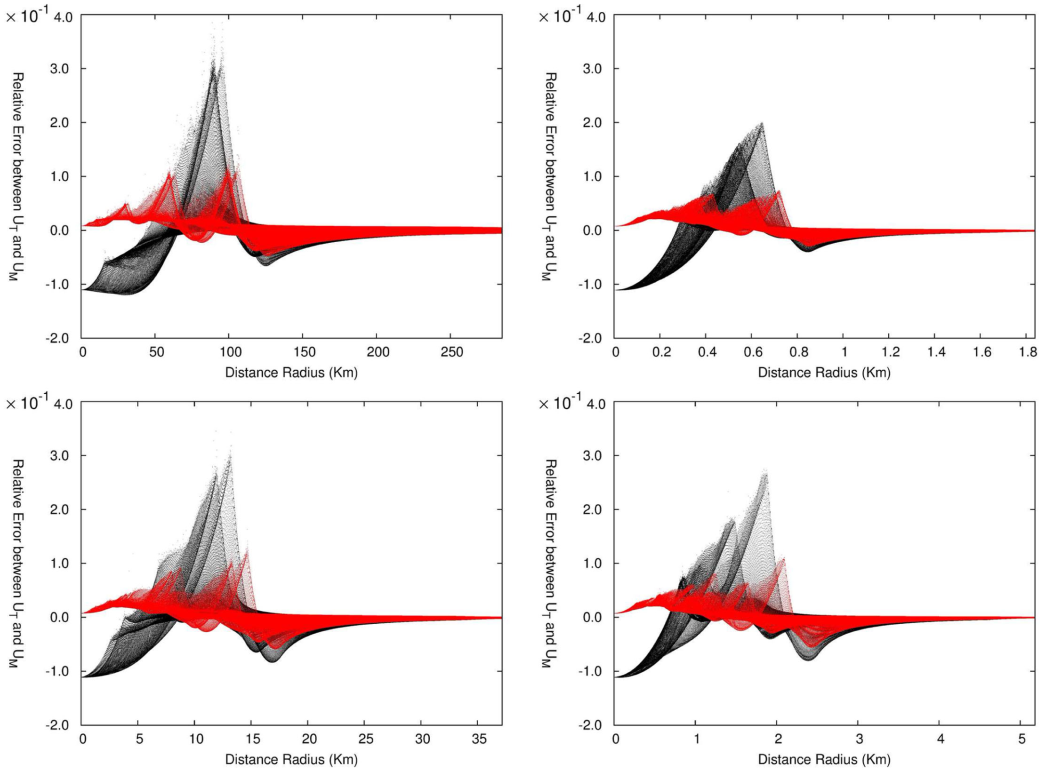

Figure 11. Relative error of the gravitational potential of the polyhedral model $U(r)$ in the latitude $\lambda \in (0^\circ, 360^\circ)$ with the model Mascon 1 (black asterisks) or the model Mascon 3 (red asterisks) for the asteroids (216) Kleopatra, (4769) Castalia, (433) Eros and (4179) Toutatis from the top to the bottom, respectively.

exterior gravitational potential and acceleration components of a homogeneous polyhedron with constant density. To build the polyhedral model of each studied asteroid in this paper, we used the surface observed by the *NEAR* mission in the case of (433) Eros, or Small Body Radar Shape Models in the case of (216) Kleopatra, (4769) Castalia and (4179) Toutatis. We have performed a series of tests for the asteroids Kleopatra, Castalia, Eros and Toutatis and compared the effectiveness and accuracy of our code with the classical polyhedron method of Tsoulis (2012). This calculation was done at each point of a grid uniformly spaced in the equatorial plane with a width twice the length of the asteroid subdivided in 500 shares.

We found very good agreement between our determination of the attraction expressions close to the surface, and the same determination by the classical polyhedron method. In a general way, we found that the major difference in the potential and its gradient occurs between the three models in the interior of the body and at the edges, where the distance with the body's centre of mass is greater. This is particularly notable for very irregular and highly elongated bodies like Kleopatra. The potential estimated close to surface by our model is more accurate than the results presented by Werner & Scheeres (1997) even if we compare them with the same potential estimated by the Tsoulis method. Furthermore, the potential calculated very close to the body's surface when we divide the tetrahedron in three parts significantly increases the accuracy. With a higher number of faces, as in the case of the bottom panel of the Fig. 11, the difference does not seem very significant. Finally, we have found that in terms of CPU time requirements our code considerably reduce the computation time with respect to the polyhedron classical method, even if we divide the tetrahedron in three parts.

We can conclude that this new approach of the Mascons solve two of the three deficiencies found by Werner & Scheeres (1997). Indeed very close to the surface the convergence is closer to the true gravity field for a given shape. This approach provides gravitational accuracy consistent with the accuracy of the shape determination. However, on the surface of the studied objects, it is advantageous to divide the body in three volume layers or more, while at a distance of 2.5 times the mean radius, the Mascon 1 model seems to be more appropriate. In a further work, we will test the new approach in integrations of the motion of a probe close to the target body of a future space mission.

Table 1. CPU time needed on a Pentium 3.10 GHz and 2.27 GHz computers.

Asteroid	(216) kleopatra		(433) Eros		(4769) Castalia		(4179) Toutatis	
CPU speed	3.10 GHz	2.27 GHz	3.10 GHz	2.27 GHz	3.10 GHz	2.27 GHz	3.10 GHz	2.27 GHz
Mascon 1	9 m 14 s	11 m 45 s	17 m 02 s	21 m 17 s	9 m 16 s	11 m 32 s	69 m 26 s	113 m 12 s
Mascon 3	11 m 37 s	19 m 48 s	25 m 55 s	36 m 28 s	11 m 39 s	19 m 46 s	105 m 07 s	194 m 58 s
Tsoulis	190 m 50 s	458 m 55 s	469 m 30 s	801 m 4 s	191 m 30 s	460 m 36 s	2338 m 12 s	4630 m 12 s

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