Effective theory for trapped few-fermion systems

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We apply the general principles of effective field theories to the construction of effective interactions suitable for few- and many-body calculations in a no-core shell model framework. We calculate the spectrum of systems with three and four two-component fermions in a harmonic trap. In the unitary limit, we find that three-particle results are within 10% of known semianalytical values even in small model spaces. The method is very general, and can be readily extended to other regimes, more particles, different species (e.g., protons and neutrons in nuclear physics), or more-component fermions (as well as bosons). As an illustration, we present calculations of the lowest-energy three-fermion states away from the unitary limit and find a possible inversion of parity in the ground state in the limit of trap size large compared to the scattering length. Furthermore, we investigate the lowest positive-parity states for four fermions, although we are limited by the dimensions we can currently handle in this case.

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I. INTRODUCTION

The properties of strongly interacting Fermi gases have been the object of great interest in recent years. Feshbach resonances allow the tuning of the interaction between trapped particles so that one can study the evolution from a dilute Fermi gas to a Bose-Einstein condensate. In three-dimensional optical lattices one can reach the low-tunneling regime where each site is an essentially isolated harmonic trap occupied by few fermions [1,2]. This opens up a new window into the study of few-body systems with two-body scattering lengths a_2 that are large compared to the range r_0 of the interaction, $|a_2| \gg r_0$.

Large-$|a_2|$ systems are of particular theoretical interest when particle momenta Q are small compared to $1/r_0$, because then they exhibit universal behavior, that is, the system properties depend essentially only on a_2 (and for bosons or multicomponent fermions, also on a three-body parameter), but not on the details of the interaction. (For a review, see Ref. [3].) The presence of a harmonic trap introduces another parameter: The trap frequency ω or, equivalently, the trap length $b=1/\sqrt{\mu\omega}$, where μ is the reduced mass of two particles. As long as $b \gg r_0$, the trapped system should still exhibit universal behavior, which for $b \lesssim |a_2|$ could be significantly different from that of the untrapped system. In the unitary limit $|a_2| \to \infty$, the untrapped two-body system has a bound state at zero energy, and a collection of two-state fermions is characterized purely by the parameter that sets the size of the system.

While large-$|a_2|$ systems have been popular in atomic physics mainly in the past decade, they have been investigated in nuclear physics since its beginning. The two-nucleon (NN) system has two S-wave channels where $|a_2| \gg r_0$: The scattering lengths are about 5 fm and -20 fm in the ${}^3S_1$ and ${}^1S_0$ channels, respectively, compared with a range of about 2 fm. Here we combine two methods previously developed to deal with untrapped particles—effective field theory (EFT) and no-core shell model (NCSM)—in order to provide solutions of few-fermion systems in a harmonic trap.

Since we are interested in the long-range dynamics, we can approximate the complicated short-range physics as a series of contact interactions that are delta functions with an increasing number of derivatives. This can be formulated as a nonrelativistic EFT where observables are expressed in an expansion in powers of Qr_0 (assuming, for simplicity, that the size of all scattering parameters except for the scattering length is set by r_0). The EFT in the untrapped two-body sector [4] reproduces the effective-range expansion and is equivalent to a pseudopotential, but can be extended to more-body systems [5,6]. It can be shown that in leading order the Hamiltonian for two-component fermions consists of a single, nonderivative, two-body contact interaction between the different components. For systems of identical bosons or more-component fermions, a nonderivative, three-body contact interaction is also present at this order. The EFT for short-range forces and its generalization for the exchange of light quanta are reviewed in Ref. [7].

The NCSM is a powerful many-body technique for solving the Schrödinger equation for A strongly interacting particles, where the many-body basis states are constructed using harmonic-oscillator (HO) wave functions. In nuclear physics, NCSM is used to describe properties of light nuclei without adjustable parameters [8,9]. Starting from interactions that describe the NN scattering phase shifts and selected few-nucleon properties with high accuracy, a unitary transformation is used to construct effective interactions in truncated spaces of a finite number of energy excitations on top of a minimum configuration. For the two-body problem, the maximum number of excitations is equivalent to the number $N_{\max}$ of shells included. After a truncation where higher-body terms in the effective interaction are neglected—the so-called cluster approximation—numerical diagonalization allows a good description of nuclear energy

spectra and other properties. Unfortunately, there is no *a priori* justification for the cluster approximation in the context of phenomenological interactions.

We have recently proposed [10] a combination of NCSM and EFT in which the cluster approximation is seen as a consequence of the EFT power counting. Instead of performing a unitary transformation on phenomenological interactions, we simply solve the EFT within the truncated space. We determine its parameters from some binding energies and then predict other bound-state properties by extrapolation to the limits $\omega \to 0$ and $N_{\max} \to \infty$. We have successfully applied the method in leading order to systems with $A \leq 6$. However, application beyond leading order becomes cumbersome because of the increased number of parameters that would have to be adjusted to properties of light nuclei. This motivates us to devise a more flexible approach, which will allow us to determine two-body parameters from two-body data.

In this paper, we present the new method and apply it to the case of two-component fermions in a harmonic trap. The two- [11] and three- [12,13] body systems with a pseudopotential in a trap have already been studied. We provide an alternative, explicit solution for these systems in a NCSM basis where ω is kept fixed but $N_{\max}$ is made large. This allows us to test the accuracy of our approach in a nontrivial system. In addition, we present a first solution of the four-body system. The paper is organized as follows. We introduce our approach to the renormalization of the many-body problem in Sec. II, and present our results in Sec. III. Conclusions and perspectives for future applications are discussed in Sec. IV.

II. MANY-BODY PROBLEM AND RENORMALIZATION OF THE INTERACTION

Here we consider a system of A identical two-component fermions of mass m, where the two-body scattering length a_2 is large compared to the range r_0 of the interaction, $|a_2| \gg r_0$. This system is assumed to be trapped in a harmonic potential of frequency ω, whose length parameter is also sufficiently large, $b \gg r_0$. We allow various values of the ratio b/a_2 ranging from the unitary limit $b/|a_2| \to 0$ to the untrapped case $b/|a_2| \to \infty$. (Note that we use units such that $\hbar = c = 1$.)

Under these conditions, details of the interparticle potential are irrelevant, and in leading order can be replaced by a nonderivative two-body contact interaction between different components. Without loss of generality we can refer to the two components as spin-1/2 states, in which case the contact interaction acts only in the ${}^1S_0$ channel. We denote its interaction strength by C_0. We start with the Hamiltonian for trapped particles,

$$H_A = \sum_{i=1}^{A} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 r_i^2 \right) + C_0 \sum_{[i<j]} \delta^{(3)}(\vec{r}_i - \vec{r}_j), \qquad (1)$$

where $\vec{r}_i$ and $\vec{p}_i$ represent the position and momentum, respectively, of particle i, and $[i<j]$ denotes a pair of particles with opposite spin. This Hamiltonian can be written as a sum of relative and center-of-mass (c.m.) motion, so that the energy of the system is the sum of internal (E) and c.m. ($E_{\text{c.m.}}$) terms:

$$E_{\text{tot}} = E_{\text{c.m.}} + E. \qquad (2)$$

The c.m. motion is that of a simple HO, so in the following we focus on the relative motion.

The internal Hamiltonian is not well defined as it stands, since the delta function is singular. Renormalization is necessary: After a truncation of the Hilbert space with a cutoff in energy or momentum, C_0 is taken to depend on the cutoff in such a way that finite, cutoff-independent results are obtained for observables. The contact interaction thus represents high-energy physics not incorporated explicitly within the space where the Hamiltonian is diagonalized. Note that part of the effect of the high-energy physics comes from its interference with low-energy physics. Here, it is simply that we need to account for high HO levels, so in addition to the high-energy cutoff C_0 also depends on ω. Because ω is an energy scale much below the cutoff energy, we expect a mild dependence on ω, which we observed in Ref. [10]. The role of ω resembles that of the pion mass in chiral EFT [7]: They set the scale of the corresponding long-range potentials and, through renormalization, contaminate all short-distance parameters. The many-body problem cannot in general be solved analytically, but renormalization can be incorporated easily in numerical calculations, which are formulated from the outset within a finite space.

These issues can be made explicit in the two-body system, where the relative motion of two particles of opposite spin is described by

$$H_2 = \frac{\omega}{2} \left[b^2 p^2 + \frac{r^2}{b^2} + 2\mu C_0 b^2 \delta^{(3)}(\vec{r}) \right] \qquad (3)$$

in terms of the relative coordinate $\vec{r}$ and the reduced mass $\mu = m/2$.

In the conventional NCSM approach, one chooses the model space by truncating in the number of shells included in the calculation. More precisely, one considers only HO states with the principal quantum number $N = 2n + l$ smaller than a given value $N_{\max}$. The effective Hamiltonian is then constructed via a unitary transformation U ($H_2^{\text{eff}} = U H_2 U^\dagger$), designed so that the lowest D eigenvalues of H_2 are exactly reproduced by H_2^{eff} (D is the dimension of the model space). Of course, the transformed Hamiltonian no longer has the form of the initial interaction and, in particular, additional nonlocal terms are induced even if one starts with a contact interaction.

The alternative approach proposed in Ref. [10] is based on principles of EFT, and it constructs the interaction between particles making use of power counting. Thus, in each model space one preserves the form of the interaction, as dictated by the power counting, and one determines the strength of each interaction strength as a function of $N_{\max}$ and ω so that some observables are exactly reproduced. For example, one can fix $C_0 = C_0(N_{\max}, \omega)$ such that the two-body ground-state energy is fixed.

In this paper we propose an intermediate approach. Thus, as in Ref. [10], we consider only the terms dictated by power counting, but instead of adjusting their strength to reproduce observables in the few-body system, we reproduce the ω-dependent energies of two-body states, as in the conven-

tional NCSM approach. The number of states whose energies are to be reproduced is fixed by the number of coupling constants to be adjusted, and not by the dimension of the model space, as in the conventional approach involving a unitary transformation. We note that, according to the general principles of EFT, even if initially we start with only a contact interaction, each truncation is going to induce additional correction terms. Effectively, the truncation of the space induces effective range, shape, and other parameters, which can be adjusted to the appropriate values as more derivatives of delta functions are included.

To renormalize the interaction, we consider the relative Hamiltonian (3) and solve the corresponding Schrödinger equation,

$$\left[b^2p^2+\frac{r^2}{b^2}+2\mu C_0 b^2\delta^{(3)}(\vec{r})\right]\psi(\vec{r})=2\frac{E}{\omega}\psi(\vec{r}), \tag{4}$$

in a finite model space defined by $0\leq 2n\leq N_{\max}$. Since the delta function acts only on S waves, we consider here only those waves. The solution $\psi(\vec{r})$ can be expanded in the complete set of HO wave functions

$$\psi(\vec{r})=\sum_{n=0}^{N_{\max}/2}A_n\phi_n(r), \tag{5}$$

with $\phi_n(r)$ the S-state solutions of the unperturbed HO,

$$\phi_n(r)=\frac{1}{\sqrt{4\pi}}\left(\frac{2n!}{b^3\Gamma(n+3/2)}\right)^{1/2}e^{-r^2/2b^2}L_n^{(1/2)}\left(\frac{r^2}{b^2}\right) \tag{6}$$

in terms of generalized Laguerre polynomials. One can easily verify that the eigenvalues of Eq. (4) are given by the consistency condition

$$\frac{2\pi b}{\mu C_0(N_{\max},\omega)}=-\frac{2}{\pi^{1/2}}\sum_{n=0}^{N_{\max}/2}\frac{L_n^{(1/2)}(0)}{(2n+3/2)-E/\omega}. \tag{7}$$

At this point, the coupling constant $C_0(N_{\max},\omega)$ is undetermined. We can fix the value of $C_0(N_{\max},\omega)$ at each $N_{\max}$ by requiring that we reproduce a given measured level $\tilde{E}(\omega)$ of two trapped fermions [2]. However, neglecting range corrections, such a level is determined by the two-body scattering length, a_2. The dimensionless interaction strength

$$\gamma_0(N_{\max},b/a_2)=\frac{\mu}{2\pi b}C_0(N_{\max},\omega) \tag{8}$$

can only depend on ω via the dimensionless ratio b/a_2. In fact, the experimental energy values are very close to the theoretical prediction of a pseudopotential, given [11] by the transcendental equation

$$\frac{\Gamma(3/4-E/2\omega)}{\Gamma(1/4-E/2\omega)}=\frac{b}{2a_2}. \tag{9}$$

[For discussions about the limits of applicability of Eq. (9), see Ref [14].] The solutions of Eq. (9) come in levels, and we can use any of the low-energy states to fix the value of $C_0(N_{\max},\omega)$. Here we use the most natural choice, the ground state. In Fig. 1 we plot the running of $\gamma_0(N_{\max},b/a_2)$ with Λb for several values of the b/a_2 ratio, where

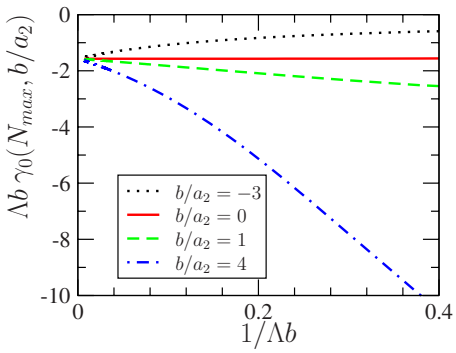

FIG. 1. (Color online) Running of the scaled interaction strength $\Lambda b\gamma_0=\mu\Lambda C_0/2\pi$ with the inverse of the ultraviolet cutoff (in units of the HO length) $1/\Lambda b$, for selected values of the ratio b/a_2.

$$\Lambda=\sqrt{2\mu(N_{\max}+3/2)\omega} \tag{10}$$

is the ultraviolet momentum cutoff defined in Ref. [10]. As the cutoff increases at fixed b, $\Lambda b\gamma_0=\mu\Lambda C_0/2\pi$ approaches $-\pi/2$ at a rate that depends on a_2, just as in the continuum case [4].

Once we adjust the strength of the two-body term for each $N_{\max}$ value (and, in general, ω), we can perform few- and many-body calculations. The many-body model spaces for $A>2$ are chosen so that the two-body truncation is included consistently in relative coordinates [8]. Note that if one wants negative-parity states, one has to truncate the many-body space to odd $N_{\max}$. In this case, we use a two-body interaction that is adjusted to the largest even number below $N_{\max}$, that is, $N_{\max}-1$.

Results for the energies of few-fermion systems are in general complicated functions of $N_{\max}$ and ω. In the unitary regime the dependence on ω gets simpler. When $b/|a_2|\to 0$, the two-body spectrum of Eq. (9) is given by $E_n/\omega=1/2+2n$, where $n=0,1,\ldots$. We see in Fig. 1 that for a large range of ultraviolet cutoffs $\Lambda b\gamma_0$ remains flat, so $\gamma_0(N_{\max},0)\simeq-\pi/2\sqrt{2N_{\max}}$. Since both μ and b disappear from Eq. (3) upon a rescaling $r\to b\rho$ (with ρ a dimensionless variable), the energy can only be proportional to ω. More generally, rescaling all coordinates in Eq. (1) will ensure that ω exactly factors out in all the many-body eigenenergies. This was to be expected, given that for $|a_2|\to\infty$ the only energy parameter is ω, which also sets the size of the system. It has been argued [15] that the proportionality constant between E and ω is related to the short-distance scaling exponent γ of the wave function. The three-fermion spectrum has been solved semianalytically in Ref. [13], while the ground-state energies for $A=2\to 22$ have recently been calculated in the case of a short-range potential [16].

In the opposite limit $b/|a_2|\to\infty$, the trap is removed. Equation (9) gives expected results. There is a single two-body bound state of energy $E_0=-1/2\mu a_2^2$, if the two-body interaction is sufficiently strong, $b/a_2\to\infty$. All other states have energies $E_n/\omega=-1/2+2n$, where $n=1,2,\ldots$, and correspond to scattering states. When $b/a_2\to-\infty$, the interaction is weak and the spectra of all few-body systems are expected to approach those of noninteracting fermions in a HO. This

can be seen from Eq. (7): As $b/a_2 \to -\infty$ the two-body ground-state energy approaches $3\omega/2$, leading to a divergence from the $n=0$ contribution and to $C_0(N_{\max}, \omega) \to 0$. In this limit all few-body energies should be set by occupation of HO levels. For example, all other two-body levels are given by the poles in the right-hand side of Eq. (7). Whether few-body bound states exist when the interaction is sufficiently attractive will be discussed below.

In any case, at the end of the calculation we extrapolate to the limit $N_{\max} \to \infty$. As we show in the next section, this extrapolation is relatively smooth. Our approach can be straightforwardly extended to higher orders. At the two-body level, we add to Eq. (1) interactions proportional to derivatives of the delta function, starting with an S-wave range correction of relative order $O(Qr_0)$ [4]. A P-wave interaction (the most important between like components) appears at $O((Qr_0)^3)$. For spin-1/2 fermions contact three- and more-body forces act in P and higher waves, so these forces are of relative order $O((Qr_0)^5)$ or higher [5]. Many-body properties are thus to a high degree determined by two-body forces alone. The two-body parameters can be determined from an extension of Eq. (9) to non-negligible-range interactions [14].

III. RESULTS

In this section, we apply the method introduced in the preceding section to the description of systems of three and four identical fermions in a harmonic trap. We consider the unitary regime ($|a_2| \to \infty$) as well as a general nonvanishing b/a_2 value, for both positive and negative scattering lengths.

In the three-body calculation, we have employed two numerical methods. In the first stage, we have used the three-body code in relative coordinates employed in Ref. [10]. Thus, to handle three identical fermions of spin-1/2, we allow interaction only in the ${}^1S_0$ channel, calculating the isospin $T=3/2$ solution. In a second stage, we have developed a program that can handle fermions of arbitrary spin, dropping the isospin from the possible quantum numbers. Both codes produce the same results for fermions of spin-1/2, but the latter can be extended to larger $N_{\max}$. We should point out that in order to correctly antisymmetrize the three-body system, we include all possible l states in each model space. Therefore, although we show below only results for the levels that are affected by the contact interaction, we also obtain energy levels that are left unchanged. (For example, states with $j^\pi = \frac{1}{2}^+$ can be obtained from addition of two $l=1$ relative angular momenta and total spin-1/2 or -3/2; however, in such channels the contact interaction is not present since it appears only for $l=0$.)

Figure 2 presents the main results of our investigation of the unitary regime. We show the dependence of the lowest-energy levels (labeled by their spin and parity, j^π) in units of ω as function of the ultraviolet cutoff $N_{\max}$. We have verified numerically that, indeed, ω factorizes, as we have argued in the preceding section. The running of the low-lying energies is fairly smooth when one increases the ultraviolet cutoff, and allows an extrapolation to the limit $N_{\max} \to \infty$. However, we find that the running depends on the state considered. In order to qualify this statement, we assume a running of the form

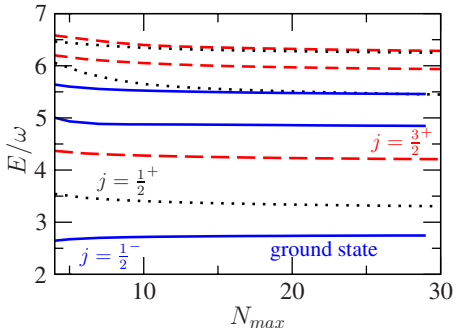

FIG. 2. (Color online) Dependence on the ultraviolet cutoff $N_{\max}$ of selected low-lying energies (in units of the HO frequency) of three harmonically trapped fermions of spin-1/2 in the unitary regime ($b/a_2=0$). We present the lowest three states of $j^\pi = \frac{1}{2}^-$ (continuous curves), $j^\pi = \frac{1}{2}^+$ (dotted curves), and $j^\pi = \frac{3}{2}^+$ (dashed curves).

$$E = E_\infty + \frac{E_c}{(N_{\max} + 3/2)^\alpha}, \tag{11}$$

with E_∞, E_c, and α fitting parameters. In a previous publication [10], we assumed a running for the energy of the many-body system of the form $E_0 + E_1/\Lambda$. This was motivated by the running in the continuum two-body system. Here we carry out a more detailed investigation of the running of the three-body solution with the ultraviolet cutoff. Allowing α to vary in the fit, we find values between 0.5 and 2.0. We find that, in general, the values of E_∞ are not very sensitive to the value of α, but can depend on the features of the running and the values of $N_{\max}$ included in the fit. For example, while in Fig. 2 the first $j^\pi = \frac{1}{2}^-$ excited state appears to be smooth, a closer examination reveals some structure, as shown in Fig. 3. In this figure we exhibit also three possibilities for fitting, where we include all points, only points with $N_{\max} \geq 15$, or only points with $N_{\max} \geq 21$. Since Eq. (11) is valid for large $N_{\max}$, our best guess for α is obtained in the case when we consider only $N_{\max} \geq 21$. In this case we find $\alpha \simeq 1.474$, and the extrapolated value for the energy of the state is $E_\infty/\omega \simeq 4.83$.

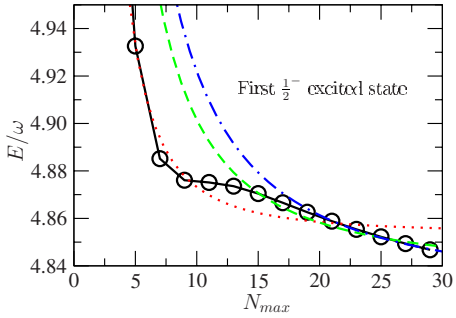

FIG. 3. (Color online) Running of the first $\frac{1}{2}^-$ excited state (circles), and three fits with the functional form in Eq. (11) taking into account all points (dotted line), only points with $N_{\max} \geq 15$ (dashed line), and only points with $N_{\max} \geq 21$ (dot-dashed line).

In Tables I–III we present our estimates for E_∞ for the states shown in Fig. 2. In our approach, there are two sources of errors: On one hand, we neglect higher-order terms in the expansion of the interaction (terms involving derivatives of contact interactions) and, on the other hand, we perform a fit of the form of Eq. (11) to obtain the large-$N_{\max}$ limit. The variation with $N_{\max}$ allows us to estimate errors associated with missing terms; thus, for low-lying states, even fairly small values of $N_{\max}$ produce results within about 10% of the extrapolated values. On the other hand, we find that, in general, the extrapolation errors coming from fits to large values of $N_{\max}$ are very small. For the levels shown in Tables I–III these extrapolation errors are beyond the last digit shown, except for second level in Table I, where the error is 2 in the last digit.

In the limit $N_{\max}\to\infty$, we should recover the semianalytical solutions obtained in Ref. [13] under the assumption of a two-body pseudopotential:

$$\frac{E}{\omega} = 1 + s + 2q, \tag{12}$$

where $q=0,1,\ldots$ and $s\geq 1.77$ is the (real) solution of a transcendental equation. The agreement between our numerical method and the semianalytical values, also shown in Tables I–III, is remarkable and provides a confirmation of the reliability of our method. As a side result, we note that the short-range scaling exponent [15] we obtain for three particles is $\gamma\simeq -0.24$. (While the virial theorem [15] is not satisfied necessarily in each model space, we expect it to be satisfied in the large-$N_{\max}$ limit.)

The successful test in the unitary regime encourages us to extend the application of our method for arbitrary values of the b/a_2 ratio. Experimentally, one can vary the scattering length of trapped particles by means of a magnetic field and, in principle, obtain a large range of ratios b/a_2. Thus, for illustration, we have considered a range of b/a_2 values, and fixed the coupling constant in each model space so that the ground-state energy given by Eq. (9) is always exactly reproduced.

In Fig. 4 we show the running of the lowest $j^\pi=\frac{1}{2}^-$ (ground state) energy of the trapped three-fermion system for $b/a_2=\pm 1$ in comparison with the unitary limit. We obtain different shifts of the energy level with respect to the unitary solution depending on the sign of a_2. If $a_2>0$, the level is pushed downward in energy with respect to the unitary value, while if $a_2<0$ the level is pushed upward. This behavior is a reflection of the same type of behavior in the two-body system: While in the unitary limit the two-body

TABLE I. Comparison between the results of the present approach (E_∞/ω) and of the semianalytical formula from Ref. [13] [Eq. (12)], for $j^\pi=\frac{1}{2}^-$.

n	l	q	s	Eq. (12)	E_∞/ω
0	1	0	1.77	2.77	2.76
0	1	1	1.77	4.77	4.83
1	1	0	4.36	5.36	5.39

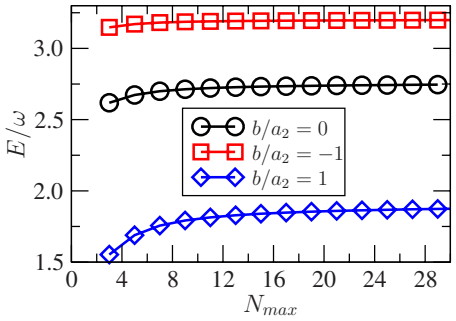

FIG. 4. (Color online) Ground-state energies (in units of the HO frequency) of the three-fermion system in a harmonic trap as function of the cutoff $N_{\max}$ for $b/a_2=0$ (circles), $b/a_2=-1$ (squares), and $b/a_2=1$ (diamonds).

ground state is $E_0/\omega=1/2$, the corresponding energy for $b/a_2=1$ decreases to $E_0/\omega\simeq -0.34$, and for $b/a_2=-1$ increases to $E_0/\omega\simeq 0.9$.

In Fig. 5 we consider both $j^\pi=\frac{1}{2}^-$ and $j^\pi=\frac{1}{2}^+$ states, and plot the lowest-energy level in each case as a function of b/a_2. We show only the extrapolated values E_∞ obtained by means of Eq. (11), taking into account only the points with $N_{\max}\geq 21$ to reduce fitting errors. For $b/a_2\lesssim 1.5$, the ground state is the same as in the unitary limit. However, we observe a tendency for an inversion of the parity of the ground state for $b/a_2\gtrsim 1.5$, although the positive- and negative-parity levels are nearly degenerate. Note that in the region with $a_2>0$ we might have larger errors from the fitting procedure, because the interaction strength increases with increasing values of the ratio b/a_2.

As discussed in the preceding section, for weak two-body attraction the spectrum should approach that of three noninteracting fermions in a HO trap—which is itself wide. In the limit $b/a_2\to-\infty$, the lowest state is a configuration with two particles in the first S state and the third in the first P level. Figure 5 indeed suggests the asymptotic value 4ω (5ω) for the energy of the negative (positive) -parity state.

On the other hand, for strong two-body attraction the relative effect of the contact interaction increases. In order to understand the possible parity inversion, we fit the $a_2>0$ results for E/ω in Fig. 5 to a quadratic function of b/a_2 (dashed line). In the untrapped $b/a_2\to\infty$ limit we find that the lowest positive- and negative-parity levels are very close, and their energy is

$$E \simeq -\frac{1}{2\mu a_2^2}. \tag{13}$$

Thus, within errors, the system of three fermions is at the threshold for the scattering of one particle on the bound state

TABLE II. Same as in Table I, but for $j^\pi=\frac{1}{2}^+$.

n	l	q	s	Eq. (12)	E_∞/ω
0	0	0	2.17	3.17	3.17
0	0	1	2.17	5.17	5.13
1	0	0	5.13	6.13	6.15

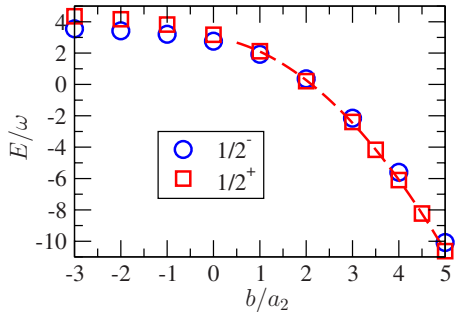

FIG. 5. (Color online) Lowest (extrapolated) energy levels for $j^\pi=\frac{1}{2}^-$ (circles) and $j^\pi=\frac{1}{2}^+$ (squares) states of the three-fermion system in a harmonic trap as function of the ratio b/a_2. Around $b/a_2\sim 1.5$, we observe an inversion of the parity of the system's ground state. The dashed line is a fit of the positive-parity points by a quadratic form.

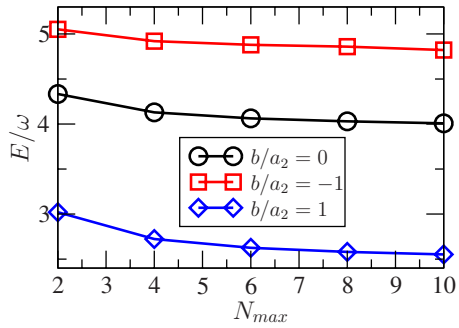

FIG. 6. (Color online) Same as Fig. 4 but for the $j^\pi=0^+$ states of the four-fermion system.

of the other two (dimer): The system is not bound, or very weakly bound, and can be viewed as composed of a dimer and an additional particle. The lack of a deep three-body bound state follows in fact from a very naive argument. The existence of a shallow two-body bound state is a consequence of a balance between the renormalized delta-function attraction and the kinetic repulsion. The addition of a third spin-1/2 fermion roughly doubles both, so we do not expect a collapse of the three-body system—as one does when the number of pair interactions grows faster than the number of particles, e.g., for identical bosons and multicomponent fermions.

The parity inversion of the ground state is therefore plausible. In the $b/a_2\to\infty$ limit, the positive-parity state likely represents the untrapped S-wave particle-dimer scattering state that one expects to dominate sufficiently close to threshold. This is consistent with results [17] for near-threshold particle-dimer scattering.

We have also solved the four-fermion problem in the trap. For now, we are limited to smaller model spaces than for three fermions because we use the REDSTICK Slater determinant code [18] that preserves only the third component of the angular-momentum projection, mixing states with good angular momentum. The dimension of the many-body basis thus increases significantly. Because we truncate the many-body configurations allowing only a maximum number of excitations on top of the minimum solution, we can eliminate spurious c.m. contributions when we compute the spectrum of the intrinsic motion. (The calculations in relative coordinates and in a properly truncated Slater determinant basis are, in fact, equivalent.) Moreover, because the Hamiltonian is rotationally invariant, we obtain eigenstates with good angular momentum (for four particles, the lowest positive-parity state has $j=0$), even though most of the individual Slater determinants that compose the many-body basis do not have good j.

TABLE III. Same as in Table I, but for $j^\pi=\frac{3}{2}^+$.

n	l	q	s	Eq. (12)	E_∞/ω
0	2	0	3.10	4.10	4.11
1	2	0	4.79	5.79	5.81
0	2	1	3.10	6.10	6.07

Figure 6 exhibits the running of the lowest positive-parity levels of four harmonically trapped spin-1/2 fermions, with the same parameters as in Fig. 4, $b/a_2=0,\pm 1$. Because of the present limitation to just a few values of $N_{\max}$ for the four-body system, our fitting procedure is likely to be subject to larger extrapolation errors, although Fig. 6 shows small variation with the ultraviolet cutoff. Thus, in the unitary limit, if in Eq. (11) we use $\alpha=1/2$, we estimate $E_\infty/\omega\sim 3.58$, while if we use $\alpha=1$, we obtain $E_\infty/\omega\sim 3.85$. (The small number of points does not allow us to obtain α reliably from the fit.) Both values are in line with previous calculations of the four-body system in the unitary limit [16,19]. From these results we can estimate the short-range scaling coefficient γ [15] for four particles between −0.65 and −0.92.

At nonvanishing b/a_2 we observe the same type of displacement in the lowest positive-parity level as for two- and three-fermion systems. However, our technical limitations in the four-body system do not allow us yet to make a more detailed investigation of a large range of ratios b/a_2. As expected, the system goes to the noninteracting case in the limit $b/a_2\to-\infty$; that is, the internal four-fermion ground-state energy is $13\omega/2$. As for three particles, we can extrapolate to the other limit $b/a_2\to\infty$ with a quadratic form in $b/a_2>0$. We find the binding energy in this limit to be about 40% below the threshold of scattering of two dimers. However, because our results are subject to large errors, we leave a more thorough analysis of four-body spectra in different regimes for future work using relative coordinates, when it should be possible to achieve larger ultraviolet cutoffs.

IV. CONCLUSIONS AND OUTLOOK

We have presented a first application of the NCSM to the description of three- and four-fermion systems in a harmonic trap. Based on the general principles of EFT, we have proposed a method to renormalize the two-body interaction to be used for few- and many-body calculations in finite model spaces. Tests of our results against semianalytical calculations in the unitary regime suggest good accuracy in estima-

tion of the energy levels of the three-fermion system. Moreover, we have extended the application of our method to arbitrary b/a_2 ratios.

There are, of course, ways to improve our results. In principle, even if a pseudopotential is a good approximation, in finite model spaces we can add a range correction to the contact term. This should improve the running of observables with the ultraviolet cutoff. Although in many cases the running is quite small, we could improve the calculations for $a_2>0$, where the interaction is stronger and requires larger model spaces to converge. (In the four-nucleon system we can follow the evolution of the contribution of different configurations. We find that the components with the lowest-energy configuration dominate for $a_2<0$, while other configurations reshuffle small contributions in each model space. However, this situation changes, with a larger contribution of the higher shells, for a stronger interaction such as for $a_2>0$. This is not surprising since a stronger interaction becomes more important across shells.) Adding higher-order terms should decrease the magnitude of E_c in Eq. (11), thereby allowing for a more precise determination of observables in few- and many-body systems.

In principle, we could extend our method even further. First, we can perform similar calculations of other observables (e.g., rms radii), with more particles, as well as for fermions of different spin. Especially for the latter it will be interesting to investigate the ability of the method to predict Efimov states in a finite model space. In the case of untrapped multistate fermions, the EFT power counting requires a contact three-body force to prevent the collapse of the three-body system [5].

Finally, the same type of approach can be considered in nuclear physics, where addition of a c.m. harmonic term to the intrinsic Hamiltonian produces effectively the same type of two-body relative Hamiltonian as in Eq. (3) [20].

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