

Quantum states, thermodynamic limits, and entropy in M theory

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We discuss the matching of the BPS part of the spectrum for a (super)membrane, which gives the possibility of getting the membrane's results via string calculations. In the small coupling limit of M theory the entropy of the system coincides with the standard entropy of type IIB string theory (including the logarithmic correction term). The thermodynamic behavior at a large coupling constant is computed by considering M theory on a manifold with a topology $T^2 \times R^9$. We argue that the finite temperature partition functions (brane Laurent series for $p \neq 1$) associated with the BPS p -brane spectrum can be analytically continued to well-defined functionals. It means that a finite temperature can be introduced in brane theory, which behaves like finite temperature field theory. In the limit $p \rightarrow 0$ (point particle limit) it gives rise to the standard behavior of thermodynamic quantities.

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I. INTRODUCTION

There are deep connections between a fundamental (super)membrane and (super)string theory. In particular, it has been shown that the Bogomol'nyi-Prasad-Sommerfield (BPS) spectrum of states for type IIB string on a circle is in correspondence with the BPS spectrum of fundamental compactified supermembrane [1,2]. Brane thermodynamics can indicate nontrivial information about microscopic degrees of freedom and the behavior of quantum systems at high temperature. Finite temperature M theory defined on a manifold with topology $T^2 \times R^9$, at small and large string coupling constant regimes, has been considered recently in [3,4]. In the small radius of compactification limit M theory recovers the type IIB superstring thermodynamics. In that case the critical temperature coincides with the Hagedorn temperature [3]. There is a first order phase transition at a temperature less than the Hagedorn temperature with a large latent heat leading to a gravitational instability [5].

The purpose of the present paper is to consider the above mentioned problems, comparing small and large coupling regimes by considering M theory on a manifold with topology $T^2 \times R^9$, where one of the sides of the T^2 torus is the Euclidean time direction (fermions obey antiperiodic boundary conditions). We turned to the problem of asymptotic density of quantum states for fundamental p -branes already initiated in [6–9].

This paper is organized as follows: In Sec. II the light-

cone Hamiltonian formalism for membranes wrapped on a torus is summarized. The small coupling limit of M theory is considered in Sec. III, while the limit of large coupling constant is analyzed in Sec. IV. We calculate the entropy associated with a string and argue that there is an interesting possibility allowing for a finite temperature being introduced into the brane theory. Section V summarizes our findings and discusses the relevant results.

II. TOROIDAL MEMBRANES

Let us consider the light-cone Hamiltonian formalism for membranes wrapped on a torus in Minkowski space. A compactification of M theory with $(-, +)$ spin structure, having the topology $T^2 \times R^9$, assumes that the dimensions X^{11}, X^{10} are compactified on a torus with radii R_{10}, R_{11} and two spatial membrane directions wind around this torus. The single-valued functions on the torus $X^{10}(\sigma, \rho), X^{11}(\sigma, \rho)$, where $\sigma, \rho \in [0, 2\pi)$, are the membrane world-volume coordinates:

$$X^{10}(\sigma, \rho) = m_0 R_{10} \sigma + \tilde{X}^{10}(\sigma, \rho),$$

$$X^{11}(\sigma, \rho) = R_{11} \rho + \tilde{X}^{11}(\sigma, \rho). \quad (1)$$

The eleven bosonic coordinates are $\{X^0, X^i, X^{10}, X^{11}\}$ and the transverse coordinates $X^i(\sigma, \rho)$, $i = 1, 2, \dots, 8$ are all single-valued. The transverse coordinates can be expanded in a complete basis of functions on the torus, namely

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$$X^i(\sigma, \rho) = \sqrt{\alpha'} \sum_{k, \ell} X_{(k, \ell)}^i e^{ik\sigma + i\ell\rho},$$

$$P^i(\sigma, \rho) = \frac{1}{(2\pi)^2 \sqrt{\alpha'}} \sum_{k, \ell} P_{(k, \ell)}^i e^{ik\sigma + i\ell\rho}. \quad (2)$$

In these equations $\alpha' = (4\pi^2 R_{11} T_2)^{-1}$, while T_2 is the membrane tension. The membrane Hamiltonian in light-cone formalism [3, 10–12] is $H = H_0 + H_{\text{int}}$, where for bosonic modes of membrane the Hamiltonian takes the form:

$$\alpha' H_0 = 8\pi^4 \alpha' T_2^2 R_{10}^2 R_{11}^2 m^2$$

$$+ \frac{1}{2} \sum_{\mathbf{n}} [P_{\mathbf{n}}^i P_{-\mathbf{n}}^i + \omega_{k\ell}^2 X_{\mathbf{n}}^i X_{-\mathbf{n}}^i], \quad (3)$$

$$\alpha' H_{\text{int}} = \frac{1}{4g_A^2} \sum (\mathbf{n}_1 \times \mathbf{n}_2)(\mathbf{n}_3 \times \mathbf{n}_4) X_{\mathbf{n}_1}^i X_{\mathbf{n}_2}^j X_{\mathbf{n}_3}^i X_{\mathbf{n}_4}^j. \quad (4)$$

In Eqs. (3) and (4) $\mathbf{n} \equiv (k, \ell)$, $\mathbf{n} \times \mathbf{n}' = k\ell' - \ell k'$, $g_A^2 \equiv R_{11}^2 (\alpha')^{-1} = 4\pi^2 R_{11}^3 T_2$, $\omega_{k\ell} = (k^2 + m^2 \ell^2 R_{10}^2 R_{11}^{-2})^{1/2}$, and $(m, k, \ell, k', \ell') \in \mathbb{Z}$. The interaction term (4) depends on the type IIA string coupling g_A . Mode operators, related to basic functions $X^i(\sigma, \rho), P^i(\sigma, \rho)$, are

$$X_{(k, \ell)}^i = \frac{1}{i\sqrt{2}\omega_{(k, \ell)}} [\alpha_{(k, \ell)}^i + \tilde{\alpha}_{(-k, -\ell)}^i],$$

$$P_{(k, \ell)}^i = \frac{1}{\sqrt{2}} [\alpha_{(k, \ell)}^i - \tilde{\alpha}_{(-k, -\ell)}^i], \quad (5)$$

$$(X_{(k, \ell)}^i)^\dagger = X_{(-k, -\ell)}^i, \quad (P_{(k, \ell)}^i)^\dagger = P_{(-k, -\ell)}^i, \quad (6)$$

and $\omega_{(k, \ell)} \equiv \text{sgn}(k)\omega_{k\ell}$. The canonical commutation relations read

$$[X_{(k, \ell)}^i, P_{(k', \ell')}^j] = i\delta_{k+k'}\delta_{\ell+\ell'}\delta^{ij},$$

$$[\alpha_{(k, \ell)}^i, \alpha_{(k', \ell')}^j] = \omega_{(k, \ell)}\delta_{k+k'}\delta_{\ell+\ell'}\delta^{ij}. \quad (7)$$

The similar relations hold for the $\tilde{\alpha}_{(k, \ell)}^i$. The mass operator becomes

$$M^2 = 2p^+ p^- - (p^i)^2 - p_{10}^2 = 2(H_0 + H_{\text{int}}) - (p^i)^2 - p_{10}^2. \quad (8)$$

The Hamiltonian of the membrane is nonlinear, but there are two situations where one can simplify this Hamiltonian (we shall consider these cases in the next sections):

- (i) *The limit $g_A \rightarrow 0$.*
- (ii) *The other limit of large g_A .*

III. ZERO TORUS AREA LIMIT OF M THEORY

The zero torus area limit of M theory on \mathbb{T}^2 leads to the asymptotic $g_A \rightarrow 0$ at fixed (R_{10}/R_{11}) . In M theory it gives a

ten-dimensional type IIB string. More precisely, it has been shown [1, 12] that quantum states of M theory describe the (p, q) strings bound states of type IIB superstring.

Let us consider string theory in Euclidean space (time coordinate X^0 is compactified on a circle of circumference β). The presence of coordinates compactified on circles gives rise to winding string states. The string single-valued function $X^0(\sigma, \tau)$ admits an expansion:

$$X^0(\sigma, \tau) = x^0 + 2\alpha' p^0 \tau + 2R_0 w_0 \sigma + \tilde{X}(\sigma, \tau), \quad (9)$$

where $p^0 = \ell_0(R_0)^{-1}$, $\ell_0, m_0 \in \mathbb{Z}$. The Hamiltonian and the level matching constraints become

$$H = \alpha' p_i^2 + \frac{m_0^2 R_0^2}{\alpha'} + \frac{\alpha' \ell_0^2}{R_0^2} + 2(N_L + N_R - a_L - a_R) = 0,$$

$$N_L - N_R = \ell_0 m_0, \quad (10)$$

where a_L, a_R are the normal ordering constants, which represent the vacuum energy of the $(1+1)$ -dimensional field theory. In the case of type II superstring the number operators in the $m_0 = \pm 1$ sector read

$$N_L = \sum_{n=1}^{\infty} \left[\alpha_{-n}^i \alpha_n^i + \left(n - \frac{1}{2} \right) S_{-n}^a S_n^a \right],$$

$$N_R = \sum_{n=1}^{\infty} \left[\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \left(n - \frac{1}{2} \right) \tilde{S}_{-n}^a \tilde{S}_n^a \right], \quad (11)$$

where $a = 1, \dots, 8$. The normal-ordering constants are the same as in the NS sector of the NSR formulation, i.e. $a_L = a_R = 1/2$.

A. The entropy in type II string theory

To begin our discussion of the entropy in string theory we recall that the semiclassical quantization of p -branes, compactified on a manifold with topology $\mathfrak{M} = \mathbb{T}^p \times \mathbb{R}^{D-p}$, leads to the “number operators” $N_{\mathbf{n}}$ with $\mathbf{n} = (n_1, \dots, n_p) \in \mathbb{Z}^p$. Therefore, let us consider multi-component versions of the classical generating functions for partition functions, namely

$$\mathfrak{G}_{\pm}(z) = \prod_{\mathbf{n} \in \mathbb{Z}^p \setminus \{0\}} [1 \pm \exp(-z\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g}))]^{\pm \Lambda}, \quad (12)$$

where $\Re z > 0$, $\Lambda > 0$, $\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g})$ is given by $\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g}) = (\sum_{\ell} a_{\ell}(n_{\ell} + g_{\ell})^2)^{1/2}$, g_{ℓ} , and a_{ℓ} are some real numbers.

In the context of thermodynamics of fundamental p -branes, classical generating functions $\mathfrak{G}_{\pm}(z)$ can be regarded as a partition function associated to Fermi (or Bose) modes, where $z \equiv \beta$ is the inverse temperature. In order to calculate the thermodynamic quantities we need first to know the total number of quantum states which can be described by the functions $\Omega_{\pm}(N)$ defined by

$$\mathcal{K}_{\pm}(t) = \sum_{N=0}^{\infty} \Omega_{\pm}(N) t^N \equiv \mathfrak{G}_{\pm}(-\log t), \quad (13)$$

where $t < 1$, and N is a total quantum number. The Laurent inversion formula associated with the above definition has the form

$$\Omega_{\pm}(N) = \frac{1}{2\pi i} \oint dt t^{-N-1} \mathcal{K}_{\pm}(t), \quad (14)$$

where the contour integral is taken on a small circle about the origin. The p -dimensional Epstein zeta function $Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)$ associated with the quadratic form $\varphi[\mathbf{a}(\mathbf{n} + \mathbf{g})] = (\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g}))^2$ for $\Re z > p$ is given by the formula

$$Z_p \left| \frac{\mathbf{g}}{\mathbf{h}} \right| (z, \varphi) = \sum'_{\mathbf{n} \in \mathbb{Z}^p} (\varphi[\mathbf{a}(\mathbf{n} + \mathbf{g})])^{-z/2} e^{2\pi i(\mathbf{n}, \mathbf{h})}, \quad (15)$$

where $(\mathbf{n}, \mathbf{h}) = \sum_{i=1}^p n_i h_i$, h_i are real numbers and the prime on Σ' means to omit the term $\mathbf{n} = -\mathbf{g}$ if all the g_i are integers. For $\Re z < p$, $Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)$ is understood to be the analytic continuation of the right-hand side of the Eq. (15). The functional equation for $Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)$ reads

$$Z_p \left| \frac{\mathbf{g}}{\mathbf{h}} \right| (z, \varphi) = \frac{\pi^{(1/2)(2z-p)} \Gamma\left(\frac{p-z}{2}\right)}{(\det \mathbf{a})^{1/2} \Gamma\left(\frac{z}{2}\right)} \times e^{-2\pi i(\mathbf{g}, \mathbf{h})} Z_p \left| \frac{\mathbf{h}}{-\mathbf{g}} \right| (p-z, \varphi^*), \quad (16)$$

and $\varphi^*[\mathbf{a}(\mathbf{n} + \mathbf{g})] = \sum_{\ell} a_{\ell}^{-1} (n_{\ell} + g_{\ell})^2$. Equation (16) gives the analytic continuation of the zeta function. Note that $Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)$ is an entire function in the complex z plane except for the case when all the h_i are integers. In this case $Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)$ has a simple pole at $z=p$ with residue $A(p) = 2\pi^{p/2} [(\det \mathbf{a})^{1/2} \Gamma(p/2)]^{-1}$, which does not depend on the winding numbers g_{ℓ} . Furthermore one has $Z_p|_{\mathbf{g}}^{\mathbf{g}}(0, \varphi) = -1$.

By means of the asymptotic expansion of $\mathcal{K}_{\pm}(t)$ for $t \rightarrow 1$, which is equivalent to the $\mathfrak{G}_{\pm}(z)$ expansion for small z , one arrives at a complete asymptotic limit of $\Omega_{\pm}(N)$ [6–9]:

$$\Omega_{\pm}(N)_{N \rightarrow \infty} = \mathcal{C}_{\pm}(p) N^{[2\Lambda Z_p|_{\mathbf{g}}^{\mathbf{g}}(0, \varphi) - p - 2]/2(1+p)} \exp\left\{\frac{1+p}{p} [\Lambda A(p) \Gamma(1+p) \zeta_{\pm}(1+p)]^{1/(1+p)} N^{p/(1+p)}\right\} [1 + \mathcal{O}(N^{-\kappa_{\pm}})], \quad (17)$$

$$\mathcal{C}_{\pm}(p) = [\Lambda A(p) \Gamma(1+p) \zeta_{\pm}(1+p)]^{[1 - 2\Lambda Z_p|_{\mathbf{g}}^{\mathbf{g}}(0, \varphi)]/(2p+2)} \frac{\exp[\Lambda(d/dz) Z_p|_{\mathbf{g}}^{\mathbf{g}}(z, \varphi)|_{(z=0)}]}{[2\pi(1+p)]^{1/2}}, \quad (18)$$

where $\zeta_{-}(z) \equiv \zeta_R(z)$ is the Riemann zeta function, $\zeta_{+}(z) = (1 - 2^{1-z})\zeta_R(z)$, $\kappa_{\pm} = p/(1+p) \min(\mathcal{C}_{\pm}(p)/p - \delta/4, 1/2 - \delta)$, and $0 < \delta < 2/3$. Using Eqs. (17) and (18) and assuming linear Regge trajectories, i.e. the mass formula $M^2 = N$ for the number of brane states of mass M to $M + dM$, one can obtain the asymptotic density for (super) p -brane states.

In fact, for linear Regge-like trajectories the partition function always diverges. This IR divergence in the partition function might be regularized by some effects of brane theory, for example, like imposing U-duality (see, for example, [13]) or choosing nonlinear behavior of Regge trajectory (say, $M^{(1+p)/p}$ or something similar).

These results can be used in the context of the brane method's calculation of the ground state degeneracy of systems with quantum numbers of certain BPS extreme black holes [14–17]. The brane picture gives the entropy in terms of partition functions $\mathfrak{G}_{\pm}(z)$ for a gas of species of massless quanta. In fact for unitary conformal theories of fixed central charge c Eq. (17) represents the degeneracy of the state $\Omega(N)$ with momentum N and for $N \rightarrow \infty$ one has [4, 18]

$$\mathcal{S}(N) = \log \Omega(N) \simeq \mathcal{S}_0 + \mathcal{A}(p, c) \log(\mathcal{S}_0), \quad (19)$$

where $\mathcal{S}_0 = \mathcal{A}_0 \sqrt{cN}$ and \mathcal{A}_0 is a real number. It gives the growth of the degeneracy of BPS solitons for $N \gg 1$. Note

that in the case of zero modes the dependence of the logarithmic correction $\mathcal{A}(p, c)$ on an embedding spacetime can be eliminate [19].

IV. LARGE STRING COUPLING LIMIT

We now focus on the case (ii) mentioned at the end of Sec. II by letting the constant g_A being large. In this limit R_{10}, R_{11} are large with fixed (R_{10}/R_{11}) and the nonlinear interacting Hamiltonian is multiplied by the small constant g_A^{-2} so that it can be considered perturbatively. In the leading order of perturbative theory in g_A^{-2} the interaction term can be dropped and the solution of the membrane equations of motion takes the form [3]

$$X^i(\sigma, \rho, \tau) = x^i + \alpha' p^i \tau + \sqrt{-\frac{\alpha'}{2}} \sum_{\mathbf{n} \neq (0,0)} \frac{e^{i\omega_{\mathbf{n}} \tau}}{\omega_{\mathbf{n}}} \times [\alpha_{\mathbf{n}}^i e^{ik\sigma + i\ell\rho} + \tilde{\alpha}_{\mathbf{n}}^i e^{-ik\sigma - i\ell\rho}]. \quad (20)$$

The momentum components in the X^{10} and X^{11} directions are $p_{10} = (\ell_{10}/R_{10})$, and $p_{11} = (\ell_{11}/R_{11})$, where $\ell_{10}, \ell_{11} \in \mathbb{Z}$. The nine-dimensional mass operator reads

$$M^2 = \frac{\ell_{10}^2}{R_{10}^2} + \frac{\ell_{11}^2}{R_{11}^2} + \frac{m_0^2 R_{10}^2}{\alpha'^2} + \frac{1}{\alpha'} \sum_{k,\ell} (\alpha_{(-k,-\ell)}^i \alpha_{(k,\ell)}^i + \tilde{\alpha}_{(-k,-\ell)}^i \tilde{\alpha}_{(k,\ell)}^i). \quad (21)$$

The level-matching conditions are [3,20] $N_\sigma^+ - N_\sigma^- = m_0 \ell_{10}$, $N_\rho^+ - N_\rho^- = \ell_{11}$, and

$$\begin{aligned} N_\sigma^+ &= \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{k\ell}} \alpha_{(-k,-\ell)}^i \alpha_{(k,\ell)}^i, \\ N_\sigma^- &= \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{\infty} \frac{k}{\omega_{k\ell}} \tilde{\alpha}_{(-k,-\ell)}^i \tilde{\alpha}_{(k,\ell)}^i, \\ N_\rho^+ &= \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \frac{\ell}{\omega_{k\ell}} [\alpha_{(-k,-\ell)}^i \alpha_{(k,\ell)}^i + \tilde{\alpha}_{(-k,-\ell)}^i \tilde{\alpha}_{(k,\ell)}^i], \\ N_\rho^- &= \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \frac{\ell}{\omega_{k\ell}} [\alpha_{(-k,-\ell)}^i \alpha_{(k,\ell)}^i + \tilde{\alpha}_{(-k,-\ell)}^i \tilde{\alpha}_{(k,\ell)}^i]. \end{aligned} \quad (22)$$

$$\quad (23)$$

Let us define the quantum oscillator operator \hat{H} as

$$\hat{H} = \sum_{k,\ell} (:\alpha_{(-k,-\ell)}^i \alpha_{(k,\ell)}^i: + :\tilde{\alpha}_{(-k,-\ell)}^i \tilde{\alpha}_{(k,\ell)}^i:), \quad (24)$$

where the annihilation operators $\alpha_{(k,\ell)}^i, \tilde{\alpha}_{(k,\ell)}^i$ are determined for $k > 0$ and $\ell \in \mathbb{Z}$, and $k = 0, \ell > 0$. In Eq. (24) the normal ordering means taking the annihilation operators to the right. The relation is (see [3]) $H = \hat{H} + 2(D-3)E$, $E = (1/2)\sum_{k,\ell} \omega_{k\ell}$, where the constant energy shift $2(D-3)E$ (E is the Casimir energy) represents the purely bosonic contribution to the vacuum energy of the $(2+1)$ -dimensional field theory. In the case of supersymmetry preserving boundary conditions for fermions the contributions to the vacuum energy coming from bosonic and fermionic fields cancel out [20,21].

In the presence of membrane excitation states with non-trivial winding numbers around the target space torus the spectrum of the light-cone Hamiltonian is discrete [3,20,21]. Let the Euclidean time coordinate X^0 play the role of X^{10} . Then fermions will obey antiperiodic boundary conditions around X^0 but periodic boundary conditions around X^{11} . In the $m_0 = \pm 1$ sector fermions are antiperiodic under the replacement $\sigma \rightarrow \sigma + 2\pi$ (while periodic under $\rho \rightarrow \rho + 2\pi$). The Hamiltonian operator becomes

$$\mathcal{H} = \frac{\ell_0^2}{R_0^2} + \frac{\ell_{11}^2}{R_{11}^2} + \frac{R_0^2}{\alpha'^2} + \frac{1}{\alpha'} (\hat{H} + 2(D-3)E), \quad (25)$$

where

$$\begin{aligned} \hat{H} = \sum_{\mathbf{n}} [& :\alpha_{-\mathbf{n}}^i \alpha_{\mathbf{n}}^i: + :\tilde{\alpha}_{-\mathbf{n}}^i \tilde{\alpha}_{\mathbf{n}}^i: + \omega_{k+1/2,\ell} (:\mathcal{S}_{-\mathbf{n}}^a \mathcal{S}_{\mathbf{n}}^a: \\ & + :\tilde{\mathcal{S}}_{-\mathbf{n}}^a \tilde{\mathcal{S}}_{\mathbf{n}}^a:)], \end{aligned} \quad (26)$$

and

$$\begin{aligned} E &= E_B + E_F = \frac{1}{2} \sum_{k,\ell} (\omega_{k\ell} - \omega_{k+1/2,\ell}), \\ \omega_{k\ell} &= \left(k^2 + \frac{\ell^2}{g_{eff}^2} \right)^{1/2}. \end{aligned} \quad (27)$$

A. Brane thermodynamics presented in M theory

Let us consider semiclassically the partition function associated with fundamental p -branes (which is known to be divergent) embedded in flat D -dimensional manifolds. For the standard quantum field model the free energy associated with bosonic (b) and fermionic (f) degrees of freedom has the form (see, for example, [8,9])

$$\begin{aligned} F^{(b,f)}(\beta) &= -\pi^p (\det \mathfrak{A})^{1/2} \int_0^\infty \frac{ds \Xi^{(b,f)}(s, \beta)}{(2s)^{(D-p+2)/2}} \Theta \left[\begin{matrix} \mathbf{g} \\ \mathbf{0} \end{matrix} \right] \\ &\quad \times (\mathbf{0} | \Omega) e^{-sM_0^2/2\pi}, \end{aligned} \quad (28)$$

where

$$\begin{aligned} \Xi^{(b)}(s, \beta) &= \theta_3 \left(0 \left| \frac{i\beta^2}{2s} \right. \right) - 1, \\ \Xi^{(f)}(s, \beta) &= 1 - \theta_4 \left(0 \left| \frac{i\beta^2}{2s} \right. \right), \end{aligned} \quad (29)$$

and $\theta_3(\nu|\tau)$ and $\theta_4(\nu|\tau) = \theta_3(\nu + \frac{1}{2}|\tau)$ are the Jacobi theta functions. Here $\mathfrak{A} = \text{diag}(R_1^{-2}, \dots, R_p^{-2})$ is a $p \times p$ matrix. The global parameters R_ℓ characterizing the non-trivial topology appear in the theory due to the fact that the coordinates $x_\ell (\ell = 1, \dots, p)$ obey the conditions $0 \leq x_\ell < 2\pi R_\ell$. The number of topological configurations of quantum fields is equal to the number of elements in group $H^1(\mathfrak{M}; \mathbb{Z}_2)$, that is, the first cohomology group with coefficients in \mathbb{Z}_2 . The multiplet $\mathbf{g} = (g_1, \dots, g_p)$ defines the topological type of field (i.e., the corresponding twist), and depends on the field type chosen in \mathfrak{M} , $g_\ell = 0$ or $1/2$. In our case $H^1(\mathfrak{M}; \mathbb{Z}_2) = \mathbb{Z}_2^p$ and so the number of topological configurations of real scalars (spinors) is 2^p .

We follow the notations and treatment of [22] and introduce the theta function with characteristics \mathbf{a}, \mathbf{b} for $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^p$,

$$\Theta \left[\begin{matrix} \mathbf{a} \\ \mathbf{b} \end{matrix} \right] (\mathbf{z} | \Omega) = \sum_{\mathbf{n} \in \mathbb{Z}^p} e^{\pi i [(\mathbf{n} + \mathbf{a}) \Omega (\mathbf{n} + \mathbf{a}) + 2(\mathbf{n} + \mathbf{a})(\mathbf{z} + \mathbf{b})]}. \quad (30)$$

In this connection $\Omega = (si/2\pi^2) \text{diag}(R_1^2, \dots, R_p^2)$.

We assume that the free energy is equivalent to a sum of the free energies of quantum fields which are present in the

modes of a p -brane. The factor $\exp(-sM_0^2/2\pi)$ in Eq. (28) should be understood as $\text{Tr} \exp(-sM^2/2\pi)$, where M is the mass operator of the brane and the trace is taken over an infinite set of Bose-Fermi oscillators N_n . The one-loop-like contribution for the (super) p -brane can be evaluated making use of the Mellin-Barnes representation for the partition function (energy integral) and in this formalism the generating function reads

$$\mathfrak{G}_{\pm}(z) = \text{Tr}[e^{-zM^2}]_{\pm} = \frac{1}{2\pi i} \int_{\Re s = s_0} ds \Gamma(s) \text{Tr}[zM^2]_{\pm}^{-s},$$

$$\text{Tr}[zM^2]_{\pm}^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \mathfrak{G}_{\pm}(tz). \quad (31)$$

One can use some subtraction procedure for the divergent terms in $\mathfrak{G}_{\pm}(z)$ in order to proceed with the regularization scheme (see for detail Ref. [23]). To simplify our calculation we set $a_{\ell} = R_{\ell} = 1$, and the final result for the free energy is [24,25]

$$F_{\pm}(\beta) \simeq -Q(D, p) \sum_{k=1}^{\infty} \frac{\Gamma\left(pk + \frac{(1-p)}{2}\right)}{\Gamma(k)} \times \zeta_{\pm}(2pk + 1 - p) x^{1+p(2k-1)}, \quad (32)$$

where $Q(D, p) = \Lambda A(p) \Gamma(p) [y(p)]^{-1-p}$, with $\Lambda = D - p - 1$, and

$$x\beta = y(p) \equiv \left[\Lambda 2^{3p-2} \pi^{(p-1)/2} \Gamma\left(\frac{p+1}{2}\right) \zeta_R(p+1) \right]^{1/2p}. \quad (33)$$

The asymptotic expansion of $\Gamma(s)$ for large value of $|s|$ has the form

$$\Gamma(s) = (2\pi)^{1/2} s^{s-1/2} e^{-s} (1 + \mathcal{O}(s^{-1})), \quad |\arg s| < \pi, \quad (34)$$

and for $p > 1$ the power series (32) is divergent for any $x > 0$.

B. The analytic continuation of a brane Laurent series and the thermodynamic limit

In principle, the power series (32) is divergent, nevertheless one can construct its analytic continuation. Let us define for $|z| < \infty$ two series

$$\mathfrak{W}_{\pm}(z) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi} \nu_{\pm}(k; p)}{\Gamma(k+1) \Gamma\left(pk + \frac{p+2}{2}\right)} \left(\frac{z}{2}\right)^{p(2k+1)+1}, \quad (35)$$

where the factors $\nu_{\pm}(k; p)$ have the form

$$\nu_{-}(k; p) = (-1)^{pk+1},$$

$$\nu_{+}(k; p) = \nu_{-}(k; p) [1 - 2^{-p(2k+1)-1}]. \quad (36)$$

For finite variable z these series converge and the convergence improves rapidly with the increasing of the integer number p . Let $z = \ell \cdot 2\pi x$, then we get the series

$$\sum_{\ell=1}^{\infty} \mathfrak{W}_{\pm}(\ell \cdot 2\pi x) = \sum_{\ell=1}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{\pi} \nu_{\pm}(k; p) (\ell \pi x)^{p(2k+1)+1}}{\Gamma(k+1) \Gamma\left(pk + \frac{p+2}{2}\right)}. \quad (37)$$

Now, if we commute the (up to now divergent) sum \sum_{ℓ} with the sum \sum_k , new extra terms of the type $x^{-1} W_{\pm}(p)$ will appear on the right-hand side of Eq. (37). Therefore, the result is

$$\begin{aligned} \sum_{\ell=1}^{\infty} \mathfrak{W}_{\pm}(\ell \cdot 2\pi x) + x^{-1} W_{\pm}(p) \\ = \sum_{k=0}^{\infty} \frac{\pi \nu_{\pm}(k; p)}{\Gamma(k+1) \Gamma\left(pk + \frac{p+2}{2}\right)} \frac{\zeta_R[-p(2k+1)]}{x^{-1-p(2k+1)}} \\ = \sin\left(\frac{\pi p}{2}\right) \sum_{k=1}^{\infty} \frac{\Gamma\left(pk + \frac{1-p}{2}\right)}{\Gamma(k)} \frac{\zeta_{\pm}(2pk + 1 - p)}{x^{-1-p(2k-1)}}, \end{aligned} \quad (38)$$

where $W_{\pm}(p)$ is an integer function of p (see, for example, [24]). In the second equality the functional equation for $\zeta_R(s)$ has been used.

The new form of $F(\beta)$ is

$$\begin{aligned} F_{-}(\beta) \simeq \frac{Q(D, p)}{\sin\left(\frac{\pi p}{2}\right)} \sum_{k=0}^{\infty} \frac{(-1)^{pk} \pi \zeta_R[-p(2k+1)]}{\Gamma(k+1) \Gamma\left(pk + \frac{p+2}{2}\right)} \\ \times \left(\frac{\beta}{y(p)}\right)^{-1-p(2k+1)}. \end{aligned} \quad (39)$$

In the string case ($p=1$) the corresponding series in Eq. (32) can be resummed into the trigonometrical form using the identities

$$\begin{aligned} \sum_{k=1}^{\infty} \zeta_{-}(2k) x^{2k} &= \frac{1}{2} (1 - \pi x \cot(\pi x)), \\ \sum_{k=1}^{\infty} \zeta_{+}(2k) x^{2k} &= \frac{\pi x}{4} \tan\left(\frac{\pi x}{2}\right). \end{aligned} \quad (40)$$

The finite radius of convergence, $|x| < 1$, of the Laurent series corresponds to the Hagedorn temperature in string thermodynamics (see for detail Ref. [23]). Using trigonometric relations, formulas (40) display a certain periodicity in the temperature. The physical meaning of that behavior is still obscure. The thermal dependence in Eqs. (39), (40), corre-

sponding to the quantum modes in two dimensions near the Hagedorn instability, can be interpreted as an indication of a vast reduction of the fundamental degrees of freedom in string theory [5].

The divergent series in Eq. (32) for the case $p > 1$, when reexpressed on the left-hand side of Eq. (38), remains well-defined for finite temperature. Note that the series (39) has smooth $\beta \rightarrow 0, \infty$ limits. For example, when $\beta \rightarrow \infty$ we get

$$F_-(\beta) \simeq \mathcal{A}_1(D, p) \beta^{-1-p} + \mathcal{A}_2(D, p) \beta^{-1-3p} + \mathcal{O}(\beta^{-1-5p}), \quad (41)$$

where $\mathcal{A}_\ell(D, p)$ ($\ell = 1, 2$) depends on the dimension of the embedding spacetime. The statistical internal energy $E = (\partial/\partial\beta)(\beta F_-(\beta))$ and the entropy $S = \beta^2(\partial/\partial\beta)F_-(\beta)$ can be easily calculated using Eq. (39). The β behavior has a similar dependence which is similar to the one found for the string case in [5, 26]. Note that in the $p \rightarrow 0$ limit (point particle limit) Eq. (39) leads to the standard thermodynamic behavior for both $F, E \sim T$.

V. CONCLUSIONS

The result (17), (18) has a universal character. We can compute state density, including the prefactors $\mathcal{C}_\pm(p)$, depending on the dimension of the embedding space. There are deep connections between strings and p -branes; at least they should be considered as different limits of a more general M theory. Indeed, string results may be obtained via membrane-

string correspondence and vice versa. Therefore, even being not a fundamental theory of p -branes it may provide new deep insights in the understanding of string theory and consistent formulation of M theory.

In this paper we had dealt with the same discrete membrane spectrum as it has been used in the membrane-string correspondence. We analyzed the logarithmic correction to the entropy in the small string coupling limit of M theory. Note that in the large string coupling limit of M theory, compactified on manifold with topology $T^2 \otimes R^9$, the analytic continuation of a brane Laurent series has been given in its explicit form. Physically, it means that a finite temperature can be introduced in the theory and a membrane (if it can be quantized semiclassically) behaves like an ideal gas of quantum modes, which corresponds to a field theory at finite temperature (zero critical temperature). Finally note that in the limit $p \rightarrow 0$ (point particle limit) the standard behavior of thermodynamic quantities has been obtained.

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