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R. da Rocha and J. M. Hoff da Silva

Citation: *Journal of Mathematical Physics* **48**, 123517 (2007); doi: 10.1063/1.2825840

View online: <http://dx.doi.org/10.1063/1.2825840>

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From Dirac spinor fields to eigenspinoren des ladungskonjugationsoperators

R. da Rocha^{a)}

*Centro de Matemática, Computação e Cognição, Universidade Federal do ABC,
09210-170, Santo André, São Paulo, Brazil and Instituto de Física “Gleb Wataghin,”
Universidade Estadual de Campinas, Unicamp 13083-970 Campinas, São Paulo, Brazil*

J. M. Hoff da Silva^{b)}

*Instituto de Física Teórica, Universidade Estadual Paulista,
Rua Pamplona 145 01405-900 São Paulo, São Paulo, Brazil*

(Received 20 August 2007; accepted 27 November 2007;
published online 19 December 2007)

Dual-helicity eigenspinors of the charge conjugation operator [eigenspinoren des ladungskonjugationsoperators (ELKO) spinor fields] belong—together with Majorana spinor fields—to a wider class of spinor fields, the so-called flagpole spinor fields, corresponding to the class (5), according to Lounesto spinor field classification based on the relations and values taken by their associated bilinear covariants. There exists only six such disjoint classes: the first three corresponding to Dirac spinor fields, and the other three, respectively, corresponding to flagpole, flag-dipole, and Weyl spinor fields. This paper is devoted to investigate and provide the necessary and sufficient conditions to map Dirac spinor fields to ELKO, in order to naturally extend the standard model to spinor fields possessing mass dimension 1. As ELKO is a prime candidate to describe dark matter, an adequate and necessary formalism is introduced and developed here, to better understand the algebraic, geometric, and physical properties of ELKO spinor fields, and their underlying relationship to Dirac spinor fields. © 2007 American Institute of Physics.
[DOI: [10.1063/1.2825840](https://doi.org/10.1063/1.2825840)]

I. INTRODUCTION

Eigenspinoren des ladungskonjugationsoperators (ELKO) spinor fields¹ represent an extended set of Majorana spinor fields, describing a nonstandard Wigner class of fermions, in which the charge conjugation and the parity operators commute, rather than anticommute.^{1–3} Further, ELKO accomplishes dual-helicity eigenspinors of the spin-1/2 charge conjugation operator, and carry mass dimension 1, besides having nonlocal properties. In order to find an adequate mathematical formalism for representing dark matter by a spinor field associated with mass dimension 1, Ahluwalia-Khalilova and Grumiller have just ushered the ELKO (Ref. 1) into quantum field theory, and it has also given rise to subsequent applications in cosmology. ELKO is a representative of a neutral fermion described by a set of four spinor fields, two of which are identified to massive McLennan-Case (Majorana) spinor fields,^{4,5} and other two which were not known yet. Another surprising character involving ELKO is that its Lagrangian possesses interaction neither with standard model fields nor with gauge fields, which endows ELKO to be a prime candidate to describe dark matter,^{6–8} which has recent observational confirmation.⁹ Likewise, the Higgs boson can interact with ELKO, and it also could be tested at LHC.

^{a)}Electronic mail: roldao.rocha@ufabc.edu.br, roldao@ifi.unicamp.br.

^{b)}Electronic mail: hoff@ift.unesp.br.

¹ELKO is the german acronym for dual-helicity eigenspinors of the charge conjugation operator (Ref. 1).

In the low-energy limit, ELKO behaves as a representation of the Lorentz group. However, all spinor fields in Minkowski spacetime can be given—from the classical viewpoint²—as elements of the carrier spaces of the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representations of $SL(2, \mathbb{C})$. Lounesto, in the classification of spinor fields, proved that any spinor field belongs to one of the six classes found by him.^{20,21} Such an *algebraic* classification is based on the values assumed by their bilinear covariants, the Fierz identities, aggregates, and boomerangs.^{20–22} Lounesto spinor field classification has wide applications in cosmology and astrophysics (via ELKO, for instance, see Refs. ^{1,2,6,22–24}), and in general relativity: it was recently demonstrated that the Einstein-Hilbert, Einstein-Palatini, and Holst³ actions can be derived from the quadratic spinor lagrangian (that describes supegravity),^{26,27} when the three classes of Dirac spinor fields, under Lounesto spinor field classification, are considered.²⁸ It was also shown²² that ELKO represents a larger class of Majorana spinor fields, and that those spinor fields cover one of the six classes in Lounesto spinor field classification. ELKO possesses an intrinsic and genuine geometric structure behind, and a great variety of geometrical and algebraic concepts, and their applications in physics and mathematical-physics,²⁹ e.g., the formalism of Penrose twistors, flagpoles, and flag dipoles,^{30–37} can be unified, described, and generalized via this formalism.

One of the main purposes of this paper is to analyze and investigate the underlying equivalence between Dirac spinor fields (DSFs) and ELKO, i.e., under which conditions a DSF can be led to an ELKO, since they are inherently distinct and represent disjoint classes in Lounesto spinor field classification. For instance, while the latter belongs to class (5) under such classification, the former is a representative of spinor fields of types (1)–(3). In addition, when acting on ELKO, the parity P and charge conjugation C operators *commute* and $P^2 = -1$, while when acting upon Dirac spinor fields, such operators *anticommute* and $P^2 = 1$. Besides, CPT equals $+1$ and -1 , respectively, for DSFs and ELKO. Any invertible map that takes Dirac particles and leads to ELKO is also capable to make mass dimension transmutations, since DSFs present mass dimension $3/2$, instead of mass dimension 1 associated with ELKO. The main physical motivation of this paper is to provide the initial prerequisites to construct a natural extension of the standard model (SM) in order to incorporate ELKO, and consequently a possible description of dark matter^{1,2,6} in this context.

The paper is organized as follows: after briefly presenting some essential algebraic preliminaries in Sec. II, we introduce in Sec. III the bilinear covariants together with the Fierz identities. Also, the Lounesto classification of spinor fields is presented together with the definition of ELKO spinor fields,¹ showing that ELKO is indeed a flagpole spinor field with opposite (dual) helicities.^{1,2,22} In Sec. IV the mapping from Dirac spinor fields to ELKO is widely investigated in detail.^{38–42}

II. PRELIMINARIES

Let V be a finite n -dimensional real vector space and V^* denotes its dual. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^n \Lambda^k(V)$ of multivectors over V . $\Lambda^k(V)$ denotes the space of the antisymmetric k -tensors, isomorphic to the k -forms vector space. Given $\psi \in \Lambda(V)$, $\tilde{\psi}$ denotes the *reversion*, an algebra antiautomorphism given by $\tilde{\psi} = (-1)^{[k/2]} \psi$ ($[k]$ denotes the integer part of k). If V is endowed with a nondegenerate, symmetric, bilinear map $g: V^* \times V^* \rightarrow \mathbb{R}$, it is possible to extend g to $\Lambda(V)$. Given $\psi = \mathbf{u}^1 \wedge \cdots \wedge \mathbf{u}^k$ and $\phi = \mathbf{v}^1 \wedge \cdots \wedge \mathbf{v}^l$, for $\mathbf{u}^i, \mathbf{v}^j \in V^*$, one defines $g(\psi, \phi) = \det(g(\mathbf{u}^i, \mathbf{v}^j))$ if $k=l$ and $g(\psi, \phi) = 0$ if $k \neq l$. The projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n$, $\psi_k \in \Lambda^k(V)$, on its p -vector part is given by $\langle \psi \rangle_p = \psi_p$. Given $\psi, \phi, \xi \in \Lambda(V)$, the *left contraction* is defined implicitly by $g(\psi \lrcorner \phi, \xi) = g(\phi, \tilde{\psi} \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $\mathbf{v} \lrcorner a = 0$. The *right contraction* is analogously defined by $g(\psi \llcorner \phi, \xi) = g(\phi, \psi \wedge \tilde{\xi})$. Both contractions are related by $\mathbf{v} \lrcorner \psi = -\hat{\psi} \llcorner \mathbf{v}$. The Clifford

²It is well known that spinors have three different, although equivalent, definitions: the operatorial, the classical, and the algebraic one (Refs. ^{10–19}).

³The Holst action is shown to be equivalent to the Ashtekar formulation of quantum gravity (Ref. ²⁵).

product between $\mathbf{w} \in V$ and $\psi \in \Lambda(V)$ is given by $\mathbf{w}\psi = \mathbf{w} \wedge \psi + \mathbf{w} \lrcorner \psi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with the Clifford product is denoted by $\mathcal{C}\ell(V, g)$ or $\mathcal{C}\ell_{p,q}$; the Clifford algebra associated with $V \simeq \mathbb{R}^{p,q}$, $p+q=n$. In what follows \mathbb{R} and \mathbb{C} denote, respectively, the real and complex numbers.

III. BILINEAR COVARIANTS AND ELKO SPINOR FIELDS

This section is devoted to recall the bilinear covariants, using the program introduced in Ref. 22, which we briefly recall here. In this article all spinor fields live in Minkowski spacetime $(M, \eta, D, \tau_\eta, \uparrow)$. The manifold $M \simeq \mathbb{R}^4$, η denotes a constant metric, where $\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, D denotes the Levi-Civita connection associated with η , M is oriented by the four-volume element τ_η and time oriented by \uparrow . Here $\{x^\mu\}$ denotes global coordinates in the Einstein-Lorentz gauge, naturally adapted to an inertial reference frame $\mathbf{e}_0 = \partial/\partial x^0$. Let $\mathbf{e}_i = \partial/\partial x^i$, $i=1, 2, 3$. Also, $\{\mathbf{e}_\mu\}$ is a section of the frame bundle $\mathbf{P}_{\text{SO}_{1,3}}(M)$ and $\{\mathbf{e}^\mu\}$ is its reciprocal frame satisfying $\eta(\mathbf{e}^\mu, \mathbf{e}_\nu) := \mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$. Classical spinor fields carrying a $D^{(1/2,0)} \oplus D^{(0,1/2)}$, or $D^{(1/2,0)}$, or $D^{(0,1/2)}$ representation of $\text{SL}(2, \mathbb{C}) \simeq \text{spin}_{1,3}^e$ are sections of the vector bundle $\mathbf{P}_{\text{spin}_{1,3}^e}(M) \times_\rho \mathbb{C}^4$, where ρ stands for the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ [or $D^{(1/2,0)}$ or $D^{(0,1/2)}$] representation of $\text{SL}(2, \mathbb{C}) \simeq \text{spin}_{1,3}^e$ in \mathbb{C}^4 . Given a spinor field $\psi \in \text{sec } \mathbf{P}_{\text{spin}_{1,3}^e}(M) \times_\rho \mathbb{C}^4$ the bilinear covariants are the following sections of the exterior algebra bundle of *multivector* fields:¹⁰

$$\begin{aligned} \sigma &= \psi^\dagger \gamma_0 \psi, & \mathbf{J} &= J_\mu \mathbf{e}^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \mathbf{e}^\mu, & \mathbf{S} &= S_{\mu\nu} \mathbf{e}^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \mathbf{e}^\mu \wedge \mathbf{e}^\nu, \\ & & \mathbf{K} &= K_\mu \mathbf{e}^\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi \mathbf{e}^\mu, & \omega &= -\psi^\dagger \gamma_0 \gamma_{0123} \psi. \end{aligned} \quad (1)$$

The set $\{\gamma_\mu\}$ refers to the Dirac matrices in chiral representation [see Eq. (5)]. Also $\{\mathbf{1}_4, \gamma_\mu, \gamma_\mu \gamma_\nu, \gamma_\mu \gamma_\nu \gamma_\rho, \gamma_0 \gamma_1 \gamma_2 \gamma_3\}$ ($\mu, \nu, \rho=0, 1, 2, 3$, and $\mu < \nu < \rho$) is a basis for $\mathbb{C}(4)$ satisfying²⁰ $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \mathbf{1}_4$ and the Clifford product is denoted by juxtaposition. More details on notations can be found in Refs. 10 and 11.

Given a fixed spin frame the bilinear covariants are considered as being the following *operator* fields, for each $x \in M$, as mappings $\mathbb{C}^4 \rightarrow \mathbb{C}^4$:

$$\begin{aligned} \sigma &= \psi^\dagger \gamma_0 \psi, & \mathbf{J} &= J_\mu \gamma^\mu = \psi^\dagger \gamma_0 \gamma_\mu \psi \gamma^\mu, & \mathbf{S} &= S_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \gamma^{\mu\nu}, \\ & & \mathbf{K} &= K_\mu \gamma^\mu = \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi \gamma^\mu, & \omega &= -\psi^\dagger \gamma_0 \gamma_{0123} \psi. \end{aligned} \quad (2)$$

In the case of the electron, described by Dirac spinor fields (classes 1–3 below), \mathbf{J} is a future-oriented timelike current vector which gives the current of probability, the bivector \mathbf{S} is associated with the distribution of intrinsic angular momentum, and the spacelike vector \mathbf{K} is associated with the direction of the electron spin. For a detailed discussion concerning such entities, their relationships and physical interpretation, and generalizations, see, e.g., Refs. 20, 21, and 43–45.

The bilinear covariants satisfy the Fierz identities^{20,21,43–45}

$$\mathbf{J}^2 = \omega^2 + \sigma^2, \quad \mathbf{K}^2 = -\mathbf{J}^2, \quad \mathbf{J}_L \mathbf{K} = 0, \quad \mathbf{J} \wedge \mathbf{K} = -(\omega + \sigma \gamma_{0123}) \mathbf{S}. \quad (3)$$

A spinor field such that *not both* ω and σ are null is said to be regular. When $\omega=0=\sigma$, a spinor field is said to be *singular*.

Lounesto spinor field classification is given by the following spinor field classes,^{20,21} where in the first three classes it is implicit that $\mathbf{J}, \mathbf{K}, \mathbf{S} \neq 0$:

- (1) $\sigma \neq 0, \omega \neq 0,$
- (2) $\sigma \neq 0, \omega = 0,$
- (3) $\sigma = 0, \omega \neq 0,$
- (4) $\sigma = 0 = \omega, \mathbf{K} \neq 0, \mathbf{S} \neq 0,$
- (5) $\sigma = 0 = \omega, \mathbf{K} = 0, \mathbf{S} \neq 0,$
- (6) $\sigma = 0 = \omega, \mathbf{K} \neq 0, \mathbf{S} = 0.$

The current density \mathbf{J} is always nonzero. Types (1)–(3) spinor fields are denominated *Dirac spinor fields* for spin-1/2 particles and types (4), (5), and (6) are, respectively, called *flag-dipole*, *flagpole*,⁴ and *Weyl spinor fields*. Majorana spinor fields are a particular case of a type-(5) spinor field. It is worthwhile to point out a peculiar feature of type-(4), -(5), and -(6) spinor fields: although \mathbf{J} is always nonzero, $\mathbf{J}^2 = -\mathbf{K}^2 = 0$. It shall be seen below that the bilinear covariants related to an ELKO spinor field satisfy $\sigma = 0 = \omega$, $\mathbf{K} = 0$, $\mathbf{S} \neq 0$, and $\mathbf{J}^2 = 0$. Since Lounesto proved that there are *no* other classes based on distinctions among bilinear covariants, ELKO spinor fields must belong to one of the disjoint six classes.

Types (1), -(2), and -(3) DSFs have different algebraic and geometrical characters, and we would like to emphasize the main differing points. For more details, see, e.g., Refs. 20 and 21. Recall that if the quantities $P = \sigma + \mathbf{J} + \gamma_{0123}\omega$ and $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$ are defined,^{20,21} in type-(1) DSF we have $P = -(\omega + \sigma\gamma_{0123})^{-1}\mathbf{K}Q$ and also $\psi = -i(\omega + \sigma\gamma_{0123})^{-1}\psi$. In type-(2) DSF, P is a multiple of $1/2\sigma(\sigma + \mathbf{J})$ and looks like a proper energy projection operator, commuting with the spin projector operator given by $\frac{1}{2}(1 - i\gamma_{0123}\mathbf{K}/\sigma)$. Also, $P = \gamma_{0123}\mathbf{K}Q/\sigma$. Furthermore, in type-(3) DSF, $P^2 = 0$ and $P = \mathbf{K}Q/\omega$. The introduction of the spin-Clifford bundle makes it possible to consider all the geometric and algebraic objects—the Clifford bundle, spinor fields, differential form fields, operators, and Clifford fields—as being elements of a unique unified formalism. It is well known that spinor fields have three different, although equivalent, definitions: the operatorial, the classical, and the algebraic one.⁴⁷ In particular, the operatorial definition allows us to factor—up to sign—the DSF ψ as $\psi = (\sigma + \omega\gamma_{0123})^{-1/2}R$, where $R \in \text{spin}_{1,3}^e$. Denoting $\mathbf{K}_k = \psi\gamma_k\tilde{\psi}$, where $\tilde{\psi}$ denotes the reversion of ψ , the set $\{\mathbf{J}, K_1, K_2, K_3\}$ is an orthogonal basis of $\mathbb{R}^{1,3}$. On the other hand, in classes (4)–(6)—where $\sigma = \bar{\psi}\psi = 0 = \omega = \bar{\psi}\gamma_5\psi$ —the vectors $\{\mathbf{J}, K_1, K_2, K_3\}$ no longer form a basis and collapse into a null line.^{20,21} In such case only the boundary term is non-null. Finally, to a Weyl spinor field ξ [type (6)] with bilinear covariants \mathbf{J} and \mathbf{K} , two Majorana spinor fields $\psi_{\pm} = \frac{1}{2}(\xi + C(\xi))$ can be associated, where C denotes the charge conjugation operator. Penrose flagpoles are implicitly defined by the equation $\sigma + \mathbf{J} + i\mathbf{S} - i\gamma_{0123}\mathbf{K} + \gamma_{0123}\omega = \frac{1}{2}(\mathbf{J} \mp i\mathbf{S}\gamma_{0123})$.^{20,21} For a physically useful discussion regarding the disjoint classes (5) and (6) see, e.g., Ref. 48. The fact that two Majorana spinor fields ψ_{\pm} can be written in terms of a Weyl type-(6) spinor field $\psi_{\pm} = \frac{1}{2}(\xi + C(\xi))$ is an “accident” when the (Lorentzian) spacetime has $n=4$ —the present case—or $n=6$ dimensions. The more general assertion concerns the property that two Majorana, and more generally ELKO spinor fields ψ_{\pm} can be written in terms of a *pure spinor* field—hereon denoted by u —as $\psi_{\pm} = \frac{1}{2}(u + C(u))$. It is well known that Weyl spinor fields are pure spinor fields when $n=4$ and $n=6$. When the complexification of $\mathbb{C} \otimes \mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$ is considered, one can consider a maximal totally isotropic subspace N of $\mathbb{C}^{1,3}$, by the Witt decomposition, where $\dim_{\mathbb{C}} N = 2$. Pure spinors are defined by the property $xu = 0$ for all $x \in N \subset \mathbb{C}^{1,3}$.¹⁵ In this context, Penrose flags can be defined by the expression $\text{Re}(iu\bar{u})$.¹⁶

Now, the algebraic and formal properties of ELKO spinor fields, as defined in Refs. 1, 2, 6, and 22, are briefly explored. An ELKO Ψ corresponding to a plane wave with momentum $p = (p^0, \mathbf{p})$ can be written, without loss of generality, as $\Psi(p) = \lambda(\mathbf{p})e^{-ip \cdot x}$ (or $\Psi(p) = \lambda(\mathbf{p})e^{ip \cdot x}$) where

$$\lambda(\mathbf{p}) = \begin{pmatrix} i\Theta\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad (4)$$

and given the rotation generators denoted by \mathfrak{J} , Wigner’s spin-1/2 time reversal operator Θ satisfies $\Theta\mathfrak{J}\Theta^{-1} = -\mathfrak{J}^*$. Hereon, as in Ref. 1, the Weyl representation of γ^μ is used, i.e.,

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix}, \quad -\gamma_k = \gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad (5)$$

where

⁴Such spinor fields are constructed by a null 1-form field current and an also null 2-form field angular momentum, the “flag” (Ref. 46).

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

σ_i are the Pauli matrices. Also,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^{0123} = -i\gamma_{0123} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{pmatrix}. \quad (7)$$

ELKO spinor fields are eigenspinors of the charge conjugation operator C , i.e., $C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p})$, for

$$C = \begin{pmatrix} \mathbf{O} & i\Theta \\ -i\Theta & \mathbf{O} \end{pmatrix} K. \quad (8)$$

The operator K is responsible for the C-conjugation of Weyl spinor fields appearing on the right. The plus sign stands for *self-conjugate* spinors, $\lambda^S(\mathbf{p})$, while the minus yields *anti-self-conjugate* spinors, $\lambda^A(\mathbf{p})$. Explicitly, the complete form of ELKO spinor fields can be found by solving the equation of helicity $(\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})\phi^\pm = \pm\phi^\pm$ in the rest frame and subsequently make a boost, to recover the result for any \mathbf{p} .¹ Here $\hat{\mathbf{p}} := \mathbf{p}/\|\mathbf{p}\|$. The four spinor fields are given as follows:

$$\lambda_{\{\mp, \pm\}}^{S/A}(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \left(1 \mp \frac{\mathbf{p}}{E+m} \right) \lambda_{\{\mp, \pm\}}^{S/A}(\mathbf{0}), \quad (9)$$

where

$$\lambda_{\{\mp, \pm\}}(\mathbf{0}) = \begin{pmatrix} \pm i\Theta[\phi^\pm(\mathbf{0})]^* \\ \phi^\pm(\mathbf{0}) \end{pmatrix}. \quad (10)$$

Note that, since $\Theta[\phi^\pm(\mathbf{0})]^*$ and $\phi^\pm(\mathbf{0})$ have opposite helicities, ELKO cannot be an eigenspinor field of the helicity operator, and indeed carries both helicities. In order to guarantee an invariant real norm, as well as positive definite norm for two ELKO spinor fields, and negative definite norm for the other two, the ELKO dual is given by

$$\overline{\lambda}_{\{\mp, \pm\}}^{S/A}(\mathbf{p}) = \pm i[\lambda_{\{\pm, \mp\}}^{S/A}(\mathbf{p})]^\dagger \gamma^0. \quad (11)$$

Omitting the subindex of the spinor field $\phi_L(\mathbf{p})$, which is denoted hereon by ϕ , the left-handed spinor field $\phi_L(\mathbf{p})$ can be represented by

$$\phi = \begin{pmatrix} \alpha(\mathbf{p}) \\ \beta(\mathbf{p}) \end{pmatrix}, \quad \alpha(\mathbf{p}), \beta(\mathbf{p}) \in \mathbb{C}. \quad (12)$$

Now using Eq. (2) it is possible to *calculate* explicitly the bilinear covariants for ELKO spinor fields⁵ as follows:

$$\overset{\circ}{\sigma} = \lambda^\dagger \gamma_0 \lambda = 0, \quad \overset{\circ}{\omega} = -\lambda^\dagger \gamma_0 \gamma_{0123} \lambda = 0, \quad (13)$$

$$\overset{\circ}{\mathbf{J}} = \overset{\circ}{J}_\mu \gamma^\mu = \lambda^\dagger \gamma_0 \gamma_\mu \lambda \gamma^\mu \neq 0, \quad (14)$$

$$\overset{\circ}{\mathbf{K}} = \overset{\circ}{K}_\mu \gamma^\mu = \lambda^\dagger i \gamma_{123} \gamma_\mu \lambda \gamma^\mu = 0, \quad (15)$$

⁵All the details are presented in Ref. 22.

$$\mathring{\mathbf{S}} = \frac{1}{2} \mathring{S}_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \lambda^\dagger \gamma_0 i \gamma_{\mu\nu} \lambda \gamma^{\mu\nu} \neq 0. \quad (16)$$

From the formulas in Eqs. (14) and (15) it is trivially seen that that $\mathbf{JK}=0$. Also, from Eq. (14) it follows that $\mathbf{J}^2=0$, and it is immediate that all Fierz identities introduced by the formulas in Eq. (3) are trivially satisfied.

It is useful to choose $i\Theta=\sigma_2$, as in Ref. 1, in such a way that it is possible to express

$$\lambda = \begin{pmatrix} \sigma_2 \phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}. \quad (17)$$

Now, any flagpole spinor field is an eigenspinor field of the charge conjugation operator,^{20,21} which explicit action on a spinor ψ is given by $\mathcal{C}\psi=-\gamma^2\psi^*$. Indeed using Eq. (17) it follows that

$$-\gamma^2\lambda^* = \begin{pmatrix} \sigma_2\phi^* \\ -\sigma_2\sigma_2\phi \end{pmatrix} = \lambda.$$

Once the definition of ELKO spinor fields is recalled, we return to the previous discussion about Penrose flagpoles. Here we extend the definition of the Penrose poles, and we can prove that they are given in terms of an ELKO spinor field by the expression $\frac{1}{2}\langle\lambda(\gamma_{0123}\tilde{\lambda})\rangle_1$, and further, Penrose flags F can also be written in terms of ELKO, as $F=\frac{1}{2}\langle\lambda(\gamma_{0123}\tilde{\lambda})\rangle_2$. This assertion can be demonstrated following a reasoning analogous as the one exposed in Refs. 16 and 18.

IV. WHICH ARE THE DIRAC SPINOR FIELDS THAT CAN BE LED TO ELKO?

In this section we are interested in analyzing a matrix $M \in \mathbb{C}(4)$ that defines the transformation from an *a priori* arbitrary DSF to an ELKO spinor field, i.e.,

$$M\psi = \lambda. \quad (18)$$

It shall be proved that not all DSFs can be led to ELKO, but only a subset of the three classes—under Lounesto classification—of DSFs restricted to some conditions. Explicitly we have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \phi_R(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix} = \begin{pmatrix} \epsilon\sigma_2\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad (19)$$

where $\epsilon=\pm 1$ and $M_{ij} \in \mathbb{C}(2)$, ($i, j=1, 2$). We are particularly interested to investigate the conditions imposed on DSFs that turn them to be led to ELKO spinor fields. Taking into account that $\phi_R(\mathbf{p})=\chi\phi_L(\mathbf{p})$, where $\chi=(E+\sigma\cdot\mathbf{p})/m$ and $\kappa\psi=\psi^*$ the following system is obtained:

$$M_{11}\chi + M_{12} = \epsilon\sigma_2\kappa, \quad (20)$$

$$M_{21}\chi + M_{22} = 1.$$

Then, writing explicitly the entries of $M=[m_{pq}]_{p,q=1}^4$, Eq. (20) reads

$$\chi m_{11} + m_{13} = 0, \quad \chi m_{31} + m_{33} = 1,$$

$$\chi m_{12} + m_{14} = -i\kappa\epsilon, \quad \chi m_{32} + m_{34} = 0, \quad (21)$$

$$\chi m_{21} + m_{23} = i\kappa\epsilon, \quad \chi m_{41} + m_{43} = 0,$$

$$\chi m_{22} + m_{24} = 0, \quad \chi m_{42} + m_{44} = 1,$$

in such way that the matrix M can be written in the form

$$M = \begin{pmatrix} m_{11} & m_{12} & -\chi m_{11} & -i\epsilon\kappa - \chi m_{12} \\ m_{21} & m_{22} & i\epsilon\kappa - \chi m_{21} & -\chi m_{22} \\ m_{31} & m_{32} & 1 - \chi m_{31} & -\chi m_{32} \\ m_{41} & m_{42} & -\chi m_{41} & 1 - \chi m_{42} \end{pmatrix}. \quad (22)$$

In order to have the product $(-i\epsilon\kappa - m_{12})(i\epsilon\kappa - \chi m_{21})$ equal to $(i\epsilon\kappa - \chi m_{21})(-i\epsilon\kappa - m_{12})$, which can be useful in further calculations, we take $m_{12} = -m_{21}$. From now on, in order to completely fix the matrix M , the *ansatz*

$$\begin{aligned} m_{11} = m_{22} = 0 = m_{32} = m_{41}, \\ m_{31} = m_{42} = 1 = m_{12} \end{aligned} \quad (23)$$

is regarded, and M is written as

$$M = \begin{pmatrix} 0 & 1 & 0 & -i\epsilon\kappa - \chi \\ -1 & 0 & i\epsilon\kappa + \chi & 0 \\ 1 & 0 & 1 - \chi & 0 \\ 0 & 1 & 0 & 1 - \chi \end{pmatrix}. \quad (24)$$

Note that such matrix is not unitary, and since $\det M \neq 0$, there exists [see Eq. (18)] M^{-1} such that $\bar{\psi} = M^{-1}\lambda$. Besides, it is immediate to note that

$$\bar{\psi} := \psi^\dagger \gamma^0 = \lambda^\dagger (M^{-1})^\dagger \gamma^0, \quad (25)$$

such that $\bar{\psi}$ can be related to the ELKO dual by

$$\bar{\psi} = \mp i\lambda_{\{\mp, \pm\}}^{S/A} \gamma^0 (M^{-1})^\dagger \gamma^0. \quad (26)$$

In what follows, the matrix M establishes necessary conditions on the Dirac spinor fields under which the mapping given by Eq. (18) is satisfied. However, the *ansatz* in Eq. (24) has just an illustrative role. In fact, for any matrix satisfying Eq. (22), there are corresponding constraints on the components of DSFs. Hereafter, we shall calculate the conditions to the case where $\mathbf{p}=0$ (and consequently $\chi=1$), since a Lorentz boost can be implemented on the rest frame in the constraints. Anyway, without loss of generality, the conditions to be found on DSFs must hold in all referentials and, in particular, in the rest frame corresponding to $\mathbf{p}=0$.

Substituting Eq. (18) in the definition given by Eq. (1) we have

$$\begin{aligned} \mathring{\sigma} = \psi^\dagger M^\dagger \gamma_0 M \psi, \quad \mathring{\mathbf{J}} = \mathring{J}_\mu \gamma^\mu = \psi^\dagger M^\dagger \gamma_0 \gamma_\mu M \psi \gamma^\mu, \quad \mathring{\mathbf{S}} = \mathring{S}_{\mu\nu} \gamma^{\mu\nu} = \frac{1}{2} \psi^\dagger M^\dagger \gamma_0 i \gamma_{\mu\nu} M \psi \gamma^{\mu\nu}, \\ \mathring{\mathbf{K}} = \mathring{K}_\mu \gamma^\mu = i \psi^\dagger M^\dagger \gamma_0 \gamma_{0123} \gamma_\mu M \psi \gamma^\mu, \quad \mathring{\omega} = -\psi^\dagger M^\dagger \gamma_0 \gamma_{0123} M \psi. \end{aligned} \quad (27)$$

These new bilinear covariants—expressed in terms of DSFs—are related to ELKO spinor fields, and by the definition of type-(5) spinor fields under Lounesto classification, they automatically satisfy the conditions $\mathring{\sigma}=0=\mathring{\omega}$, $\mathring{\mathbf{K}}=0$, and $\mathring{\mathbf{S}} \neq 0$. Types (1), (2), and (3) of DSFs satisfy $\mathbf{K} \neq 0$, but when they are transformed in ELKO spinor fields via the action of M , they must satisfy $\mathring{\mathbf{K}} = \mathring{K}^\mu \gamma_\mu = 0$. As $\{\gamma_\mu\}$ is a basis of $\mathbb{R}^{1,3}$, each one of the components \mathring{K}^μ must equal zero, i.e.,

$$\mathring{K}_0 = \psi^\dagger M^\dagger \gamma_0 i \gamma_{0123} \gamma_0 M \psi = \psi^\dagger \left[\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \otimes \mathbb{I} \right] \psi = 0, \quad (28)$$

where $a := -(1 + i\epsilon\kappa)$. The other components read

$$\mathring{K}_1 = \psi^\dagger M^\dagger \gamma_0 i \gamma_{0123} \gamma_1 M \psi = -\psi^\dagger \left[\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \otimes \sigma_1 \right] \psi = 0, \quad (29)$$

$$\mathring{K}_2 = \psi^\dagger M^\dagger \gamma_0 i \gamma_{0123} \gamma_2 M \psi = \psi^\dagger \left[\begin{pmatrix} 2 & a \\ a & -a \end{pmatrix} \otimes \sigma_2 \right] \psi = 0, \quad (30)$$

$$\mathring{K}_3 = \psi^\dagger M^\dagger \gamma_0 i \gamma_{0123} \gamma_3 M \psi = \psi^\dagger \left[\begin{pmatrix} 0 & -a \\ a^* & 0 \end{pmatrix} \otimes \sigma_3 \right] \psi = 0. \quad (31)$$

After all, denoting $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$ ($\psi_r \in \mathbb{C}, r=1, \dots, 4$), we have the following simultaneous conditions for Eqs. (28)–(31), respectively:

$$\begin{aligned} 0 &= \operatorname{Re}(\psi_1^* \psi_3) + \operatorname{Re}(\psi_2^* \psi_4), \\ 0 &= \operatorname{Re}(\psi_2^* \psi_3) + \operatorname{Re}(\psi_1^* \psi_4), \\ 0 &= \operatorname{Im}(\psi_1^* \psi_4) - \operatorname{Im}(\psi_2^* \psi_3) - 2 \operatorname{Im}(\psi_3^* \psi_4) - 2 \operatorname{Im}(\psi_1^* \psi_2), \\ 0 &= \operatorname{Re}(\psi_1^* \psi_3) - \operatorname{Re}(\psi_2^* \psi_4). \end{aligned} \quad (32)$$

These constraints must hold for type-(1), -(2), and -(3) DSFs. Note that the first and the last conditions together mean $\operatorname{Re}(\psi_1^* \psi_3) = 0$ and $\operatorname{Re}(\psi_2^* \psi_4) = 0$. In what follows we obtain the extra necessary and sufficient conditions for each class of DSFs.

A. Additional conditions on class-(2) Dirac spinor fields

Type-(2) DSFs satisfy by definition the condition

$$\omega = -\psi^\dagger \gamma_0 \gamma_{0123} \psi = -\psi_1^* \psi_3 - \psi_3^* \psi_1 + \psi_2^* \psi_4 + \psi_4^* \psi_2 = 0. \quad (33)$$

Besides, the conditions obtained from $\mathring{\mathbf{K}}=0$, we also have in this case the additional condition

$$\mathring{\sigma} = \psi^\dagger M^\dagger \gamma_0 M \psi = \psi^\dagger \left[\begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix} \otimes i\sigma_2 \right] \psi = \operatorname{Re}(\psi_1^* \psi_4) + \operatorname{Im}(\psi_2^* \psi_3) = 0. \quad (34)$$

B. Additional conditions on class-(3) Dirac spinor fields

Class-(3) Dirac spinor fields satisfy—by definition—the condition

$$\sigma = \psi^\dagger \gamma_0 \psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 = 0. \quad (35)$$

Apart from the conditions obtained from $\mathring{\mathbf{K}}=0$, we also have for this class the additional condition

$$\mathring{\omega} = -\psi^\dagger M^\dagger \gamma_0 \gamma_{0123} M \psi = \psi^\dagger \left[\begin{pmatrix} 2 & a \\ a & 0 \end{pmatrix} \otimes \sigma_2 \right] \psi = \operatorname{Im}(\psi_1^* \psi_4) - \operatorname{Im}(\psi_2^* \psi_3) - 2 \operatorname{Im}(\psi_1^* \psi_2) = 0. \quad (36)$$

C. Additional conditions on class-(1) Dirac spinor fields

After the action of the matrix M , class-(1) DSFs must obey all the conditions given by Eqs. (32), (34), and (36). Note that if one relaxes the condition given by Eq. (34) or Eq. (36), DSFs of types (3) and (2) are, respectively, obtained.

TABLE I. Additional conditions, in components, for class- (1), -(2), and -(3) Dirac spinor fields.

| Class | Additional conditions |
|-------|--|
| (1) | $\psi_{2a}(\psi_{3a} - \psi_{3b}) + \psi_{2b}(\psi_{3a} + \psi_{3b}) = 0 = \psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a}$ |
| (2) | $\psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a} = 0 = \psi_{2a}\psi_{3a} + \psi_{2b}\psi_{3b} + \psi_{1a}\psi_{4a} + \psi_{1b}\psi_{4b}$ |
| (3) | $\psi_{2a}(\psi_{3a} - \psi_{3b}) + \psi_{2b}(\psi_{3a} + \psi_{3b}) = 0$ and $(\psi_{1a}\psi_{4b} - \psi_{1b}\psi_{4a}) - (\psi_{2a}\psi_{3b} - \psi_{2b}\psi_{3a}) - 2(\psi_{3a}\psi_{4b} - \psi_{3b}\psi_{4a}) - 2(\psi_{1a}\psi_{2b} - \psi_{1b}\psi_{2a}) = 0$ |

Using the decomposition $\psi_j = \psi_{j_a} + i\psi_{j_b}$ [where $\psi_{j_a} = \text{Re}(\psi_j)$ and $\psi_{j_b} = \text{Im}(\psi_j)$] it follows that $\text{Re}(\psi_i^* \psi_j) = \psi_{i_a}\psi_{j_a} + \psi_{i_b}\psi_{j_b}$ and $\text{Im}(\psi_i^* \psi_j) = \psi_{i_a}\psi_{j_b} - \psi_{i_b}\psi_{j_a}$ for $i, j = 1, \dots, 4$. So, in components, the conditions in common for all types of DSFs are

$$\psi_{1a}\psi_{3a} + \psi_{1b}\psi_{3b} = 0, \tag{37}$$

$$\psi_{2a}\psi_{4a} + \psi_{2b}\psi_{4b} = 0, \tag{38}$$

and the additional conditions for each case are summarized in Table I above.

V. CONCLUDING REMARKS AND OUTLOOKS

Once the matrix M —leading an arbitrary DSF to an ELKO—has been introduced, we proved that it can be written in the general form given by Eq. (22), without loss of generality. The *ansatz* given by Eq. (22) is useful to illustrate and explicitly exhibit how to obtain the necessary conditions on the components of a DSF—under Lounesto spinor field classification—in order for it be led to an ELKO spinor field. In the case of a type-(1) DSF, as accomplished in Sec. IV C, there are six conditions, from the definition of ELKO ($\overset{\circ}{\sigma} = 0 = \overset{\circ}{\omega} = \overset{\circ}{K}^\mu$), and then the equivalence class of type-(1) DSFs that can be led to ELKO spinor fields can be written in the form⁶

$$\psi = \begin{pmatrix} \psi_1 \\ f_1(\psi_1) \\ f_2(\psi_1) \\ f_3(\psi_1) \end{pmatrix}, \tag{39}$$

where f_i are complex scalar functions of the component $\psi_1 \in \mathbb{C}$ of ψ , obtainable—using the implicit function theorem—through the conditions given in Eqs. (37) and (38), and also those given by Table I. For a general and arbitrary *ansatz*, the equivalence class of type-(1) DSFs that can be led to ELKO spinor fields, via the matrix M , are given by

$$\psi = \begin{pmatrix} \psi_1 \\ g_1(M)(\psi_1) \\ g_2(M)(\psi_1) \\ g_3(M)(\psi_1) \end{pmatrix}, \tag{40}$$

where each $g_i(M)$ is a complex scalar function of the component $\psi_1 \in \mathbb{C}$ of ψ . Such scalar functions depend explicitly on the form of M , and to a fixed but arbitrary M there corresponds other six conditions analogous to Eqs. (37) and (38), and also those given by Table I. All these conditions obtained by the *ansatz* are general, and illustrate the general procedure of finding the conditions.

⁶Among the three equivalent definitions of spinor fields, viz., the classical, algebraic, and operatorial, here the classical one—where a spinor is an element that carries the representation space of the group $\text{spin}_+(1,3)$ —is regarded.

Regarding Secs. IV A and IV B, for the equivalence class of type-(2) and -(3) DSFs that are led to ELKO spinor fields, it is only demanded five conditions, instead of six, since, respectively, $\hat{\sigma}=0$, $\hat{\omega} \neq 0$ and $\hat{\sigma} \neq 0$, $\hat{\omega}=0$. In both cases, the most general form of the DSFs is given by

$$\psi = \begin{pmatrix} \psi_{1a} + i\psi_{1b} \\ \psi_{2a} + i\psi_{2b} \\ \psi_{3a} + i\psi_{3b} \\ \psi_{4a} + i\psi_{4b} \end{pmatrix} = \begin{pmatrix} \psi_{1a} + i\psi_{1b} \\ \psi_{2a} + ih_1(M)(\psi_{1a}, \psi_{1b}, \psi_{2a}) \\ h_2(M)(\psi_{1a}, \psi_{1b}, \psi_{2a}) + ih_3(M)(\psi_{1a}, \psi_{1b}, \psi_{2a}) \\ h_4(M)(\psi_{1a}, \psi_{1b}, \psi_{2a}) + ih_5(M)(\psi_{1a}, \psi_{1b}, \psi_{2a}) \end{pmatrix}, \quad (41)$$

where each $h_A(M)$ ($A=1, \dots, 5$) is a M matrix-dependent real scalar function of the (real) components ψ_{1a} , ψ_{1b} , and ψ_{2a} of ψ .

One of the main physical motivations here is that dark matter, which can be described by ELKO,⁶ interacts very weakly with SM particles, and the task is how to extend SM in order to incorporate ELKO. This approach can be of prime importance in *a posteriori* investigation about the dynamical aspects and about the standard model in ELKO context. Once we know the behavior of DSFs in the context of SM, and also the particular subsets of the equivalence classes of DSFs that can be led to ELKO, it is natural to ask whether it is now possible to extend SM using ELKO. Our paper is the first attempt—to our knowledge—to accomplish this purpose, and a new and physically alluring branch on standard model extensions and cosmology is proposed for further promising investigations.

ACKNOWLEDGMENTS

The authors are very grateful to Professor Dharamvir Ahluwalia-Khalilova for important comments about this paper, and to JMP referee for pointing out elucidating and enlightening viewpoints. Roldão da Rocha thanks Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) (PDJ 05/03071-0) and J. M. Hoff da Silva thanks CAPES-Brazil for financial support. R. da Rocha thanks Professor Dharamvir Ahluwalia-Khalilova for private communication on the subject.

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