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M. Aparicio Alcalde, G. Flores Hidalgo, and N. F. Svaiter

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# The two-loop massless $(\lambda/4!)\varphi^4$ model in nontranslational invariant domain

M. Aparicio Alcalde[a)]
*Centro Brasileiro de Pesquisas Físicas-CBPF, Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil*

G. Flores Hidalgo[b)]
*Instituto de Física Teórica-IFT/UNESP, Rua Pamplona 145, São Paulo, SP, 01405-900, Brazil*

N. F. Svaiter[c)]
*Centro Brasileiro de Pesquisas Físicas-CBPF, Rua Dr. Xavier Sigaud 150, Rio de Janeiro, RJ, 22290-180, Brazil*

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We study the $(\lambda/4!)\varphi^4$ massless scalar field theory in a four-dimensional Euclidean space, where all but one of the coordinates are unbounded. We are considering Dirichlet boundary conditions in two hyperplanes, breaking the translation invariance of the system. We show how to implement the perturbative renormalization up to two-loop level of the theory. First, analyzing the full two and four-point functions at the one-loop level, we show that the bulk counterterms are sufficient to render the theory finite. Meanwhile, at the two-loop level, we must also introduce surface counterterms in the bare Lagrangian in order to make finite the full two and also four-point Schwinger functions. © *2006 American Institute of Physics.*
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## I. INTRODUCTION

In this paper we are interested to show how to implement the renormalization procedure up to two-loop level in the massless $(\lambda/4!)\varphi^4$ scalar field theory, defined in a four-dimensional Euclidean space with one compactified dimension. Our aim is to shed light on the renormalization procedure in a system defined in a domain where translational symmetry is broken, which must be done, for example, in the high temperature dimensional reduced quantum chromodynamics (QCD), defined in a finite region.

Quantum chromodynamics is a non-Abelian Yang-Mills theory with gauge group SU(3). Since it is assumed that the fermions of the theory transform according to the fundamental representation of the gauge group, each flavor of quark is a triplet of the color group SU(3). Gauge bosons transform according to the adjoint representation. The interaction between the quarks is mediated by the gluons. Due to the non-Abelian structure of the theory, the gluons couple not only with the quarks but have also cubic and quartic self-interaction. The self-interaction of the gluons provides the antiscreening of the color charge in QCD. This is responsible for asymptotic freedom and presumably confinement.

The confinement-deconfinement phase transition in QCD may occur in usual matter at sufficiently high temperature or if it is strongly compressed.[1–3] In ultrarelativistic heavy ion collisions, we expect that the plasma of quarks and gluons can be produced. We would like to stress that, although nonequilibrium processes occur in the quark-gluon plasma in the heavy ion collisions, for

a)Electronic mail: aparicio@cbpf.br
b)Electronic mail: gflores@ift.unesp.br
c)Electronic mail: nfuxsvai@cbpf.br

 

simplicity in a first approximation we can assume a static situation. Just after the collision hot and compressed nuclear matter is confined in a small region of the space and in such circumstances the volume and surface effects become very important.

In the above described physical situation there are two important points: the first one is that the thermodynamic limit of the infinite volume system cannot be used and therefore finite volume effects should be investigated and taken into account. The second point is that the quark-gluon plasma exists in a situation of high temperature, where using the Matsubara formalism to describe high temperature QCD, dimensional reduction must occur.[4–7] Dimensional reduction is based on the Appelquist-Carrazone decoupling theorem.[8] From a more fundamental theory, the effective Lagrangian density of this theory can be obtained as some low-energy limit of the fundamental theory where the heavy modes have been removed. There are some interesting physical situations where the decoupling theorem can be used. First, for scalar fields without spontaneous symmetry breaking. Second, in quantum electrodynamics, where a derivative expansion of the photon effective action can be obtained by integrating out the fermionic fields. Also in QCD at least in lowest order in perturbation theory the decoupling theorem works. The decoupling theorem is not valid, for example, in spontaneous broken gauge theories. It is important to stress that the nonvalidity of the decoupling theorem means that low-energy experiments can provide information about the high energy physics.

Going back to the heavy-ions collision situation, we can assume that the following scenario appears: in the situation where dimensional reduction occurs, we have an effective theory for the gluons field and also finite size effects for these bosonic fields. To shed light on the renormalization procedure in systems defined in domain where translational symmetry is broken, as for example, the high temperature dimensional reduced QCD, in this paper we are interested to investigate scalar models, impose classical boundary condition over the fields. We hope that this study will give us some insight over the most interesting and also more complicated situation as the one mentioned above. Therefore, in this paper we analyze how to implement the perturbative renormalization up to two-loop level of $(\lambda/4!)\varphi^4$ massless scalar field model defined in a four-dimensional Euclidean space with one compactified dimension.

Finite size effects and the presence of macroscopic structures in different field theory models have been extensively studied in the literature. The critical behavior of the $O(N)$ model in the presence of a surface was a target of intense investigations.[9] The same $O(N)$ model was studied in two different geometries: the periodic cube and the cylinder along one dimension (the time) and finite and periodic in the $(d-1)$ remaining dimensions by Brezin and Zinn-Justin.[10] Finite size effects in QCD[11] and also in different field theory models have also been extensively studied in the literature. Assuming periodic or antiperiodic boundary conditions for bosonic and fermionic models, respectively, the translation symmetry is maintained, and surface effects are avoided. Therefore, to avoid surface effects, quantum fields defined in manifolds with periodic or anti-periodic boundary conditions in the spatial section were preferred by many authors.[12] Nevertheless, the case of boundaries conditions that break the translational symmetry deserves our attention.

In the case of hard boundary conditions as, for example, Dirichlet-Dirichlet (DD) or Neumann-Neumann (NN), the translational invariance is lacking. This fact makes the Feynman diagrams harder to compute than in an unbounded space. Moreover the renormalization program is implemented in a different way from unbounded or translational invariance systems since some surface divergence appears.[13] For translational invariant systems, one can use the momentum space representation, which is a more convenient framework to analyze the ultraviolet divergences of a theory. Translational invariance is preserved for momentum conservation conditions. For nontranslational invariant systems a more convenient representation for the $n$-point Schwinger functions is a mixed momentum coordinate space.

Fosco and Svaiter considered the anisotropic scalar model in a $d$-dimensional Euclidean space, where all but one of the coordinates are unbounded. Translational invariance along the bounded coordinate which lies in the interval $[0,L]$ is broken because the choice of boundary condition chosen for the hyperplanes at $z=0$ and $z=L$. Two different possibilities of boundary conditions were considered: (DD) and also (NN), and the renormalization of the two-point function was achieved in the one-loop approximation.[14] Further, the renormalization of the four-point function was achieved in the one-loop approximation by Caicedo and Svaiter.[15] Finally Svaiter[16] studied the renormalization of the $(\lambda/4!)\varphi^4$ massless scalar field model in the one-loop approximation in finite size systems assuming that the system is in thermal equilibrium with a reservoir. Also, still studying surface, edge, and corners effects, Rodrigues and Svaiter[17] analyzed first the renormalized vacuum fluctuations associated with a massless real scalar field, confined in the interior or a rectangular infinitely long waveguide. A closed form of the analytic continuation of the local zeta function in the interior of the waveguide was obtained and a detailed study of the surface and edge divergences was presented. Next, these authors[18] studied the renormalized stress tensor associated with an electromagnetic field in the interior of a rectangular infinitely long waveguide.

In this paper we will consider an interacting massless scalar model, in a four-dimensional Euclidean space, where the first three coordinates are unbounded and the last one lies in the interval $[0,L]$. We analyze only DD boundary conditions. First, we present an algebraic expression in coordinate space for the free propagator which let us identify the divergences of the $n$-point Schwinger functions for the interacting theory. This algebraic expression agrees with the result obtained by Lukosz.[19] We would like to stress that instead of assuming hard boundary conditions, some authors assumed soft boundary conditions and also treated the boundary as a quantum mechanical object.[20] Here, we prefer to keep hard classical boundary conditions.

The organization of the paper is as follows: In Sec. II we discuss the slab configurations, obtaining some important expressions for the free propagator in order to understand some procedures in the divergence identification. In Sec. III the regularization program is implemented in the one-loop approximation. In Sec. IV the regularization program is implemented in the two-loop approximation. Section V contains our conclusions. In the Appendix, an expression for the free propagator is introduced. Throughout this paper we use $\hbar = c = 1$.

## II. CLASSICAL BOUNDARY CONDITIONS AND SOME PROPERTIES OF THE FREE PROPAGATOR

Let us consider a neutral scalar field with a $(\lambda\varphi^4)$ self-interaction, defined in a $d$-dimensional Minkowski space-time. The vacuum persistence functional is the generating functional of all vacuum expectation value of time-ordered products of the theory. The Euclidean field theory can be obtained by analytic continuation to imaginary time allowed by the positive energy condition for the relativistic field theory. In the Euclidean field theory, we have the Euclidean counterpart for the vacuum persistence functional, that is, the generating functional of complete Schwinger functions. The $(\lambda\varphi^4)_d$ Euclidean theory is defined by these Euclidean Green's functions. The Euclidean generating functional $Z(h)$ is formally defined by the following functional integral:

$$Z(h) = \int [\mathrm{d}\varphi]\exp\left(-S_0 - S_I + \int d^dx h(x)\varphi(x)\right), \tag{1}$$

where the action that describes a free scalar field is

$$S_0(\varphi) = \int d^dx\left(\frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m_0^2\varphi^2(x)\right), \tag{2}$$

and the interacting part, defined by the non-Gaussian contribution, is

$$S_I(\varphi) = \int d^dx\frac{\lambda}{4!}\varphi^4(x). \tag{3}$$

In Eq. (1), $|\mathrm{d}\varphi|$ is a translational invariant measure, formally given by $|\mathrm{d}\varphi| = \Pi_x\, \mathrm{d}\varphi(x)$. The terms $\lambda$ and $m_0^2$ are, respectively, the bare coupling constant and mass squared of the model.

Finally, $h(x)$ is a smooth function that we introduce to generate the Schwinger functions of the theory by means of functional derivatives. Note that we are using the same notation for functionals and functions, for example, $Z(h)$ instead of the usual notation $Z[h]$.

In the weak-coupling perturbative expansion, we perform a formal perturbative expansion with respect to the non-Gaussian terms of the action. As a consequence of this formal expansion, all the $n$-point unrenormalized Schwinger functions are expressed in a power series of the bare coupling constant $g_0$. Let us summarize how to perform the weak-coupling perturbative expansion in the $(\lambda\varphi^4)_d$ theory. The Gaussian functional integral $Z_0(h)$ associated with the Euclidean generating functional $Z(h)$ is

$$Z_0(h) = \mathcal{N}\int [\mathrm{d}\varphi]\exp\left(-\frac{1}{2}\varphi K\varphi + h\varphi\right). \tag{4}$$

We are using a compact notation and the first term on the right-hand side of Eq. (4) is given by

$$\varphi K\varphi = \int d^dx \int d^dy\, \varphi(x)K(m_0;x,y)\varphi(y). \tag{5}$$

The term that couples linearly the field with the external source is

$$h\varphi = \int d^dx\, \varphi(x)h(x). \tag{6}$$

As usual $\mathcal{N}$ is a normalization factor and the symmetric kernel $K(m_0;x,y)$ is defined by

$$K(m_0;x,y) = (-\Delta + m_0^2)\delta^d(x-y), \tag{7}$$

where $\Delta$ denotes the Laplacian in the Euclidean space $R^d$. As usual, the normalization factor is defined using the condition $Z_0(h)|_{h=0}=1$. Therefore $\mathcal{N}=(\det(-\Delta+m_0^2))^{1/2}$ but, in the following, we are absorbing this normalization factor in the functional measure. It is convenient to introduce the inverse kernel, i.e., the free two-point Schwinger function $G_0(m_0;x-y)$ which satisfies the identity

$$\int d^dz\, G_0(m_0;x-z)K(m_0;z-y) = \delta^d(x-y). \tag{8}$$

Since Eq. (4) is a Gaussian functional integral, simple manipulations, performing only Gaussian integrals, gives

$$\int [\mathrm{d}\varphi]e^{-S_0+\int d^dx h(x)\varphi(x)} = \exp\left[\frac{1}{2}\int d^dx \int d^dy\, h(x)G_0^{(2)}(m_0;x-y)h(y)\right]. \tag{9}$$

Therefore, we have an expression for $Z_0(h)$ in terms of the inverse kernel $G_0^{(2)}(m_0;x-y)$, i.e., in terms of the free two-point Schwinger function. This construction is fundamental to perform the weak-coupling perturbative expansion with the Feynman diagramatic representation of the perturbative series. The non-Gaussian contribution in a perturbation with regard to the remaining terms of the action. It is important to point out that the weak-coupling perturbative expansion can be defined in arbitrary geometries, and classical boundary conditions must be implemented in the two-point Schwinger function. Another way is to restrict the space of functions that appear in the functional integral.

We are interested in studying finite size systems, where the translational invariance is broken. In this situation, we are analyzing the perturbative renormalization for the $(\lambda/4!)\varphi^4$ massless scalar field model, in the two-loop approximation. Therefore, let us assume boundary conditions over the plates for the massless field $\varphi(x)$. For simplicity we are assuming Dirichlet-Dirichlet boundary conditions, i.e.,

$$\varphi(\vec{r},z)\big|_{z=0} = \varphi(\vec{r},z)\big|_{z=L} = 0, \tag{10}$$

for the free field. Since the translational invariance is not preserved, let us use a Fourier expansion of the fields in the following form:

$$\varphi(\vec{r},z) = \frac{1}{(2\pi)^{(d-1)/2}} \int d^{d-1}p \sum_n \phi_n(\vec{p}) e^{i\vec{p}.\vec{r}} u_n(z), \tag{11}$$

where the set $u_n(z)$ are the orthonormalized eigenfunctions associated to the operator $-d^2/\mathrm{d}z^2$, $[-d/\mathrm{d}z^2 u_n(z) = k_n^2 u_n(z)]$, and $k_n = n\pi/L$, $n=1,2,\ldots$ . The orthonormal set corresponding to the eigenfunctions of the Hermitian operator $-d^2/\mathrm{d}z^2$ defined on a finite interval is given by

$$u_n(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi z}{L}\right), \; n = 1,2, \ldots \; . \tag{12}$$

These eigenfunctions satisfy the completeness and orthonormality relations, i.e.,

$$\sum_n u_n(z) u_n^*(z') = \delta(z - z') \tag{13}$$

and

$$\int_0^L \mathrm{d}z\, u_n(z) u_{n'}^*(z) = \delta_{n,n'} . \tag{14}$$

Since we are interested in performing the weak coupling expansion, let us first write the free two-point Schwinger function. This free two-point Schwinger function can be expressed in the following form:

$$G_0^{(2)}(\vec{r},z,z') = \frac{1}{(2\pi)^{d-1}} \int d^{d-1}p \sum_n e^{i\vec{p}\cdot\vec{r}} u_n(z) u_n^*(z') G_{0,n}(\vec{p}), \tag{15}$$

where $G_{0,n}(\vec{p})$ is given by

$$G_{0,n}(\vec{p}) = (\vec{p}^2 + k_n^2 + m^2)^{-1} . \tag{16}$$

Next, we will present some properties of the two-point free Schwinger function in order to understand the behavior of the interacting field theory in the presence of macroscopic structures. Therefore, in order to understand some procedures used in the identification of the divergences in the Schwinger functions that will appear in the next section, let us analyze some properties of the free two-point Schwinger function. Substituting Eq. (12) and Eq. (16) in Eq. (15) we get that the free propagator $G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2)$ can be written as

$$G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z_1}{L}\right) \sin\left(\frac{n\pi z_2}{L}\right) \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{i\vec{p}.(\vec{r}_1 - \vec{r}_2)}}{\left(\vec{p}^2 + \left(\frac{n\pi}{L}\right)^2 + m^2\right)} . \tag{17}$$

The next step is to show that the two-point free Schwinger function can be written in terms of the variables: $r_{12}, z_{12}^-$ and finally $z_{12}^+$, where $r_{12} = (|\vec{r}_1 - \vec{r}_2|)/L$, $z_{12}^- = (z_1 - z_2)/L$, and $z_{12}^+ = (z_1 + z_2)/L$, respectively. Working in the four-dimensional case and also in the massless situation, a straightforward calculation (see the Appendix) gives us that $G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2)$ can be written as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{16\pi^2L^2}\sum_{k=-\infty}^{\infty}\left\{\frac{1}{\left(k-\frac{|z_{12}^-|}{2}\right)^2+\left(\frac{r_{12}}{2}\right)^2}-\frac{1}{\left(k-\frac{|z_{12}^+|}{2}\right)^2+\left(\frac{r_{12}}{2}\right)^2}\right\}. \tag{18}$$

The former expression for the two-point Schwinger function was obtained also by Lukosz[19] using the image method. Performing the summations in Eq. (18) (see the Appendix), it is possible to find a closed expression for $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$. We get

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{\sinh(\pi r_{12})}{16\pi L^2 r_{12}}\left\{\frac{\sin\left(\frac{\pi z_1}{L}\right)\sin\left(\frac{\pi z_2}{L}\right)}{\left[\sinh^2\left(\frac{\pi r_{12}}{2}\right)+\sin^2\left(\frac{\pi z_{12}^-}{2}\right)\right]\left[\sinh^2\left(\frac{\pi r_{12}}{2}\right)+\sin^2\left(\frac{\pi z_{12}^+}{2}\right)\right]}\right\}. \tag{19}$$

It is not difficult to show that the two-point Schwinger function $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ satisfies the following properties:

(i) The free two-point Schwinger function is not negative, i.e., $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)\geqslant 0$, for $z_1,z_2\in[0,L]$ and $\vec{r}_1$, $\vec{r}_2\in\mathcal{R}^3$, since we are working in a Euclidean space.

(ii) The free two-point Schwinger function is zero when one of its points are evaluated on the boundaries

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,0,z_2)=G_0^{(2)}(\vec{r}_1-\vec{r}_2,L,z_2)=G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,0)=G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,L)=0,$$

since we are assuming Dirichlet boundary conditions.

(iii) The free two-point Schwinger function contain the usual bulk divergences, i.e., when $(\vec{r}_1,z_1)=(\vec{r}_2,z_2)$, it is singular. From Eq. (18) we can identify three singular terms. Splitting the free two-point Schwinger function in the singular and regular terms we have

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{4\pi^2L^2}\left\{\frac{1}{(z_{12}^-)^2+r_{12}^2}-\frac{1}{(z_{12}^+)^2+r_{12}^2}-\frac{1}{(2-z_{12}^+)^2+r_{12}^2}\right\}$$

$$+\frac{1}{4\pi^2L^2}\left\{\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{(2k-|z_{12}^-|)^2+r_{12}^2}-\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{(2k-|z_{12}^+|)^2+r_{12}^2}\right\}. \tag{20}$$

The first term on the right-side of the last equation, is singular only when $\vec{r}_1=\vec{r}_2$ and $z_1=z_2$. This is the term that carries the usual bulk divergences. The second term is singular only when $z_1=z_2=0$ and $\vec{r}_1=\vec{r}_2$. The third term is singular only when $z_1=z_2=L$ and $\vec{r}_1=\vec{r}_2$. These two terms mentioned previously carries surface divergences. Finally the two last terms do not have singularities.

(iv) When $|\vec{r}_1-\vec{r}_2|/L\gg 1$ the free propagator behaves like

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)\sim\frac{1}{2\pi L^2}\frac{e^{-\pi r_{12}}}{r_{12}}\sin\left(\frac{\pi z_1}{L}\right)\sin\left(\frac{\pi z_2}{L}\right), \tag{21}$$

which shows an exponential convergence behavior.

(v) The integral of the variable $\{\vec{r},z\}$ on a neighborhood around $\{\vec{r}',z'\}$ of the free propagator is finite, i.e., $\int_R d^3r\, d_z\, G_0^{(2)}(\vec{r}-\vec{r}',z,z')<\infty$. See Fig. 1.

Property (v) allows us to show that the external legs of the Feynman diagrams do not create divergences. Let us suppose we have the integral corresponding to some Feynman diagram,

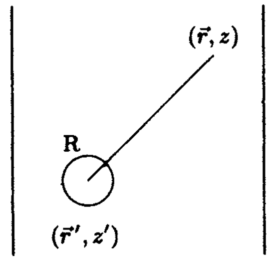


FIG. 1. The integral of the variable $\{\vec{r},z\}$ on a neighborhood around $\{\vec{r}';z'\}$.

$$\int_R d^3r\, \mathrm{d}z\, G_0^{(2)}(\vec{r}-\vec{r}',z,z')F(\vec{r}',z'), \tag{22}$$

where $G_0^{(2)}(\vec{r}-\vec{r}',z,z')$ is some external leg and $F(\vec{r}',z')$ describes the remainder part of the diagram. Now in order to proceed we must use the following statement: for two continuous and positives functions $f(\vec{x})$ and $g(\vec{x})$ defined in a finite region $R$ with the exception of the point $\vec{x}_1$ where $f(\vec{x})$ diverges, then the integral $I=\int_R d^dx f(\vec{x})g(\vec{x})$ is finite, if and only if $I'=\int_V d^dx f(\vec{x})$ is finite on some neighborhoods $V$ of the point $\vec{x}_1$. With the property (v) and the statement before we can see that external legs from the Feynman diagrams do not generate divergences.

## III. REGULARIZED TWO- AND FOUR-POINT SCHWINGER FUNCTIONS AT ONE-LOOP ORDER

In this section we identify the divergent contribution in the two- and four-point Schwinger function at one-loop level. Essentially we use Eq. (20) in the 1PI diagrams of the Green functions considering their external legs, and the integrations in the coordinate space. We write Eq. (20) as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{4\pi^2L^2}\left[\frac{1}{(z_{12}^-)^2+r_{12}^2}-\frac{1}{(z_{12}^+)^2+r_{12}^2}-\frac{1}{(2-z_{12}^+)^2+r_{12}^2}+h(r_{12},z_1,z_2)\right], \tag{23}$$

where $h(r_{12},z_1,z_2)$ is given by

$$h(r_{12},z_1,z_2)=\frac{1}{4}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{\left(k-\frac{|z_{12}^-|}{2}\right)^2+\left(\frac{r_{12}}{2}2\right)}-\frac{1}{4}\sum_{\substack{k=-\infty\\k\neq 0,1}}^{\infty}\frac{1}{\left(k-\frac{z_{12}^+}{2}\right)^2+\left(\frac{r_{12}}{2}2\right)}. \tag{24}$$

From the property (iii) we see that the three first contributions on the right-hand side of Eq. (23) have singularities. Otherwise, the last term is finite in the whole domain where we defined the model. After this brief introduction, we are able to study the interacting theory. Let us start analyzing the tadpole diagram, displayed in Fig. 2, from which we can write the expression for the

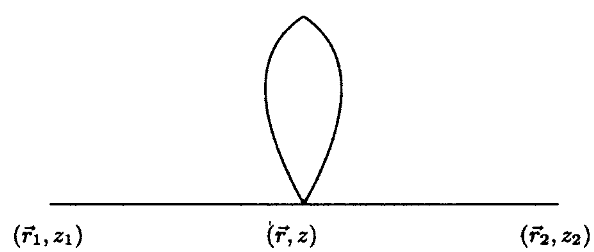


FIG. 2. The two-point function at one-loop level.

one-loop two-point Schwinger function $G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$. We have that

$$G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{\lambda}{2}\int d^3r\,\mathrm{d}z\,G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(0,z,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z). \tag{25}$$

In the following we are generalizing the results obtained by Fosco and Svaiter.[14] Let us begin studying the quantity $G_0^{(2)}(0,z,z)$ that appears in the tadpole defined in Eq. (25). From Eq. (20) we get that $G_0^{(2)}(0,z,z)$ can be written as

$$G_0^{(2)}=(0,z,z)=\frac{1}{4\pi^2L^2}\left[A-\frac{1}{(2z/L)^2}-\frac{1}{(2-2z/L)^2}+\sum_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{1}{(2k)^2}-\sum_{\substack{k=-\infty\\k\neq0,1}}^{\infty}\frac{1}{(2k-2z/L)^2}\right], \tag{26}$$

where $A$ is given by

$$A=\lim_{(z_1,\vec{r}_1)\to(z_2,\vec{r}_2)}\frac{L^2}{(z_1-z_2)^2+|\vec{r}_1-\vec{r}_2|^2}$$

$$=\lim_{\Lambda\to\infty}\frac{L^2S_4}{8\pi^2}\Lambda^2, \tag{27}$$

$S_d=[2\pi^{d/2}/\Gamma(d/2)]$ and $\Lambda$ is an ultraviolet cutoff. In the same way, from Eq. (26) by performing the summations, we-get for $G_0^{(2)}(0,z,z)$,

$$G_0^{(2)}(0,z,z)=\frac{1}{4\pi^2L^2}\left[A+\frac{\pi^2}{12}-\frac{\pi^2}{4}\frac{1}{\sin^2(\pi z/L)}\right]. \tag{28}$$

Substituting Eq. (27) in Eq. (28) we obtain

$$G_0^{(2)}(0,z,z)=\lim_{\Lambda\to\infty}\frac{S_4}{32\pi^4}\Lambda^2+\frac{1}{48L^2}-\frac{1}{16L^2}\frac{1}{\sin^2(\pi z/L)}. \tag{29}$$

The first term in Eq. (29) is a bulk divergence. Substituting Eq. (29) in Eq. (25) we get

$$\begin{aligned}G_1^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=&\lim_{\Lambda\to\infty}\frac{\lambda S_4}{64\pi^4}\Lambda^2\int_R d^3r\,\mathrm{d}z\,G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)\\&+\frac{\lambda}{96L^2}\int_R d^3r\,\mathrm{d}z\,G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)\\&-\frac{\lambda}{32L^2}\int_R d^3r\,\mathrm{d}z\frac{G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)}{\sin^2(\pi z/L)}\end{aligned} \tag{30}$$

The first term on the right-hand side carries a bulk divergence. The second term is finite. To see this we analyze the integral by sectors. Therefore we have

$$\int_R d^3r\,\mathrm{d}z\,G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)=\int_{R_1}+\int_{R_2}+\int_{R_3}+\int_{R_4}+\int_{R_5}, \tag{31}$$

where each integral is defined in different regions displayed in Fig. 3, where the points $(\vec{r}_1,z_1)$ and $(\vec{r}_2,z_2)$ are the centers of the regions $R_1$ and $R_2$, respectively. Using the property (v) we have that the integrals on $R_1$ and $R_2$ are finite. Since the free propagators $G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)$ and $G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z)$ presented in Eq. (31) do not have divergences on $R_3$ and this region is compact, then the integral on $R_3$ is finite. The integrals defined in regions $R_4$ and $R_5$ also are finite since from the property (iv) the propagator decreases exponentially when one of its points becomes far from the

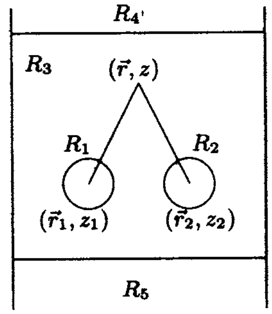


FIG. 3. Regions of integration $R_i$.

other. Thus the integral defined by Eq. (31) is finite. Finally we must study the third integral on the right-hand side of Eq. (30). Note that the term $1/\sin^2(\pi z/L)$ diverges when $z$ is evaluated on the boundaries.

Nevertheless this integral is convergent, because the products of $G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)$ and $G_0^{(2)}\times(\vec{r}_2-\vec{r},z_2,z)$ take away the divergence. Using Eq. (19), we have that the third integral on the right-hand side of Eq. (30) is finite. Therefore the one-loop two-point Schwinger function only has bulk divergence.

Our next step is to analyze the four-point Schwinger function in the one-loop level (see Fig. 4). Since the free propagator only has singularities when its two points are equal or also when the two points joined are evaluated at the boundaries, we continue our analysis of the integrals only in the domains where the two external points of the free propagators take the same values. The complete four-point function at one-loop level is given by

$$G_1^{(4)}(\vec{r}_1,z_1,\vec{r}_2,z_2,\vec{r}_3,z_3,\vec{r}_4,z_4) = \frac{\lambda^2}{2}\int d^{d-1}r\int d^{d-1}r'\int_0^L \mathrm{d}z\int_0^L \mathrm{d}z'\; G_0^{(2)}(\vec{r}_1-\vec{r},z_1,z)G_0^{(2)}(\vec{r}_2-\vec{r},z_2,z) \times\big[G_0^{(2)}(\vec{r}-\vec{r}',z,z')\big]^2 G_0^{(2)}(\vec{r}_3-\vec{r}',z_3,z')G_0^{(2)}(\vec{r}_4-\vec{r}',z_4,z'). \tag{32}$$

For simplicity, in Fig. 5 we define three different regions between the boundaries. The first one, $R_1$ is concerned when $\{\vec{r}',z'\}$ is close to $\{\vec{r},z\}$. In this region the contribution coming from $[G_0^{(2)}(\vec{r}-\vec{r}',z,z')]^2$ is singular. Nevertheless, we still must analyze if this divergent behavior will appear in the integral defined by Eq. (32). We will show that the singularities will appear only as bulk divergences. In the region $R_2$ $(z,z'\to 0$ and $\vec{r}'\to\vec{r})$ the term $[G_0^{(2)}(\vec{r}-\vec{r}',z,z')]^2$ is also divergent. As we will see, this divergent behavior disappears when we compute the complete four-point function at one-loop order, defined by Eq. (32). In the region $R_3$ $(z,z'\to L$ and $\vec{r}'\to\vec{r})$ the situation is identical as in the region $R_2$. Using the same argument that we used before to analyze the convergence of the integral denned by Eq. (22), we can study the convergence of the integral defined by Eq. (32) with the amputated external legs. Therefore we must study Eq. (32) with the external legs amputated. Therefore we must study the quantity $\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'[G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2$. Substituting Eq. (23) in the former equation we get

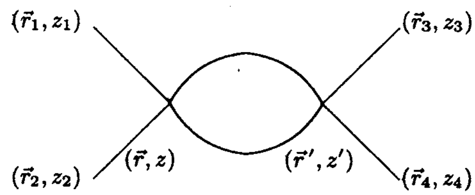


FIG. 4. The four-point function at one loop.

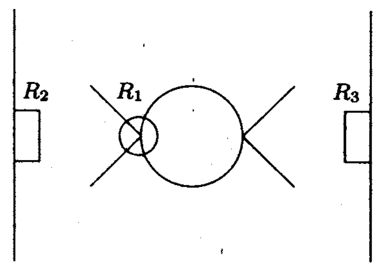


FIG. 5. Regions of integration for the four-point function.

$$\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\big[G_0^{(2)}(\vec{r}'-\vec{r},z',z)\big]^2=\frac{1}{(4\pi^2L^2)^2}\big(I_1+I_2+I_3+I_4+I_5\big)+\text{finite part},\qquad(33)$$

where the integrals $I_i, i=1,2,\ldots$, are given by

$$I_1=\int d^3r'\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{1}{\big[(z_{12}^-)^2+r_{12}^2\big]^2},\qquad(34)$$

$$I_2=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\left[-\frac{1}{\big[(z_{12}^+)^2+r_{12}^2\big]^2}+\frac{1}{\big[(2-z_{12}^+)^2+r_{12}^2\big]^2}\right],\qquad(35)$$

$$I_3=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\left[-\frac{2}{\big[(z_{12}^-)^2+r_{12}^2\big]\big[(z_{12}^+)^2+r_{12}^2\big]}-\frac{2}{\big[(z_{12}^-)^2+r_{12}^2\big]\big[(2-z_{12}^+)^2+r_{12}^2\big]}\right]\qquad(36)$$

$$I_4=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{2}{\big[(z_{12}^+)^2+r_{12}^2\big]\big[(2-z_{12}^+)^2+r_{12}^2\big]},\qquad(37)$$

$$I_5=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\left[\frac{1}{(z_{12}^-)^2+r_{12}^2}-\frac{1}{(z_{12}^+)^2+r_{12}^2}-\frac{1}{(2-z_{12}^+)+r_{12}^2}\right]h(r_{12},z_1,z_2).\qquad(38)$$

Let us investigate each term of Eq. (33). The integral $I_1$ must be analyzed only in the region $R_1$. For this purpose we need an auxiliary result. We can prove that a continuous and positive function $f(x)$ which does not have singularities except for $x=0$, and $M=\int_{-\vec{\epsilon}}^{\vec{\epsilon}}d^dx f(w^2)$ where $w^2=|\vec{w}|^2$, then there exist $\epsilon'$ such that $M=S_d\int_0^{\epsilon'}\mathrm{d}w\, w^{d-1}f(w^2)$ where $\epsilon<\epsilon'<\sqrt{d}\epsilon$. Then we get

$$I_1=\int_{R_1}d^3r'\,\mathrm{d}z'\frac{1}{[(z_{12}^-)^2+r_{12}^2]^2}=\int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}}d^3r'\int_{z-\epsilon}^{z+\epsilon}\mathrm{d}z'\frac{1}{[(z-z')^2+|\vec{r}-\vec{r}'|^2]^2}$$

$$=\int_{-\vec{\epsilon}}^{\vec{\epsilon}}\frac{d^4w}{w^4}=S_4\int_0^{\epsilon'}\mathrm{d}w\frac{w^3}{w^4}=S_4\ln w\,\Big|_0^{\epsilon'}.\qquad(39)$$

Therefore $I_1$ contributes with a bulk divergence of the type as the one that appears in the theory without boundaries. In the usual renormalization procedure, the contribution coming from $I_1$ can be eliminated by the usual counterterms. Concerning the contribution coming from $I_2$ we have that the first term $1/[(z_{12}^+)^2+r_{12}^2]^2$ is not singular in the region $R_1$. In the region $R_2$, using the same auxiliary result that we used before, we can obtain an upper bound to the contribution coming from this term. We get

$$\int_{R_2} d^3r\,\mathrm{d}z\,d^3r\,\mathrm{d}z'\frac{1}{[(z+z')^2+|\vec{r}-\vec{r}'|^2]^2} < \int d^3r\int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r'\int_0^{\epsilon}\int_0^{\epsilon}\mathrm{d}z\,\mathrm{d}z'\frac{1}{[z^2+z'^2+|\vec{r}-\vec{r}'|^2]^2}$$

$$<\frac{1}{4}\int d^3r\int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^5w\frac{1}{w^2}=\frac{1}{12}S_5\epsilon'^3\int_{R'} d^3r. \tag{40}$$

Since the region $R'\subset R_2$ is finite this integral is convergent. Next, let us analyze the term $1/[(2-z_{12}^{+})^2+r_{12}^2]^2$ of $I_2$ in the region $R_3$. Since the behavior of the field in each plates (for $z=0$ and $z=L$) is the same, then the analysis follows the same lines as previous ones and therefore this contribution is also finite. To study $I_3$, we consider first the term $2/[(z_{12}^{-})^2+r_{12}^2][(z_{12}^{+})^2+r_{12}^2]$. This expression must be studied in the regions $R_1$ and $R_2$, respectively. In $R_1$ we can see that the convergence of

$$\int_{R_1} d^3r dz d^3r' dz'\,\frac{1}{[(z-z')^2+|\vec{r}-\vec{r}'|^2]\underbrace{[(z+z')^2+|\vec{r}-\vec{r}'|^2]}_{\text{finite in } R_1}}, \tag{41}$$

depends on the convergence of

$$\int_{R_1} d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{1}{(z-z')^2+|\vec{r}-\vec{r}'|^2}. \tag{42}$$

From the above arguments we have that Eq. (42) can be written as

$$\int_{-\vec{\epsilon}}^{\vec{\epsilon}}\frac{d^4w}{w^2}=S_4\int_0^{\epsilon'}\mathrm{d}w\,w=\frac{S_4\epsilon'^2}{2}, \tag{43}$$

thus Eq. (42) gives a finite contribution. Now we consider the first term of $I_3$ in the region $R_2$. For this purpose we will use the following property. Let us take a continuous and positive function $f(x)$ which does not have singularities except for $x=0$, and $N=\int_0^{\vec{\epsilon}}\int_0^{\vec{\epsilon}} d^l y d^m z f(y^2+z^2)$ then there exist $\epsilon'$ in such a way that $N=(S_{l+m+2}/S_{l+1}S_{m+1})\int_0^{\epsilon'}\mathrm{d}w\,w^{l+m+1}f(w^2)$ where $\epsilon'>0$. Using this property, we have for the first term of Eq. (35), in the region $R_2$, that

$$\int d^3r\int_0^{\epsilon}\mathrm{d}z\int_z^{z+\epsilon}\mathrm{d}z'\int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r'\frac{1}{[(z-z')^2+|\vec{r}-\vec{r}'|^2][(z+z')^2+|\vec{r}-\vec{r}'|^2]}$$

$$<\int d^3r\int_0^{\epsilon}\mathrm{d}z\int_0^{\epsilon}\mathrm{d}u\int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^3v\frac{1}{(u^2+v^2)(z^2+u^2+v^2)}$$

$$<\frac{1}{4}S_4\int d^3r\int_0^{\epsilon}\mathrm{d}z\int_0^{\epsilon'}\mathrm{d}w\frac{w}{(z^2+w^2)}=\frac{S_3S_4}{2S_1S_2}\epsilon''. \tag{44}$$

Therefore the first term of $I_3$ is also finite in $R_2$. The second term $2/[((z_{12}^{-})^2+r_{12}^2)((2-z_{12}^{+})^2+r_{12}^2)]$ in $I_3$ must be analyzed also in the regions $R_1$ and $R_3$. This analysis follows the same lines as the last case, therefore the contribution coming from this term is also finite.

We have now to study the term $I_4$. Note that $2/[((z_{12}^{+})^2+r_{12}^2)((2-z_{12}^{+})^2+r_{12}^2)]$ must be analyzed in the regions $R_2$ and $R_3$, respectively. Let us start with the region $R_2$. Using previous arguments we have that the convergence of $I_4$ depends on the convergence of the following expression:

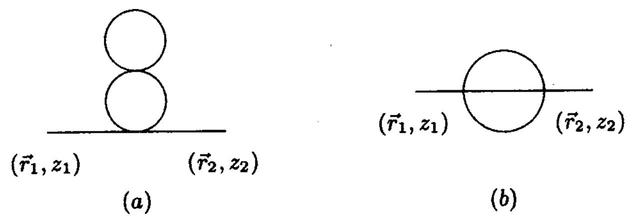


FIG. 6. Two-point Schwinger functions at two-loop level.

$$\int_0^{\epsilon}\int_0^{\epsilon} dz\, dz' \int_{\vec{r}-\vec{\epsilon}}^{\vec{r}+\vec{\epsilon}} d^3r' \frac{1}{z^2+z'^2+|\vec{r}-\vec{r}'|^2} = \frac{S_5}{4}\int_0^{\epsilon'} dw \frac{w^4}{w^2} = \frac{S_5}{12}\epsilon'^3, \tag{45}$$

which is finite. In the region $R_3$ our analysis follows the same lines as in the region $R_2$, thus the integral in the region. $R_3$ is also finite.

Using the same argument that we used before, it is not difficult to show that the contribution coming from $I_5$ is also finite. We conclude that the integrals given by Eq. (32) only have bulk divergences. In this way we can conclude that at the one-loop level the bulk counterterms are sufficient to render the complete connected Schwinger functions finite. In the next section we will identify the divergent contribution in the connected two-point Schwinger functions at the two-loop order.

## IV. THE DIVERGENCES IN THE TWO-POINT SCHWINGER FUNCTIONS AT TWO-LOOP LEVEL

In this section we will generalize some results obtained by Fosco and Svaiter[14] and also by Caicedo and Svaiter.[15] We will identify the divergent contribution in the connected two-point Schwinger functions at the two-loop order. The diagrams that we are interested to analyze are displayed in Fig. 6. The expression that corresponds to Fig. 6(a) is given by

$$\frac{\lambda^2}{4}\int d^3r'\, dz'\, d^3r\, dz\, G_0^{(2)}(\vec{r}_1-\vec{r}',z_1,z')\left[G_0^{(2)}(\vec{r}'-\vec{r},z',z)\right]^2 G_0^{(2)}(0,z,z) G_0^{(2)}(\vec{r}_2-\vec{r}',z_2,z'). \tag{46}$$

Since the external legs in Eq. (46) do not contribute to generate divergences, let us consider only the following integral:

$$\int d^3r\, dz [G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2 G_0^{(2)}(0,z,z). \tag{47}$$

Replacing Eq. (29) in Eq. (47) we get

$$\begin{aligned}\int d^3r\, dz [G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2 G_0^{(2)}(0,z,z) = {} & \lim_{\Lambda\to\infty}\frac{S_4}{32\pi^4}\Lambda^2\int d^3r\, dz [G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2 \\ & + \frac{1}{48L^2}\int d^3r\, dz [G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2 \\ & - \frac{1}{16L^2}\int d^3r\, dz [G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2 \frac{1}{\sin^2(\pi z/L)}.\end{aligned} \tag{48}$$

The first term and the second one in Eq. (48) can be renormalized introducing only bulk counterterms. The most interesting behavior appears in the last term of this equation. Note that this unrenormalized quantity contains only bulk divercences, since the contribution coming from

$[G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2$ cancels the surface divergent behavior generated by the $1/\sin^2(\pi z/L)$ term. Nevertheless, after the introduction of a bulk counterterm to render the contribution $[G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^2$ finite between the plates, surface divergences appear. Thus this surface divergence must be renormalized. After the introduction of surface and bulk counterterms, the finite contribution coming from the last term of Eq. (48), up to a finite renormalization constant, is given by

$$\frac{1}{16L^2}\int d^3r\,\mathrm{d}z\left[(G_0^{(2)}(\vec{r}'-\vec{r},z',z))^2-\frac{1}{(4\pi^2L^2)^2}\frac{1}{[(z_{12}^-)^2+r_{12}^2]^2}\right]\left[\frac{1}{\sin^2(\pi z/L)}-\frac{L^2}{(\pi z)^2}-\frac{L^2}{\pi^2(L-z)^2}\right]. \tag{49}$$

Therefore this term contains an overlapping between bulk and surface counterterms.

We still must analyze the sunset diagram. The expression corresponding to Fig. 6(b) is given by

$$\frac{\lambda^2}{6}\int d^3r'\,\mathrm{d}z'\,d^3r\,\mathrm{d}z\,G_0^{(2)}(\vec{r}_1-\vec{r}',z_1z')\big[G_0^{(2)}(\vec{r}'-\vec{r},z',z)\big]^3G_0^{(2)}(\vec{r}_2-\vec{r}',z_2,z'). \tag{50}$$

Again, the external legs do not contribute to generate divergences, and therefore let us study the amputated diagram, i.e., without external legs. We have

$$\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'[G_0^{(2)}(\vec{r}'-\vec{r},z',z)]^3=\frac{1}{(4\pi^2L^2)^3}(I_1+I_2+\cdots+I_{12})+\text{finite part}, \tag{51}$$

where

$$I_1=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{1}{[(z_{12}^-)^2+r_{12}^2]^3}, \tag{52}$$

$$I_2=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{1}{[(z_{12}^+)^2+r_{12}^2]^3}, \tag{53}$$

$$I_3=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{1}{[(2-z_{12}^+)^2+r_{12}^2]^3}, \tag{54}$$

$$I_4=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{2}{[(z_{12}^-)^2+r_{12}^2]^2[(z_{12}^+)^2+r_{12}^2]}, \tag{55}$$

$$I_5=\int d^3r\,\mathrm{d}z\,d^3r'\mathrm{d}z'\frac{2}{[(z_{12}^-)^2+r_{12}^2]^2[(2-z_{12}^+)^2+r_{12}^2]}, \tag{56}$$

$$I_6=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{2}{[(z_{12}^-)^2+r_{12}^2][(z_{12}^+)^2+r_{12}^2]^2}, \tag{57}$$

$$I_7=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{2}{[(z_{12}^-)^2+r_{12}^2][(2-z_{12}^+)^2+r_{12}^2]^2}, \tag{58}$$

$$I_8=\int d^3r\,\mathrm{d}z\,d^3r'\,\mathrm{d}z'\frac{2}{[(z_{12}^+)^2+r_{12}^2]^2[(2-z_{12}^+)^2+r_{12}^2]}, \tag{59}$$

$$I_9 = \int d^3r\, \mathrm{d}z\, d^3r'\, \mathrm{d}z' \frac{2}{[(z_{12}^+)^2 + r_{12}^2][(2 - z_{12}^+)^2 + r_{12}^2]^2}, \tag{60}$$

$$I_{10} = \int d^3r\, \mathrm{d}z\, d^3r'\, \mathrm{d}z' \frac{6}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2][(2 - z_{12}^+)^2 + r_{12}^2]}, \tag{61}$$

$$I_{11} = \int d^3r\, \mathrm{d}z\, d^3r'\, \mathrm{d}z' \left[\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right]^2 h(r_{12}, z_1, z_2), \tag{62}$$

$$I_{12} = \int d^3r\, \mathrm{d}z\, d^3r'\, \mathrm{d}z' \left[\frac{1}{(z_{12}^-)^2 + r_{12}^2} - \frac{1}{(z_{12}^+)^2 + r_{12}^2} - \frac{1}{(2 - z_{12}^+)^2 + r_{12}^2}\right] h^2(r_{12}, z_1, z_2), \tag{63}$$

Let us analyze each contribution coming from each term of Eq. (51). The first integral $I_1$ given by Eq. (52) is divergent in $R_1$. In general we can show that

$$\int_{R_1} d^3r'\, \mathrm{d}z' \frac{1}{[(z_{12}^-)^2 + r_{12}^2]^n} = \begin{cases} \text{finite}, & n < 2, \\ \infty, & n \geqslant 2. \end{cases} \tag{64}$$

Using the above result we can see that the integrals $I_3, I_4$ and the first integral of $I_{11}$ are divergent. These integrals contain bulk divergences which must be removed introducing bulk counterterms. Next let us analyze the contribution coming from the integral $I_2$ in the region $R_2$. Using previous arguments and considering the external legs we get

$$\int_{R_2} d^3r\, \mathrm{d}z\, \mathrm{d}^3r'\, \mathrm{d}z' \frac{zz'}{[(z_{12}^+)^2 + r_{12}^2]^3} < \int d^3r \int_{-\vec{\epsilon}}^{\vec{\epsilon}} \int_0^{\epsilon} \int_0^{\epsilon} d^3u\, \mathrm{d}z\, \mathrm{d}z' \frac{zz'}{(z^2 + z'^2 + w^2)^3} < \frac{S_7}{S_2^2} \epsilon' \int d^3r. \tag{65}$$

Therefore this term gives a finite contribution to Eq. (50). The contribution from the integral $I_6$ to the integral must be studied in region $R_2$. In this case we must consider the external legs, and the property: let us take a function $f(x,y)$ positive which does not have singularities except for $(x,y)=(0,0), I=\int_0^{\epsilon}\int_0^{\epsilon}\mathrm{d}x\,\mathrm{d}y\, f(x,y)$ then, $I<\int_0^{\epsilon}\mathrm{d}x\int_x^{x+\epsilon}\mathrm{d}y\, f(x,y)+\int_0^{\epsilon}\mathrm{d}y\int_y^{y+\epsilon}\mathrm{d}y f(x,y)$, we get

$$\int_{R_2} d^3r\, \mathrm{d}z\, d^3r'\, \mathrm{d}z' \frac{zz'}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2]^2} < 2 \int d^3r \int_{-\vec{\epsilon}}^{\vec{\epsilon}} d^3w \int_0^{\epsilon} \mathrm{d}z \int_0^{\epsilon} \mathrm{d}u \frac{z(z+u)}{(u^2 + w^2)(u^2 + z^2 + w^2)^2}. \tag{66}$$

From the above arguments we have that the contribution from the integral $I_6$ is smaller than

$$S_4 \int d^3r \int_0^{\epsilon} \mathrm{d}z \int_0^{\epsilon'} \mathrm{d}s \frac{z^2 s}{(s^2 + z^2)^2} + 2S_3 \int d^3r \int_0^{\epsilon} \mathrm{d}z \int_0^{\epsilon} \mathrm{d}u \int_0^{\epsilon'} \mathrm{d}s \frac{zus^2}{(u^2 + s^2)(u^2 + s^2 + z^2)^2}$$

$$< \frac{S_4 S_5}{S_2 S_3}\epsilon'' \int d^3 r + 2\frac{(S_5)^2}{(S_2)^2 S_3}\int d^3 r \int_0^{\epsilon'''} \mathrm{d}w \frac{w^4}{w^4} < \left(\frac{S_4 S_5}{S_2 S_3}\epsilon'' + 2\frac{(S_5)^2}{(S_2)^2 S_3}\epsilon'''\right)\int d^3 r. \qquad (67)$$

We conclude that the integral $I_6$ is finite. Also integrating the contribution coming from the term $I_8$ on $R_2$ we get

$$\int_{R_2} d^3 r dz d^3 r' dz' \frac{1}{[(z_{12}^+)^2 + r_{12}^2]^2 \underbrace{[(2 - z_{12}^+)^2 + r_{12}^2]}_{\text{finite}}}. \qquad (68)$$

Using the fact that the integral $\int_{R_2} d^3 r\, \mathrm{d}z\, d^3 r'\, \mathrm{d}z'\, 1/[(z_{12}^+)^2 + r_{12}^2]^2$ is finite, we have that this integral also is convergent in $R_2$. The contribution from the term $I_{10}$ on $R_2$ is given by

$$\int_{R_2} d^3 r\, \mathrm{d}z\, d^3 r'\, \mathrm{d}z' \frac{1}{[(z_{12}^-)^2 + r_{12}^2][(z_{12}^+)^2 + r_{12}^2]\underbrace{[(2 - z_{12}^+)^2 + r_{12}^2]}_{\text{finite}}}. \qquad (69)$$

Since the integral $\int_{R_2} d^3 r\, \mathrm{d}z\, d^3 r'\, \mathrm{d}z' \frac{1}{[(z_{12}^-)^2 + r_{12}^2][(x_{12}^+) + r_{12}^2]}$ is finite, then the integral defined by Eq. (69) is convergent in $R_2$. The contribution coming from the terms $I_{11}$ contain only a bulk divergence. Otherwise, the contributions coming from the terms $I_{12}$ are finite. We conclude that we need only bulk counterterms to render the integral defined by Eq. (50) finite. The same analysis can be done for the four-point Schwinger function in the two-loop approximation. We obtained that only bulk divergences appear in the full four-point function.

## V. CONCLUSIONS

In this paper we are interested to show how to implement the renormalization procedure in systems where the translational invariance is broken by the presence of macroscopic structures. For the sake of simplicity we are studying the self-interacting massless scalar field theory in a four-dimensional Euclidean space. We impose that one coordinate is defined in a compact domain, introducing two parallel mirrors where we are assuming Dirichlet-Dirichlet boundary conditions. Note that although that there are some similarities with the finite temperature field theory using the Matsubara formalism, in thermal systems there appears only bulk divergences, as for example, in the case of the system where we assume periodic boundary conditions. In nontranslational invariant systems, in general, to render the theory finite it is necessary to introduce surface counterterms.

In this work we generalize some results obtained by Fosco and Svaiter[14] and also by Caicedo and Svaiter.[15] We identify the divergences of the Schwinger functions in the massless self-interacting scalar field theory up to the two-loop approximation. First, analyzing the full two- and four-point Schwinger functions at the one-loop level, we show that the bulk counterterms are sufficient to render the theory finite. Second, at the two-loop level, we must introduce surface counterterms in the bare Lagrangian in order to make finite the full two- and also four-point Schwinger functions. The most interesting behavior appears in the "double scoop" diagram given by Eq. (46). The amputated diagram is given by Eq. (47) and we are interested in the last term of Eq. (48). This unrenormalized quantity contains only bulk divergences. Nevertheless, after the introduction of a bulk counterterm to render the contribution finite between the plates, surface divergences appear. Thus this surface divergence must be renormalized. Therefore this term contains an overlapping between bulk and surface counterterms. This procedure can be generalized to the $n$-loop level. The inclusion of the counterterm in the Lagrangian up to two-loop level with the full renormalized action and the general algorithm to identify the surface and bulk conterterms in the $n$-loop level will be left to future work.

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## APPENDIX

In this Appendix we will derive a useful representation for the free two-point Schwinger function. Starting from Eq. (17), we have that $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ is given by

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{2}{(2\pi)^{d-1}L}\sum_{n=1}^{\infty}\int d^{d-1}p\,\sin\left(\frac{n\pi z_1}{L}\right)\sin\left(\frac{n\pi z_2}{L}\right)\frac{e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}}{\left[\vec{p}^2+\left(\frac{n\pi}{L}\right)^2+m^2\right]}. \tag{A1}$$

Using the variables $u=(z_1-z_2)/L$ and $v=(z_1+z_2)/L$ defined, respectively, in the region $u\in[-1,1]$ and $v\in[0,2]$, and also making use of a trigonometric identity and performing the sum that appears in Eq. (A1) we obtain[21]

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{2}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{e^{i\vec{p}\cdot(\vec{r}_1-\vec{r}_2)}}{(\vec{p}^2+m^2)^{1/2}}\left[\frac{\cosh(L(1-|u|)(\vec{p}^2+m^2)^{1/2})}{\sinh(L(\vec{p}^2+m^2)^{1/2})}-\frac{\cosh(L(1-v)(\vec{p}^2+m^2)^{1/2})}{\sinh(L(\vec{p}^2+m^2)^{1/2})}\right]. \tag{A2}$$

Taking $m=0,d=4$, and integrating the angular part, it is possible to show that $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ can be written as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{-i}{2(2\pi)^2r'L^2}\int_0^{\infty}\mathrm{d}x(e^{ixr'}-e^{-ixr'})\left[\frac{\cosh((1-|u|)x)}{\sinh x}-\frac{\cosh((1-v)x)}{\sinh x}\right], \tag{A3}$$

where the variable $r'$ is defined by $r'\equiv(|\vec{r}_1-\vec{r}_2|)/L$. Making use of the following integral representation of the product between the gamma function and the Riemann zeta function[21]

$$\int_0^{\infty}\mathrm{d}x\frac{x^{z-1}e^{-\beta x}}{e^{px}-1}=\frac{\Gamma(z)}{p^z}\zeta\left(z,\frac{\beta}{p}+1\right), \tag{A4}$$

where $\mathrm{Re}(z)>1,\mathrm{Re}(\beta/p)>-1$ and the Riemann zeta function $\zeta(z,q)$ is defined by

$$\zeta(z,q)=\sum_{k=0}^{\infty}\frac{1}{(k+q)^z},\quad q\neq 0,-1,-2,\ldots\ , \tag{A5}$$

then, it is possible to write $G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)$ as

$$G_0^{(2)}(\vec{r}_1-\vec{r}_2,z_1,z_2)=\frac{1}{16\pi^2L^2}\left[\sum_{k=-\infty}^{\infty}\frac{1}{\left(k-\frac{|u|}{2}\right)^2+\left(\frac{r'}{2}\right)^2}-\sum_{k=-\infty}^{\infty}\frac{1}{\left(k-\frac{v}{2}\right)^2+\left(\frac{r'}{2}\right)^2}\right]. \tag{A6}$$

Finally, using the following identity:

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k-z)^2 + r^2} = \frac{\pi}{2r} \frac{\sinh(2\pi r)}{\sinh^2(\pi r) + \sin^2(\pi z)}, \tag{A7}$$

we obtain the expression for the two-point Schwinger function that we need to proceed in our analysis. Using the above equation in Eq. (A6) we get

$$G_0^{(2)}(\vec{r}_1 - \vec{r}_2, z_1, z_2) = \frac{\sinh(\pi r')}{16\pi L^2 r'} \left[ \frac{\sin\left(\frac{\pi z_1}{L}\right)\sin\left(\frac{\pi z_2}{L}\right)}{\left[\sinh^2\left(\frac{\pi r'}{2}\right) + \sin^2\left(\frac{\pi u}{2}\right)\right]\left[\sinh^2\left(\frac{\pi r'}{2}\right) + \sin^2\left(\frac{\pi v}{2}\right)\right]} \right]. \tag{A8}$$


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