Journal of Mathematical Physics

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Citation: J. Math. Phys. 51, 033501 (2010); doi: 10.1063/1.3318158
View online: http://dx.doi.org/10.1063/1.3318158
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v51/i3
Published by the AIP Publishing LLC.

JOURNAL OF MATHEMATICAL PHYSICS **51**, 033501 (2010)

Permutability of Backlund transformations for $N=2$ supersymmetric sine-Gordon

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(Received 30 September 2009; accepted 14 January 2010; published online 2 March 2010)

The permutability of two Backlund transformations is employed to construct a nonlinear superposition formula and to generate a class of solutions for the $N=2$ super sine-Gordon model. We present explicitly the one and two soliton solutions. © *2010 American Institute of Physics.* [doi:10.1063/1.3318158]

I. INTRODUCTION

Backlund transformations reduce the order of the nonlinear differential equations making the system sometimes effectively more tractable. Starting with a simple input solution, we may be able to solve for a more complicated one. In many cases, this may be very difficult to accomplish. A convenient and powerful way is to use the *permutability theorem* which provides a closed algebraic nonlinear superposition formula for the solutions.

The Backlund transformation and the Permutability theorem are employed to derive a series of consistency conditions which are satisfied by soliton solutions of certain class of integrable models. Within such class, we encounter the sine-Gordon[1] and KdV (Korteweg de Vries) (Ref. 2) equations. This framework was also applied to the $N=1$ super-KdV (Ref. 3) and super-sinh-Gordon[4] in order to derive its soliton solutions.

The $N=2$ super-sine-Gordon model was proposed in Ref. 5 and later in Ref. 6 its algebraic structure was uncovered. Certain solutions of this model have already been constructed,[7] however, they were such that involve a single Grassmann parameter. In this paper we extend the nonlinear superposition formulas for soliton solutions of the $N=2$ super-sine-Gordon model. These formulas are derived from the Backlund transformation proposed in Ref. 8 and the permutability condition which implies that the order of 2 successive Backlund transformations is irrelevant. As examples, we present explicitly the 1- and 2-soliton solutions with distinct Grassmann parameters.

Recently the Pohlmeyer reduction in $\mathrm{AdS}_n x S_n$ superstring models have been considered[9] which in the simple case of $n=2$ was shown[10] to be equivalent to the $N=2$ supersymmetric sine-Gordon.

This paper is organized as follows. In Sec. II we discuss the $N=2$ super-sine-Gordon and its Backlund transformation. In Sec. III we apply the permutability condition to derive a closed algebraic nonlinear superposition formulas involving solutions of the model. Finally in Secs. IV and V we present the 1- and 2-soliton solutions, respectively. In A we present the Backlund transformation in components. In Appendices B and C we give details for the derivation of the superposition formulas.

II. $N=2$ SUPER-SINE-GORDON—BACKLUND TRANSFORMATION

Let us start by introducing the $N=2$ superfields,[5]

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$$\phi^{\pm} = \varphi^{\pm}(z^{\pm},\bar{z}^{\pm}) + \theta^{\pm}\psi^{\mp}(z^{\pm},\bar{z}^{\pm}) + \bar{\theta}^{\pm}\bar{\psi}^{\mp}(z^{\pm},\bar{z}^{\pm}) + \theta^{\pm}\bar{\theta}^{\pm}F^{\pm}(z^{\pm},\bar{z}^{\pm}),$$

where

$$z^{\pm} = z \pm \tfrac{1}{2}\theta^{+}\theta^{-}, \quad \bar{z}^{\pm} = \bar{z} \pm \tfrac{1}{2}\bar{\theta}^{+}\bar{\theta}^{-}.$$

The superfield components $\phi^{\pm}$ can be expanded in Grassmann variables $\theta^{\pm}$ and $\bar{\theta}^{\pm}$. For instance, the component $\varphi^{\pm}(z^{\pm},\bar{z}^{\pm})$ gives rise to

$$\varphi^{\pm}(z^{\pm},\bar{z}^{\pm}) = \varphi^{\pm} \pm \tfrac{1}{2}\theta^{+}\theta^{-}\partial_z\varphi^{\pm} \pm \tfrac{1}{2}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{\bar{z}}\varphi^{\pm} + \tfrac{1}{4}\theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\partial_z\partial_{\bar{z}}\varphi^{\pm}.$$

By expanding all components of $\phi^{\pm}$, we obtain

$$\phi^{\pm} = \varphi^{\pm} + \theta^{\pm}\psi^{\mp} + \bar{\theta}^{\pm}\bar{\psi}^{\mp} \pm \tfrac{1}{2}\theta^{+}\theta^{-}\partial_z\varphi^{\pm} \pm \tfrac{1}{2}\bar{\theta}^{+}\bar{\theta}^{-}\partial_{\bar{z}}\varphi^{\pm} + \theta^{\pm}\bar{\theta}^{\pm}F^{\pm}$$
$$\pm \theta^{\pm}\bar{\theta}^{+}\bar{\theta}^{-}\tfrac{1}{2}\partial_{\bar{z}}\psi^{\mp} \pm \bar{\theta}^{\pm}\theta^{+}\theta^{-}\tfrac{1}{2}\partial_z\bar{\psi}^{\mp} + \tfrac{1}{4}\theta^{+}\theta^{-}\bar{\theta}^{+}\bar{\theta}^{-}\partial_z\partial_{\bar{z}}\varphi^{\pm}.$$

We next introduce the superderivatives,

$$D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + \tfrac{1}{2}\theta^{\mp}\partial_z, \quad \bar{D}_{\pm} = \frac{\partial}{\partial\bar{\theta}^{\pm}} + \tfrac{1}{2}\bar{\theta}^{\mp}\partial_{\bar{z}},$$

satisfying the following conditions:

$$D_{\pm}^2 = 0, \quad \bar{D}_{\pm}^2 = 0,$$

$$\{\bar{D}_{\pm},D_{\pm}\} = 0, \quad \{\bar{D}_{\pm},D_{\mp}\} = 0,$$

$$\{D_{+},D_{-}\} = \partial_z, \quad \{\bar{D}_{+},\bar{D}_{-}\} = \partial_{\bar{z}}.$$

The equations of motion for the supersymmetric sine-Gordon model with $N=2$ are given by[5]

$$\bar{D}_{\pm}D_{\pm}\phi^{\pm} = g\sin(\beta\phi^{\mp}), \tag{1}$$

where g is a mass parameter and β is the coupling constant. From now on we assume $\beta=1$ which may be reinserted by a convenient field reparametrization. In components, the equations of motion for the $N=2$ super-sine-Gordon reads

$$F^{\pm} = g\sin\varphi^{\mp},$$

$$\partial_{\bar{z}}\psi^{\mp} = g\cos\varphi^{\mp}\bar{\psi}^{\pm},$$

$$\partial_z\bar{\psi}^{\mp} = -g\cos\varphi^{\mp}\psi^{\pm},$$

$$\partial_z\partial_{\bar{z}}\varphi^{\pm} = -g\cos\varphi^{\mp}F^{\mp} - g\sin\varphi^{\mp}\psi^{\pm}\bar{\psi}^{\pm}.$$

Moreover, the chiral ϕ^{+} and the antichiral ϕ^{-} superfields satisfy the conditions

$$\bar{D}_{\pm}\phi^{\mp} = D_{\pm}\phi^{\mp} = 0. \tag{2}$$

Let us now recall the Backlund transformation for the $N=2$ super-sine-Gordon model.[8] For this purpose, consider the pair of first order differential equations,

$$D_+\phi_1^+ = D_+\phi_2^+ - \frac{8}{\kappa}\mathcal{F}\cos\left(\frac{\phi_1^- + \phi_2^-}{2}\right), \tag{3}$$

$$\bar{D}_+\phi_1^+ = -\bar{D}_+\phi_2^+ + \kappa\mathcal{G}\cos\left(\frac{\phi_1^- - \phi_2^-}{2}\right), \tag{4}$$

where $\mathcal{F}$ and $\mathcal{G}$ are fermionic auxiliary superfields and κ is an arbitrary constant. The above equation and the condition

$$(\bar{D}_+D_+ + D_+\bar{D}_+)\phi_1^+ = 0$$

leads to the equations of motion,

$$\bar{D}_+D_+\phi_2^+ = g\sin\phi_2^-,$$

provided the superfields $\mathcal{F}$ and $\mathcal{G}$ satisfy

$$\bar{D}_+\mathcal{F} = -\frac{\kappa g}{4}\sin\left(\frac{\phi_1^- - \phi_2^-}{2}\right), \quad D_+\mathcal{G} = -\frac{2g}{\kappa}\sin\left(\frac{\phi_1^- + \phi_2^-}{2}\right). \tag{5}$$

In a similar way,

$$D_-\phi_1^- = D_-\phi_2^- + \lambda\mathcal{G}\cos\left(\frac{\phi_1^+ + \phi_2^+}{2}\right), \tag{6}$$

$$\bar{D}_-\phi_1^- = -\bar{D}_-\phi_2^- - \frac{8}{\lambda}\mathcal{F}\cos\left(\frac{\phi_1^+ - \phi_2^+}{2}\right), \tag{7}$$

where λ is another arbitrary constant. Together with the condition

$$(\bar{D}_-D_- + D_-\bar{D}_-)\phi_1^- = 0,$$

yields

$$\bar{D}_-D_-\phi_2^- = g\sin\phi_2^+,$$

provided $\mathcal{G}$ and $\mathcal{F}$ satisfy

$$\bar{D}_-\mathcal{G} = \frac{2g}{\lambda}\sin\left(\frac{\phi_1^+ - \phi_2^+}{2}\right), \quad D_-\mathcal{F} = \frac{\lambda g}{4}\sin\left(\frac{\phi_1^+ + \phi_2^+}{2}\right). \tag{8}$$

Acting with D_+ in Eq. (3), $\bar{D}_+$ in (4), D_- in (6), and $\bar{D}_-$ in (7), we find

$$D_+\mathcal{F} = 0, \quad \bar{D}_+\mathcal{G} = 0, \quad D_-\mathcal{G} = 0, \quad \bar{D}_-\mathcal{F} = 0. \tag{9}$$

These last conditions allow us to rewrite the fermionic superfields into two distinct manners, i.e.,

$$\mathcal{F} = D_+\Phi_1^+ = \bar{D}_-\Phi_2^-, \tag{10}$$

$$\mathcal{G} = D_-\Phi_1^- = \bar{D}_+\Phi_2^+, \tag{11}$$

where the chiral Φ_p^+ and antichiral $\Phi_p^-, p=1,2$ superfields are defined as

$$\Phi_1^\pm = q_1^\pm(z^\pm,\bar{z}^\pm) + \theta^\pm\zeta_1^\pm(z^\pm,\bar{z}^\pm) + \bar{\theta}^\pm\zeta_2^\pm(z^\pm,\bar{z}^\pm) + \theta^\pm\bar{\theta}^\pm q_2^\pm(z^\pm,\bar{z}^\pm),$$

$$\Phi_2^{\pm} = p_1^{\pm}(z^{\pm},\bar{z}^{\pm}) + \theta^{\pm}\xi_1^{\pm}(z^{\pm},\bar{z}^{\pm}) + \bar{\theta}^{\pm}\xi_2^{\pm}(z^{\pm},\bar{z}^{\pm}) + \theta^{\pm}\bar{\theta}^{\pm}p_2^{\pm}(z^{\pm},\bar{z}^{\pm}).$$

The second equality in (10) implies

$$\zeta_1^+ = \xi_2^-, \quad q_2^+ = \partial_{\bar{z}} p_1^-, \quad p_2^- = -\partial_z q_1^+, \quad \partial_z \zeta_2^+ = -\partial_{\bar{z}} \xi_1^-, \tag{12}$$

while the second equality in (11) implies

$$\zeta_1^- = \xi_2^+, \quad p_2^+ = -\partial_z q_1^-, \quad q_2^- = \partial_{\bar{z}} p_1^+, \quad \partial_z \zeta_2^- = -\partial_{\bar{z}} \xi_1^+. \tag{13}$$

Equations (3)–(9) describe the Backlund transformation for the $N=2$ super-sine-Gordon system. In Appendix A we present these equations in components.

III. THE PERMUTABILITY CONDITION

A Backlund transformation from $\phi_0^{\pm}$ to $\phi_1^{\pm}$ is described by

$$D_+(\phi_0^+ - \phi_1^+) = -\frac{8}{\kappa_1}\mathcal{F}^{(0,1)} \cos\left(\frac{\phi_0^- + \phi_1^-}{2}\right), \tag{14}$$

$$\bar{D}_+(\phi_0^+ + \phi_1^+) = \kappa_1 \mathcal{G}^{(0,1)} \cos\left(\frac{\phi_0^- - \phi_1^-}{2}\right), \tag{15}$$

$$D_-(\phi_0^- - \phi_1^-) = \lambda_1 \mathcal{G}^{(0,1)} \cos\left(\frac{\phi_0^+ + \phi_1^+}{2}\right), \tag{16}$$

$$\bar{D}_-(\phi_0^- + \phi_1^-) = -\frac{8}{\lambda_1}\mathcal{F}^{(0,1)} \cos\left(\frac{\phi_0^+ - \phi_1^+}{2}\right), \tag{17}$$

where we have introduced the superscript indices (0,1) for the auxiliary fermionic superfields denoting its dependence in $\phi_0^{\pm}$ and $\phi_1^{\pm}$. The later, in turn, satisfy the following condition [as in (5) and (8)]:

$$\bar{D}_+\mathcal{F}^{(0,1)} = -g\frac{\kappa_1}{4}\sin\left(\frac{\phi_0^- - \phi_1^-}{2}\right), \tag{18}$$

$$D_+\mathcal{G}^{(0,1)} = -g\frac{2}{\kappa_1}\sin\left(\frac{\phi_0^- + \phi_1^-}{2}\right), \tag{19}$$

$$\bar{D}_-\mathcal{G}^{(0,1)} = g\frac{2}{\lambda_1}\sin\left(\frac{\phi_0^+ - \phi_1^+}{2}\right), \tag{20}$$

$$D_-\mathcal{F}^{(0,1)} = g\frac{\lambda_1}{4}\sin\left(\frac{\phi_0^+ + \phi_1^+}{2}\right). \tag{21}$$

The chiral conditions (9), i.e.,

$$\bar{D}_-\mathcal{F}^{(0,1)} = 0, \quad D_+\mathcal{F}^{(0,1)} = 0, \quad \bar{D}_+\mathcal{G}^{(0,1)} = 0, \quad D_-\mathcal{G}^{(0,1)} = 0,$$

are automatically satisfied using Eq. (10), i.e., expressing the fermionic superfields as derivatives of chiral superfields,

$$\mathcal{F}^{(0,1)} = D_+\Phi_1^{+(0,1)} = \bar{D}_-\Phi_2^{-(0,1)}, \quad \mathcal{G}^{(0,1)} = D_-\Phi_1^{-(0,1)} = \bar{D}_+\Phi_2^{+(0,1)},$$

where the superscript indices indicate whether the superfields $\Phi_1^{\pm}$ and $\Phi_2^{\pm}$ depend on $\phi_0^{\pm}$ and $\phi_1^{\pm}$. Acting with superderivatives D_-, $\bar{D}_-$, D_+, and $\bar{D}_+$ on Eqs. (14)–(17), respectively, we find

$$\partial_z(\phi_0^+ - \phi_1^+) = -2\gamma_1 s_{0,1}^+ c_{0,1}^- + \frac{8}{\kappa_1}\mathcal{F}^{(0,1)} D_- c_{0,1}^-, \tag{22}$$

$$\partial_{\bar{z}}(\phi_0^+ + \phi_1^+) = 2\frac{g^2}{\gamma_1}\bar{s}_{0,1}^+ \bar{c}_{0,1}^- - \kappa_1 \mathcal{G}^{(0,1)} \bar{D}_- \bar{c}_{0,1}^-, \tag{23}$$

$$\partial_z(\phi_0^- - \phi_1^-) = -2\gamma_1 s_{0,1}^- c_{0,1}^+ - \lambda_1 \mathcal{G}^{(0,1)} D_+ c_{0,1}^+, \tag{24}$$

$$\partial_{\bar{z}}(\phi_0^- + \phi_1^-) = 2\frac{g^2}{\gamma_1}\bar{s}_{0,1}^- \bar{c}_{0,1}^+ + \frac{8}{\lambda_1}\mathcal{F}^{(0,1)} \bar{D}_+ \bar{c}_{0,1}^+, \tag{25}$$

where $\gamma_1 = g\lambda_1/\kappa_1$ and

$$c_{j,k}^{\pm} = \cos\left(\frac{\phi_j^{\pm} + \phi_k^{\pm}}{2}\right), \quad s_{j,k}^{\pm} = \sin\left(\frac{\phi_j^{\pm} + \phi_k^{\pm}}{2}\right), \tag{26}$$

$$\bar{c}_{j,k}^{\pm} = \cos\left(\frac{\phi_j^{\pm} - \phi_k^{\pm}}{2}\right), \quad \bar{s}_{j,k}^{\pm} = \sin\left(\frac{\phi_j^{\pm} - \phi_k^{\pm}}{2}\right). \tag{27}$$

We now assume that the order of 2 successive Backlund transformations is irrelevant leading to the same final result. Such condition is known as the permutability theorem, i.e., $\phi_0^{\pm} \xrightarrow{\gamma_1} \phi_1^{\pm} \xrightarrow{\gamma_2} \phi_{12}^{\pm}$ and in the inverse order, $\phi_0^{\pm} \xrightarrow{\gamma_2} \phi_2^{\pm} \xrightarrow{\gamma_1} \phi_{21}^{\pm}$, does not change the final result, $\phi_{12}^{\pm} = \phi_{21}^{\pm} \equiv \phi_3^{\pm}$.

The permutability theorem applied to the Backlund equation (14) leads to

$$D_+(\phi_0^+ - \phi_1^+) = -\frac{8}{\kappa_1}\mathcal{F}^{(0,1)} c_{0,1}^-,$$

$$D_+(\phi_1^+ - \phi_3^+) = -\frac{8}{\kappa_2}\mathcal{F}^{(1,3)} c_{1,3}^-,$$

$$D_+(\phi_0^+ - \phi_2^+) = -\frac{8}{\kappa_2}\mathcal{F}^{(0,2)} c_{0,2}^-,$$

$$D_+(\phi_2^+ - \phi_3^+) = -\frac{8}{\kappa_1}\mathcal{F}^{(2,3)} c_{2,3}^-. \tag{28}$$

Taking into account that the sums of the first two and the last two equations are the same, we obtain,

$$\frac{1}{\kappa_1}\mathcal{F}^{(0,1)} c_{0,1}^- + \frac{1}{\kappa_2}\mathcal{F}^{(1,3)} c_{1,3}^- = \frac{1}{\kappa_2}\mathcal{F}^{(0,2)} c_{0,2}^- + \frac{1}{\kappa_1}\mathcal{F}^{(2,3)} c_{2,3}^-. \tag{29}$$

Similarly, from (17), we obtain

$$\bar{D}_-(\phi_0^- + \phi_1^-) = -\frac{8}{\lambda_1}\mathcal{F}^{(0,1)}\overline{c}_{0,1}^{+},$$

$$\bar{D}_-(\phi_1^- + \phi_3^-) = -\frac{8}{\lambda_2}\mathcal{F}^{(1,3)}\overline{c}_{1,3}^{+},$$

$$\bar{D}_-(\phi_0^- + \phi_2^-) = -\frac{8}{\lambda_2}\mathcal{F}^{(0,2)}\overline{c}_{0,2}^{+},$$

$$\bar{D}_-(\phi_2^- + \phi_3^-) = -\frac{8}{\lambda_1}\mathcal{F}^{(2,3)}\overline{c}_{2,3}^{+}, \tag{30}$$

leading to

$$\frac{1}{\lambda_1}\mathcal{F}^{(0,1)}\overline{c}_{0,1}^{+} - \frac{1}{\lambda_2}\mathcal{F}^{(1,3)}\overline{c}_{1,3}^{+} = \frac{1}{\lambda_2}\mathcal{F}^{(0,2)}\overline{c}_{0,2}^{+} - \frac{1}{\lambda_1}\mathcal{F}^{(2,3)}\overline{c}_{2,3}^{+}. \tag{31}$$

We propose as solution for the nonlinear superposition formula $\phi_{12}^{\pm} = \phi_{21}^{\pm} = \phi_3^{\pm}$,

$$\phi_3^{\pm} = \phi_0^{\pm} + \Gamma_{\pm} + \Delta_{\pm}, \tag{32}$$

with

$$\Gamma_{\pm}(x,y) = 2\arctan\left[\delta\tan\left(\frac{x+y}{4}\right)\right] \pm 2\arctan\left[\delta\tan\left(\frac{x-y}{4}\right)\right],$$

$$x = \phi_1^+ - \phi_2^+, \quad y = \phi_1^- - \phi_2^-,$$

$$\delta = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2}, \quad \gamma_k = g\frac{\lambda_k}{\kappa_k}. \tag{33}$$

Notice that the solution $\phi_3^{\pm}$ when the fermionic superfields are neglected is derived in Appendix B to be $\phi_3^{\pm} = \phi_0^{\pm} + \Gamma_{\pm}$. The term $\Delta_{\pm}$ comes from the contribution of the fermionic superfields and has the following form:

$$\Delta_{\pm} = \sum_{j,k=1}^{2} \Lambda_{j,k}^{\pm} f_{j,k} + \Lambda_0^{\pm} f_0,$$

$$f_{j,k} = \mathcal{F}^{(0,j)}\mathcal{G}^{(0,k)}, \quad f_0 = \mathcal{F}^{(0,1)}\mathcal{F}^{(0,2)}\mathcal{G}^{(0,1)}\mathcal{G}^{(0,2)},$$

where we have assumed the coefficients $\Lambda^{\pm}$ to be functionals of $x = (\phi_1^+ - \phi_2^+)$ and $y = (\phi_1^- - \phi_2^-)$, i.e.,

$$\Lambda_{j,k}^{\pm} = \Lambda_{j,k}^{\pm}(x,y), \quad \Lambda_0^{\pm} = \Lambda_0^{\pm}(x,y). \tag{34}$$

Observe that there are no terms like $\Lambda_1^{\pm}\mathcal{F}^{(0,1)}\mathcal{F}^{(0,2)}$ nor $\Lambda_2^{\pm}\mathcal{G}^{(0,1)}\mathcal{G}^{(0,2)}$ due to chiral equations (2). $\Lambda_{\pm}$ are determined in Appendix C, where

$$\Lambda_{1,1}^{+} = \Lambda_{2,2}^{+} = -\frac{8\mu_-}{g\eta_+\eta_-}\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_{1,2}^{+} = \frac{8\mu_-}{g\eta_+\eta_-}\left(\frac{\lambda_2}{\lambda_1}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_{2,1}^{+} = \frac{8\mu_-}{g\,\eta_+\eta_-}\left(\frac{\lambda_1}{\lambda_2}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_0^{+} = -\frac{32\mu_-}{(g\,\eta_+\eta_-)^2}\sin\left(\frac{x}{2}\right)\left[\cos\left(\frac{y}{2}\right)(a+\cos x-\cos y)-2\mu_+\cos\left(\frac{x}{2}\right)\right],$$

$$\Lambda_{1,1}^{-} = \Lambda_{2,2}^{-} = \frac{8\mu_-}{g\,\eta_+\eta_-}\cos\left(\frac{y}{2}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_{1,2}^{-} = -\frac{8\mu_-}{g\,\eta_+\eta_-}\left(\frac{\kappa_2}{\kappa_1}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_{2,1}^{-} = -\frac{8\mu_-}{g\,\eta_+\eta_-}\left(\frac{\kappa_1}{\kappa_2}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_0^{-} = -\frac{32\mu_-}{(g\,\eta_+\eta_-)^2}\sin\left(\frac{y}{2}\right)\left[\cos\left(\frac{x}{2}\right)(a-\cos x+\cos y)-2\mu_+\cos\left(\frac{y}{2}\right)\right],$$

$$\mu_{\pm} = \frac{\gamma_1}{\gamma_2} \pm \frac{\gamma_2}{\gamma_1},$$

$$a = \frac{1}{2}\left(\frac{\gamma_1^2}{\gamma_2^2}+\frac{\gamma_2^2}{\gamma_1^2}\right)+3,$$

$$\eta_{\pm} = \mu_+ - 2\cos\left(\frac{x \pm y}{2}\right).$$

A. Solution in components

In components the nonlinear superposition formula (32) yields the following expressions:

$$\varphi_3^{+} = \varphi_0^{+} + \tilde{\Gamma}_+ - \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_1^{+} - \mathcal{B}_1^{+} + \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_1^{+}\right),$$

$$\psi_3^{-} = \psi_0^{-} + F_{1,2}\psi_{1,2}^{-} + \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_2^{+} - \mathcal{B}_2^{+} + \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_2^{+}\right),$$

$$\bar{\psi}_3^{-} = \bar{\psi}_0^{-} + F_{1,2}\bar{\psi}_{1,2}^{-} + \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_3^{+} - \mathcal{B}_3^{+} + \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_3^{+}\right),$$

$$\varphi_3^{-} = \varphi_0^{-} + \tilde{\Gamma}_- + \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_1^{-} - \mathcal{B}_1^{-} - \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_1^{-}\right),$$

$$\psi_3^{+} = \psi_0^{+} + F_{1,2}\psi_{1,2}^{+} - \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_2^{-} - \mathcal{B}_2^{-} - \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_2^{-}\right),$$

$$\bar{\psi}_3^{+} = \bar{\psi}_0^{+} + F_{1,2}\bar{\psi}_{1,2}^{+} - \frac{8\mu_-}{g\,\tilde{\eta}_+\tilde{\eta}_-}\left(\mathcal{A}_3^- - \mathcal{B}_3^- - \frac{4}{g\,\tilde{\eta}_+\tilde{\eta}_-}\mathcal{C}_3^-\right),$$

where

$$\tilde{\Gamma}_{\pm} = 2\ \arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4}\right)\right] \pm 2\ \arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4}\right)\right],$$

$$\tilde{\eta}_{\pm} = \mu_+ - 2\ \cos\left(\frac{\varphi_{1,2}^+ \pm \varphi_{1,2}^-}{2}\right),$$

$$F_{1,2} = \frac{\delta}{2}\left[\frac{\sec^2\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4}\right)}{1+\delta^2\tan^2\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4}\right)} + \frac{\sec^2\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4}\right)}{1+\delta^2\tan^2\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4}\right)}\right],$$

$$\mathcal{A}_1^+ = \cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_{1,2}^-}{2}\right)(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)}),$$

$$\mathcal{A}_2^+ = -\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_{1,2}^-}{2}\right)(\zeta_1^{+(0,1)}p_2^{+(0,1)} + \zeta_1^{+(0,2)}p_2^{+(0,2)}) - \Sigma^+(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)})\psi_{1,2}^-,$$

$$\mathcal{A}_3^+ = -\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_{1,2}^-}{2}\right)(q_2^{+(0,1)}\xi_2^{+(0,1)} + q_2^{+(0,2)}\xi_2^{+(0,2)}) - \Sigma^+(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)})\bar{\psi}_{1,2}^-,$$

$$\mathcal{B}_1^+ = \sin\left(\frac{\varphi_{1,2}^-}{2}\right)\left(\frac{\lambda_2}{\lambda_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right),$$

$$\mathcal{B}_2^+ = -\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\left(\frac{\lambda_2}{\lambda_1}\zeta_1^{+(0,1)}p_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2}\zeta_1^{+(0,2)}p_2^{+(0,1)}\right) - \Omega^+\left(\frac{\lambda_2}{\lambda_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right)\psi_{1,2}^-,$$

$$\mathcal{B}_3^+ = -\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\left(\frac{\lambda_2}{\lambda_1}q_2^{+(0,1)}\xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2}q_2^{+(0,2)}\xi_2^{+(0,1)}\right) - \Omega^+\left(\frac{\lambda_2}{\lambda_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\lambda_1}{\lambda_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right)\bar{\psi}_{1,2}^-,$$

$$\mathcal{C}_1^+ = \sin\left(\frac{\varphi_{1,2}^+}{2}\right)A^+\zeta_1^{+(0,1)}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\xi_2^{+(0,2)},$$

$$\mathcal{C}_2^+ = \sin\left(\frac{\varphi_{1,2}^+}{2}\right)A^+\zeta_1^{+(0,1)}\zeta_1^{+(0,2)}(\xi_2^{+(0,2)}p_2^{+(0,1)} - \xi_2^{+(0,1)}p_2^{+(0,2)}),$$

$$\mathcal{C}_3^+ = \sin\left(\frac{\varphi_{1,2}^+}{2}\right)A^+\xi_2^{+(0,1)}\xi_2^{+(0,2)}(\zeta_1^{+(0,1)}q_2^{+(0,2)} - \zeta_1^{+(0,2)}q_2^{+(0,1)}),$$

$$\mathcal{A}_1^- = \cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_{1,2}^+}{2}\right)(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)}),$$

$$\mathcal{A}_2^- = -\cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_{1,2}^+}{2}\right)(\partial_z q_1^{+(0,1)}\xi_2^{+(0,1)} + \partial_z q_1^{+(0,2)}\xi_2^{+(0,2)}) - \Sigma^-(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)})\psi_{1,2}^+,$$

$$\mathcal{A}_3^- = \cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_{1,2}^+}{2}\right)(\zeta_1^{+(0,1)}\partial_{\bar{z}} p_1^{+(0,1)} + \zeta_1^{+(0,2)}\partial_{\bar{z}} p_1^{+(0,2)}) - \Sigma^-(\zeta_1^{+(0,1)}\xi_2^{+(0,1)} + \zeta_1^{+(0,2)}\xi_2^{+(0,2)})\bar{\psi}_{1,2}^+,$$

$$\mathcal{B}_1^- = \sin\left(\frac{\varphi_{1,2}^+}{2}\right)\left(\frac{\kappa_2}{\kappa_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right),$$

$$\mathcal{B}_2^- = -\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\left(\frac{\kappa_2}{\kappa_1}\partial_z q_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2}\partial_z q_1^{+(0,2)}\xi_2^{+(0,1)}\right) - \Omega^-\left(\frac{\kappa_2}{\kappa_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right)\psi_{1,2}^+,$$

$$\mathcal{B}_3^- = \sin\left(\frac{\varphi_{1,2}^+}{2}\right)\left(\frac{\kappa_2}{\kappa_1}\zeta_1^{+(0,1)}\partial_{\bar{z}} p_1^{+(0,2)} + \frac{\kappa_1}{\kappa_2}\zeta_1^{+(0,2)}\partial_{\bar{z}} p_1^{+(0,1)}\right) - \Omega^-\left(\frac{\kappa_2}{\kappa_1}\zeta_1^{+(0,1)}\xi_2^{+(0,2)} + \frac{\kappa_1}{\kappa_2}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\right)\bar{\psi}_{1,2}^+,$$

$$\mathcal{C}_1^- = \sin\left(\frac{\varphi_{1,2}^-}{2}\right)A^-\zeta_1^{+(0,1)}\zeta_1^{+(0,2)}\xi_2^{+(0,1)}\xi_2^{+(0,2)},$$

$$\mathcal{C}_2^- = -\sin\left(\frac{\varphi_{1,2}^-}{2}\right)A^-\xi_2^{+(0,1)}\xi_2^{+(0,2)}(\zeta_1^{+(0,2)}\partial_z q_1^{+(0,1)} - \zeta_1^{+(0,1)}\partial_z q_1^{+(0,2)}),$$

$$\mathcal{C}_3^- = -\sin\left(\frac{\varphi_{1,2}^-}{2}\right)A^-\zeta_1^{+(0,1)}\zeta_1^{+(0,2)}(\xi_2^{+(0,2)}\partial_{\bar{z}} p_1^{+(0,1)} - \xi_2^{+(0,1)}\partial_{\bar{z}} p_1^{+(0,2)}),$$

$$\Sigma^+ = \cos\left(\frac{\varphi_{1,2}^+}{2}\right)\Omega^+ - \frac{1}{2}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_{1,2}^+}{2}\right),$$

$$\Omega^+ = -\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\left[\frac{1}{\tilde{\eta}_+}\sin\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{2}\right) + \frac{1}{\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{2}\right)\right],$$

$$A^+ = \cos\left(\frac{\varphi_{1,2}^-}{2}\right)(a + \cos\,\varphi_{1,2}^+ - \cos\,\varphi_{1,2}^-) - 2\mu_+\cos\left(\frac{\varphi_{1,2}^+}{2}\right),$$

$$\Sigma^- = \cos\left(\frac{\varphi_{1,2}^-}{2}\right)\Omega^- - \frac{1}{2}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_{1,2}^+}{2}\right),$$

$$\Omega^- = -\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\left[\frac{1}{\tilde{\eta}_+}\sin\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{2}\right) - \frac{1}{\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{2}\right)\right],$$

$$A^- = \cos\left(\frac{\varphi_{1,2}^+}{2}\right)(a - \cos\,\varphi_{1,2}^+ + \cos\,\varphi_{1,2}^-) - 2\mu_+\cos\left(\frac{\varphi_{1,2}^-}{2}\right),$$

and denoted

$$\varphi_{1,2}^{\pm}=\varphi_1^{\pm}-\varphi_2^{\pm}, \quad \psi_{1,2}^{\pm}=\psi_1^{\pm}-\psi_2^{\pm}, \quad \bar{\psi}_{1,2}^{\pm}=\bar{\psi}_1^{\pm}-\bar{\psi}_2^{\pm}.$$

From the Backlund equations we get (see Appendix A)

$$\zeta_1^{+(0,k)}=-\frac{\kappa_k}{8}\frac{(\psi_0^- - \psi_k^-)}{\cos\left(\frac{\varphi_0^- + \varphi_k^-}{2}\right)}, \quad \xi_2^{+(0,k)}=\frac{1}{\kappa_k}\frac{(\bar{\psi}_0^- + \bar{\psi}_k^-)}{\cos\left(\frac{\varphi_0^- - \varphi_k^-}{2}\right)},$$

$$\partial_z q_1^{+(0,k)}=\frac{\lambda_k g}{4}\sin\left(\frac{\varphi_0^+ + \varphi_k^+}{2}\right), \quad p_2^{+(0,k)}=\frac{2g}{\kappa_k}\sin\left(\frac{\varphi_0^- + \varphi_k^-}{2}\right),$$

$$q_2^{+(0,k)}=-\frac{\kappa_k g}{4}\sin\left(\frac{\varphi_0^- - \varphi_k^-}{2}\right), \quad \partial_{\bar{z}} p_1^{+(0,k)}=\frac{2g}{\lambda_k}\sin\left(\frac{\varphi_0^+ - \varphi_k^+}{2}\right).$$

IV. 1-SOLITON SOLUTION

Setting $\phi_0^{\pm}=0$ in the Backlund equations (14)–(21) we find in components

$$\partial_{\bar{z}}\zeta_1^{+(0,1)}=-\frac{g^2}{\gamma_1}\cos\left(\frac{\varphi_1^+}{2}\right)\cos\left(\frac{\varphi_1^-}{2}\right)\zeta_1^{+(0,1)},$$

$$\partial_z\zeta_1^{+(0,1)}=\gamma_1\cos\left(\frac{\varphi_1^+}{2}\right)\cos\left(\frac{\varphi_1^-}{2}\right)\zeta_1^{+(0,1)},$$

$$\partial_{\bar{z}}\xi_2^{+(0,1)}=-\frac{g^2}{\gamma_1}\cos\left(\frac{\varphi_1^+}{2}\right)\cos\left(\frac{\varphi_1^-}{2}\right)\xi_2^{+(0,1)},$$

$$\partial_z\xi_2^{+(0,1)}=\gamma_1\cos\left(\frac{\varphi_1^+}{2}\right)\cos\left(\frac{\varphi_1^-}{2}\right)\xi_2^{+(0,1)},$$

$$\partial_{\bar{z}}\varphi_1^{\pm}=-\frac{2g^2}{\gamma_1}\sin\left(\frac{\varphi_1^{\pm}}{2}\right)\cos\left(\frac{\varphi_1^{\mp}}{2}\right),$$

$$\partial_z\varphi_1^{\pm}=2\gamma_1\sin\left(\frac{\varphi_1^{\pm}}{2}\right)\cos\left(\frac{\varphi_1^{\mp}}{2}\right).$$

Integrating the above equations we get the 1-soliton solution,

$$\psi_1^-=\frac{8}{\kappa_1}\cos\left(\frac{\varphi_1^-}{2}\right)\zeta_1^{+(0,1)}, \quad \psi_1^+=-\lambda_1\cos\left(\frac{\varphi_1^+}{2}\right)\xi_2^{+(0,1)},$$

$$\bar{\psi}_1^-=\kappa_1\cos\left(\frac{\varphi_1^-}{2}\right)\xi_2^{+(0,1)}, \quad \bar{\psi}_1^+=-\frac{8}{\lambda_1}\cos\left(\frac{\varphi_1^+}{2}\right)\zeta_1^{+(0,1)},$$

$$\varphi_1^{\pm}=2\arctan(a_1\rho_1)\pm 2\arctan(b_1\rho_1),$$

$$\zeta_1^{+(0,1)}=\xi_2^{+(0,1)}=\epsilon_1\chi_1,$$

$$\chi_1 = \frac{\rho_1}{\sqrt{(1+a_1^2\rho_1^2)(1+b_1^2\rho_1^2)}},$$

where a_1 and b_1 are arbitrary constants, ϵ_1 is a Grassmann parameter, and

$$\rho_1 = \exp\left(\gamma_1 z - \frac{g^2}{\gamma_1}\bar{z}\right).$$

The 1-soliton solution constructed in this section can be obtained from those of Ref. 7 by relating parameters since they both involve a single Grassmann parameter.

V. 2-SOLITON SOLUTION

For the 2-soliton case we obtain from the superposition formulas (32)

$$\varphi_3^+ = \varphi_3^{+(0)} + \varphi_3^{+(1)}\epsilon_1\epsilon_2,$$

$$\varphi_3^{+(0)} = 2\arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4}\right)\right] + 2\arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4}\right)\right],$$

$$\varphi_3^{+(1)} = \frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\left(\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{\lambda_2}\right)\chi_1\chi_2,$$

$$\psi_3^- = \epsilon_1\psi_3^{-(1)} + \epsilon_2\psi_3^{-(2)},$$

$$\psi_3^{-(1)} = \frac{8}{\kappa_1}F_{1,2}\cos\left(\frac{\varphi_1^-}{2}\right)\chi_1 + \frac{16}{\kappa_1\gamma_1}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\chi_1\left[\gamma_2\sin\left(\frac{\varphi_2^-}{2}\right) - \gamma_1\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_1^-}{2}\right)\right],$$

$$\psi_3^{-(2)} = -\frac{8}{\kappa_2}F_{1,2}\cos\left(\frac{\varphi_2^-}{2}\right)\chi_2 + \frac{16}{\kappa_2\gamma_2}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\chi_2\left[\gamma_1\sin\left(\frac{\varphi_1^-}{2}\right) - \gamma_2\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_2^-}{2}\right)\right],$$

$$\bar{\psi}_3^- = \epsilon_1\bar{\psi}_3^{-(1)} + \epsilon_2\bar{\psi}_3^{-(2)},$$

$$\bar{\psi}_3^{-(1)} = \kappa_1 F_{1,2}\cos\left(\frac{\varphi_1^-}{2}\right)\chi_1 + \frac{2\kappa_1}{\gamma_2}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\chi_1\left[\gamma_1\sin\left(\frac{\varphi_2^-}{2}\right) - \gamma_2\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_1^-}{2}\right)\right],$$

$$\bar{\psi}_3^{-(2)} = -\kappa_2 F_{1,2}\cos\left(\frac{\varphi_2^-}{2}\right)\chi_2 + \frac{2\kappa_2}{\gamma_1}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^-}{2}\right)\chi_2\left[\gamma_2\sin\left(\frac{\varphi_1^-}{2}\right) - \gamma_1\cos\left(\frac{\varphi_{1,2}^+}{2}\right)\sin\left(\frac{\varphi_2^-}{2}\right)\right],$$

$$\varphi_3^- = \varphi_3^{-(0)} + \varphi_3^{-(1)}\epsilon_1\epsilon_2,$$

$$\varphi_3^{-(0)} = 2\arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ + \varphi_{1,2}^-}{4}\right)\right] - 2\arctan\left[\delta\tan\left(\frac{\varphi_{1,2}^+ - \varphi_{1,2}^-}{4}\right)\right],$$

$$\varphi_3^{-(1)} = -\frac{8\mu_-}{g\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\left(\frac{\kappa_2}{\kappa_1} - \frac{\kappa_1}{\kappa_2}\right)\chi_1\chi_2,$$

$$\psi_3^+ = \epsilon_1\psi_3^{+(1)} + \epsilon_2\psi_3^{+(2)},$$

$$\psi_3^{+(1)} = -\lambda_1 F_{1,2}\cos\left(\frac{\varphi_1^+}{2}\right)\chi_1 - \frac{2\lambda_1}{\gamma_1}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\chi_1\left[\gamma_2 \sin\left(\frac{\varphi_2^+}{2}\right) - \gamma_1\cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_1^+}{2}\right)\right],$$

$$\psi_3^{+(2)} = \lambda_2 F_{1,2}\cos\left(\frac{\varphi_2^+}{2}\right)\chi_2 - \frac{2\lambda_2}{\gamma_2}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\chi_2\left[\gamma_1 \sin\left(\frac{\varphi_1^+}{2}\right) - \gamma_2\cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_2^+}{2}\right)\right],$$

$$\bar{\psi}_3^+ = \epsilon_1\bar{\psi}_3^{+(1)} + \epsilon_2\bar{\psi}_3^{+(2)},$$

$$\bar{\psi}_3^{+(1)} = -\frac{8}{\lambda_1}F_{1,2}\cos\left(\frac{\varphi_1^+}{2}\right)\chi_1 - \frac{16}{\lambda_1\gamma_2}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\chi_1\left[\gamma_1 \sin\left(\frac{\varphi_2^+}{2}\right) - \gamma_2\cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_1^+}{2}\right)\right],$$

$$\bar{\psi}_3^{+(2)} = \frac{8}{\lambda_2}F_{1,2}\cos\left(\frac{\varphi_2^+}{2}\right)\chi_2 - \frac{16}{\lambda_2\gamma_1}\frac{\mu_-}{\tilde{\eta}_+\tilde{\eta}_-}\sin\left(\frac{\varphi_{1,2}^+}{2}\right)\chi_2\left[\gamma_2 \sin\left(\frac{\varphi_1^+}{2}\right) - \gamma_1\cos\left(\frac{\varphi_{1,2}^-}{2}\right)\sin\left(\frac{\varphi_2^+}{2}\right)\right],$$

where

$$\varphi_k^{\pm} = 2\arctan(a_k\rho_k) \pm 2\arctan(b_k\rho_k),$$

$$\chi_k = \frac{\rho_k}{\sqrt{(1+a_k^2\rho_k^2)(1+b_k^2\rho_k^2)}},$$

$k=1,2$, a_k and b_k are arbitrary constants, ϵ_k is a Grassmann constant, and

$$\rho_k = \exp\left(\gamma_k z - \frac{g^2}{\gamma_k}\bar{z}\right).$$

Notice that the 2-soliton solution constructed in this section generalizes those constructed in Ref. 7) involving a single Grassmann parameter.

Both 1- and 2-soliton solutions presented above were verified to satisfy the equations of motion.

ACKNOWLEDGMENTS

L.H.Y. acknowledges support from Fapesp. J.F.G. and A.H.Z. thank CNPq for partial support.

APPENDIX A: BACKLUND TRANSFORMATION IN COMPONENTS

In order to simplify notation let us introduce $\varphi_{\pm}^{(-)} = \varphi_1^- \pm \varphi_2^-$, $\varphi_{\pm}^{(+)} = \varphi_1^+ \pm \varphi_2^+$, and similar notation for the other fields.

In components Eq. (5) becomes as follows.

(i)

$$\bar{D}_+\mathcal{F} = -\frac{\kappa g}{4}\sin\left(\frac{\phi_1^- - \phi_2^-}{2}\right),$$

$$\Downarrow$$

$$q_2^+ = -\frac{\kappa g}{4}\sin\left(\frac{\varphi_-^{(-)}}{2}\right), \quad \partial_{\bar{z}}\zeta_1^+ = -\frac{\kappa g}{8}\cos\left(\frac{\varphi_-^{(-)}}{2}\right)\bar{\psi}_-^{(+)}, \quad \partial_z\zeta_2^+ = \frac{\kappa g}{8}\cos\left(\frac{\varphi_-^{(-)}}{2}\right)\psi_-^{(+)},$$

$$\partial_{\bar{z}}\partial_z q_1^+ = \frac{\kappa g}{8}\cos\left(\frac{\varphi_-^{(-)}}{2}\right)F_-^{(-)} + \frac{\kappa g}{16}\sin\left(\frac{\varphi_-^{(-)}}{2}\right)\psi_-^{(+)}\bar{\psi}_-^{(+)}.$$

(ii)

$$D_+\mathcal{G} = -\frac{2g}{\kappa}\sin\left(\frac{\phi_1^- + \phi_2^-}{2}\right),$$

$$\Downarrow$$

$$p_2^+ = \frac{2g}{\kappa}\sin\left(\frac{\varphi_+^{(-)}}{2}\right), \quad \partial_{\bar{z}}\xi_1^+ = \frac{g}{\kappa}\cos\left(\frac{\varphi_+^{(-)}}{2}\right)\bar{\psi}_+^{(+)}, \quad \partial_z\xi_2^+ = -\frac{g}{\kappa}\cos\left(\frac{\varphi_+^{(-)}}{2}\right)\psi_+^{(+)},$$

$$\partial_{\bar{z}}\partial_z p_1^+ = -\frac{g}{\kappa}\cos\left(\frac{\varphi_+^{(-)}}{2}\right)F_+^{(-)} - \frac{g}{2\kappa}\sin\left(\frac{\varphi_+^{(-)}}{2}\right)\psi_+^{(+)}\bar{\psi}_+^{(+)}.$$

Similarly we find for (8) the following.

(i)

$$\bar{D}_-\mathcal{G} = \frac{2g}{\lambda}\sin\left(\frac{\phi_1^+ - \phi_2^+}{2}\right),$$

$$\Downarrow$$

$$q_2^- = \frac{2g}{\lambda}\sin\left(\frac{\varphi_-^{(+)}}{2}\right), \quad \partial_{\bar{z}}\zeta_1^- = \frac{g}{\lambda}\cos\left(\frac{\varphi_-^{(+)}}{2}\right)\bar{\psi}_-^{(-)}, \quad \partial_z\zeta_2^- = -\frac{g}{\lambda}\cos\left(\frac{\varphi_-^{(+)}}{2}\right)\psi_-^{(-)},$$

$$\partial_{\bar{z}}\partial_z q_1^- = -\frac{g}{\lambda}\cos\left(\frac{\varphi_-^{(+)}}{2}\right)F_-^{(+)} - \frac{g}{2\lambda}\sin\left(\frac{\varphi_-^{(+)}}{2}\right)\psi_-^{(-)}\bar{\psi}_-^{(-)}.$$

(ii)

$$D_-\mathcal{F} = \frac{\lambda g}{4}\sin\left(\frac{\phi_1^+ + \phi_2^+}{2}\right),$$

$$\Downarrow$$

$$p_2^- = -\frac{\lambda g}{4}\sin\left(\frac{\varphi_+^{(+)}}{2}\right), \quad \partial_{\bar{z}}\xi_1^- = -\frac{\lambda g}{8}\cos\left(\frac{\varphi_+^{(+)}}{2}\right)\bar{\psi}_+^{(-)}, \quad \partial_z\xi_2^- = \frac{\lambda g}{8}\cos\left(\frac{\varphi_+^{(+)}}{2}\right)\psi_+^{(-)},$$

$$\partial_{\bar{z}}\partial_z p_1^- = \frac{\lambda g}{8}\cos\left(\frac{\varphi_+^{(+)}}{2}\right)F_+^{(+)} + \frac{\lambda g}{16}\sin\left(\frac{\varphi_+^{(+)}}{2}\right)\psi_+^{(-)}\bar{\psi}_+^{(-)}.$$

From (3) and (4), we have the following.

(i)

$$D_+\phi_1^+ = D_+\phi_2^+ - \frac{8}{\kappa}\mathcal{F}\cos\left(\frac{\phi_1^- + \phi_2^-}{2}\right),$$

$$\Downarrow$$

$$\psi_-^{(-)} = -\frac{8}{\kappa}\zeta_1^+ \cos\left(\frac{\varphi_+^{(-)}}{2}\right), \quad F_-^{(+)} = -\frac{8}{\kappa}q_2^+ \cos\left(\frac{\varphi_+^{(-)}}{2}\right),$$

$$\partial_z \varphi_-^{(+)} = -\frac{4}{\kappa}\sin\left(\frac{\varphi_+^{(-)}}{2}\right)\zeta_1^+\psi_+^{(+)} - \frac{8}{\kappa}\partial_z q_1^+ \cos\left(\frac{\varphi_+^{(-)}}{2}\right).$$

(ii)

$$\bar{D}_+\phi_1^+ = -\bar{D}_+\phi_2^+ + \kappa\mathcal{G}\cos\left(\frac{\phi_1^- - \phi_2^-}{2}\right),$$

$$\Downarrow$$

$$\bar{\psi}_+^{(-)} = \kappa\xi_2^+ \cos\left(\frac{\varphi_-^{(-)}}{2}\right), \quad F_+^{(+)} = \kappa p_2^+ \cos\left(\frac{\varphi_-^{(-)}}{2}\right),$$

$$\partial_{\bar{z}}\varphi_+^{(+)} = \frac{\kappa}{2}\sin\left(\frac{\varphi_-^{(-)}}{2}\right)\xi_2^+\bar{\psi}_-^{(+)} + \kappa\partial_{\bar{z}}p_1^+ \cos\left(\frac{\varphi_-^{(-)}}{2}\right).$$

From (6) and (7), we have the following.

(i)

$$D_-\phi_1^- = D_-\phi_2^- + \lambda\mathcal{G}\cos\left(\frac{\phi_1^+ + \phi_2^+}{2}\right),$$

$$\Downarrow$$

$$\psi_-^{(+)} = \lambda\zeta_1^- \cos\left(\frac{\varphi_+^{(+)}}{2}\right), \quad F_-^{(-)} = \lambda q_2^- \cos\left(\frac{\varphi_+^{(+)}}{2}\right),$$

$$\partial_z\varphi_-^{(-)} = \frac{\lambda}{2}\sin\left(\frac{\varphi_+^{(+)}}{2}\right)\zeta_1^-\psi_+^{(-)} + \lambda\partial_z q_1^- \cos\left(\frac{\varphi_+^{(+)}}{2}\right).$$

(ii)

$$\bar{D}_-\phi_1^- = -\bar{D}_-\phi_2^- - \frac{8}{\lambda}\mathcal{F}\cos\left(\frac{\phi_1^+ - \phi_2^+}{2}\right),$$

$$\Downarrow$$

$$\bar{\psi}_+^{(+)} = -\frac{8}{\lambda}\xi_2^- \cos\left(\frac{\varphi_-^{(+)}}{2}\right), \quad F_+^{(-)} = -\frac{8}{\lambda}p_2^- \cos\left(\frac{\varphi_-^{(+)}}{2}\right),$$

$$\partial_{\bar{z}}\varphi_+^{(-)} = -\frac{4}{\lambda}\sin\left(\frac{\varphi_-^{(+)}}{2}\right)\xi_2^-\bar{\psi}_-^{(-)} - \frac{8}{\lambda}\partial_{\bar{z}}p_1^- \cos\left(\frac{\varphi_-^{(+)}}{2}\right).$$

APPENDIX B: SUPERPOSITION FORMULA

Applying the permutability theorem to Eqs. (22) and (24) after neglecting the contribution proportional to fermionic superfields, we obtain the following relations:

$$\gamma_1 s^+_{0,1} c^-_{0,1} + \gamma_2 s^+_{1,3} c^-_{1,3} = \gamma_2 s^+_{0,2} c^-_{0,2} + \gamma_1 s^+_{2,3} c^-_{2,3},$$

$$\gamma_1 s^-_{0,1} c^+_{0,1} + \gamma_2 s^-_{1,3} c^+_{1,3} = \gamma_2 s^-_{0,2} c^+_{0,2} + \gamma_1 s^-_{2,3} c^+_{2,3}.$$

Summing and subtracting the above equations, we find

$$\gamma_1[(s^+_{0,1} c^-_{0,1} \pm s^-_{0,1} c^+_{0,1}) - (s^+_{2,3} c^-_{2,3} \pm s^-_{2,3} c^+_{2,3})] + \gamma_2[(s^+_{1,3} c^-_{1,3} \pm s^-_{1,3} c^+_{1,3}) - (s^+_{0,2} c^-_{0,2} \pm s^-_{0,2} c^+_{0,2})] = 0. \tag{B1}$$

Using the identity

$$\sin a \cos b \pm \sin b \cos a = \sin(a \pm b), \tag{B2}$$

and Eqs. (26) and (27) we can rewrite (B1) as

$$\gamma_1 \left\{ \sin\left[\left(\frac{\phi_0^+ + \phi_1^+}{2} \right) \pm \left(\frac{\phi_0^- + \phi_1^-}{2} \right) \right] - \sin\left[\left(\frac{\phi_2^+ + \phi_3^+}{2} \right) \pm \left(\frac{\phi_2^- + \phi_3^-}{2} \right) \right] \right\}$$
$$+ \gamma_2 \left\{ \sin\left[\left(\frac{\phi_1^+ + \phi_3^+}{2} \right) \pm \left(\frac{\phi_1^- + \phi_3^-}{2} \right) \right] - \sin\left[\left(\frac{\phi_0^+ + \phi_2^+}{2} \right) \pm \left(\frac{\phi_0^- + \phi_2^-}{2} \right) \right] \right\} = 0.$$

Using the fact that

$$\sin a - \sin b = 2 \cos\left(\frac{a+b}{2} \right) \sin\left(\frac{a-b}{2} \right)$$

yields

$$2 \cos(Y^+ \pm Y^-)\{\gamma_1 \sin[(X^+_{1,2} \pm X^-_{1,2}) - (X^+_{3,0} \pm X^-_{3,0})] + \gamma_2 \sin[(X^+_{1,2} \pm X^-_{1,2}) + (X^+_{3,0} \pm X^-_{3,0})]\} = 0,$$

where we have denoted

$$Y^{\pm} = \frac{\phi_0^{\pm} + \phi_1^{\pm} + \phi_2^{\pm} + \phi_3^{\pm}}{4},$$

$$X^{\pm}_{j,k} = \frac{\phi_j^{\pm} - \phi_k^{\pm}}{4},$$

from where it follows that

$$(\gamma_1 + \gamma_2)\sin(X^+_{1,2} \pm X^-_{1,2})\cos(X^+_{3,0} \pm X^-_{3,0}) = (\gamma_1 - \gamma_2)\sin(X^+_{3,0} \pm X^-_{3,0})\cos(X^+_{1,2} \pm X^-_{1,2})$$

or

$$\tan(X^+_{3,0} \pm X^-_{3,0}) = \left(\frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \right) \tan(X^+_{1,2} \pm X^-_{1,2}),$$

and, therefore,

$$\left(\frac{\phi_3^+ - \phi_0^+}{4} \right) \pm \left(\frac{\phi_3^- - \phi_0^-}{4} \right) = \arctan[\delta \tan(X^+_{1,2} \pm X^-_{1,2})],$$

where $\delta = ((\gamma_1 + \gamma_2)/(\gamma_1 - \gamma_2))$. Adding and subtracting the above expressions we obtain

$$\phi_3^{\pm} = \phi_0^{\pm} + \Gamma_{\pm},$$

with

$$\Gamma_{\pm} = 2 \arctan[\delta \tan(X_{1,2}^{+} + X_{1,2}^{-})] \pm 2 \arctan[\delta \tan(X_{1,2}^{+} - X_{1,2}^{-})].$$

APPENDIX C: SOME USEFUL FORMULAE

Relations (29) and (31) can be written in matrix form,

$$\begin{pmatrix} \mathcal{F}^{(1,3)} \\ \mathcal{F}^{(2,3)} \end{pmatrix} = \frac{1}{Z}\begin{pmatrix} A & -B \\ C & -D \end{pmatrix}\begin{pmatrix} \mathcal{F}^{(0,1)} \\ \mathcal{F}^{(0,2)} \end{pmatrix}, \tag{C1}$$

where

$$A = \kappa_2\lambda_2(\bar{c}_{0,1}^{+}c_{2,3}^{-} + \bar{c}_{2,3}^{+}c_{0,1}^{-}),$$

$$B = \kappa_2\lambda_1\bar{c}_{0,2}^{+}c_{2,3}^{-} + \kappa_1\lambda_2\bar{c}_{2,3}^{+}c_{0,2}^{-},$$

$$C = \kappa_2\lambda_1\bar{c}_{1,3}^{+}c_{0,1}^{-} + \kappa_1\lambda_2\bar{c}_{0,1}^{+}c_{1,3}^{-},$$

$$D = \kappa_1\lambda_1(\bar{c}_{0,2}^{+}c_{1,3}^{-} + \bar{c}_{1,3}^{+}c_{0,2}^{-}),$$

$$Z = \kappa_2\lambda_1\bar{c}_{1,3}^{+}c_{2,3}^{-} - \kappa_1\lambda_2\bar{c}_{2,3}^{+}c_{1,3}^{-}. \tag{C2}$$

Introduce Eq. (32) into expressions (C2). Consider now the following expansions:

$$c_{k,3}^{-} = c_{k,\Gamma_-}\left(1 - \frac{\Delta_-^2}{8}\right) - \frac{\Delta_-}{2}s_{k,\Gamma_-},$$

$$\bar{c}_{k,3}^{+} = \bar{c}_{k,\Gamma_-}\left(1 - \frac{\Delta_+^2}{8}\right) + \frac{\Delta_+}{2}\bar{s}_{k,\Gamma_+},$$

where we have denoted

$$c_{k,\Gamma_-} = \cos\left(\frac{\phi_k^- + \phi_0^- + \Gamma_-}{2}\right) = c_{k,0}^{-}\sigma_+ - s_{k,0}^{-}\rho_-,$$

$$s_{k,\Gamma_-} = \sin\left(\frac{\phi_k^- + \phi_0^- + \Gamma_-}{2}\right) = s_{k,0}^{-}\sigma_+ + c_{k,0}^{-}\rho_-,$$

$$\bar{c}_{k,\Gamma_+} = \cos\left(\frac{\phi_k^+ - \phi_0^+ - \Gamma_+}{2}\right) = \bar{c}_{k,0}^{+}\sigma_- + \bar{s}_{k,0}^{+}\rho_+,$$

$$\bar{s}_{k,\Gamma_+} = \sin\left(\frac{\phi_k^+ - \phi_0^+ - \Gamma_+}{2}\right) = \bar{s}_{k,0}^{+}\sigma_- - \bar{c}_{k,0}^{+}\rho_+,$$

and

$$\sigma_{\pm} = \frac{1 \pm \delta^2 \tan\left(\frac{x+y}{4}\right)\tan\left(\frac{x-y}{4}\right)}{\sqrt{1+\delta^2 \tan^2\left(\frac{x+y}{4}\right)}\sqrt{1+\delta^2 \tan^2\left(\frac{x-y}{4}\right)}},$$

$$\rho_{\pm} = \frac{\delta\left[\tan\left(\frac{x+y}{4}\right) \pm \tan\left(\frac{x-y}{4}\right)\right]}{\sqrt{1+\delta^2 \tan^2\left(\frac{x+y}{4}\right)}\sqrt{1+\delta^2 \tan^2\left(\frac{x-y}{4}\right)}}.$$

Next, we expand the expressions for A, B, C, D, and Z in power series of f obtaining

$$A = A_0 + \sum_{j,k=1}^{2} A_{j,k} f_{j,k} + \mathcal{O}(f_0),$$

$$A_0 = \kappa_2 \lambda_2 (\bar{c}^{+}_{0,1} c_{2,\Gamma_-} + c^{-}_{0,1} \bar{c}_{2,\Gamma_+}),$$

$$A_{j,k} = \frac{1}{2}\kappa_2 \lambda_2 (c^{-}_{0,1} \bar{s}_{2,\Gamma_+} \Lambda^{+}_{j,k} - \bar{c}^{+}_{0,1} s_{2,\Gamma_-} \Lambda^{-}_{j,k}),$$

$$B = B_0 + \sum_{j,k=1}^{2} B_{j,k} f_{j,k} + \mathcal{O}(f_0),$$

$$B_0 = \kappa_2 \lambda_1 \bar{c}^{+}_{0,2} c_{2,\Gamma_-} + \kappa_1 \lambda_2 \bar{c}_{2,\Gamma_+} c^{-}_{0,2},$$

$$B_{j,k} = \frac{1}{2}(\kappa_1 \lambda_2 c^{-}_{0,2} \bar{s}_{2,\Gamma_+} \Lambda^{+}_{j,k} - \kappa_2 \lambda_1 \bar{c}^{+}_{0,2} s_{2,\Gamma_-} \Lambda^{-}_{j,k}),$$

$$C = C_0 + \sum_{j,k=1}^{2} C_{j,k} f_{j,k} + \mathcal{O}(f_0),$$

$$C_0 = \kappa_1 \lambda_2 \bar{c}^{+}_{0,1} c_{1,\Gamma_-} + \kappa_2 \lambda_1 \bar{c}_{1,\Gamma_+} c^{-}_{0,1},$$

$$C_{j,k} = \frac{1}{2}(\kappa_2 \lambda_1 c^{-}_{0,1} \bar{s}_{1,\Gamma_+} \Lambda^{+}_{j,k} - \kappa_1 \lambda_2 \bar{c}^{+}_{0,1} s_{1,\Gamma_-} \Lambda^{-}_{j,k}),$$

$$D = D_0 + \sum_{j,k=1}^{2} D_{j,k} f_{j,k} + \mathcal{O}(f_0),$$

$$D_0 = \kappa_1 \lambda_1 (\bar{c}^{+}_{0,2} c_{1,\Gamma_-} + c^{-}_{0,2} \bar{c}_{1,\Gamma_+}),$$

$$D_{j,k} = \frac{1}{2}\kappa_1 \lambda_1 (c^{-}_{0,2} \bar{s}_{1,\Gamma_+} \Lambda^{+}_{j,k} - \bar{c}^{+}_{0,2} s_{1,\Gamma_-} \Lambda^{-}_{j,k}),$$

$$Z = Z_0 + \sum_{j,k=1}^{2} Z_{j,k} f_{j,k} + \mathcal{O}(f_0),$$

$$Z_0 = \kappa_2 \lambda_1 \bar{c}_{1,\Gamma_+} c_{2,\Gamma_-} - \kappa_1 \lambda_2 c_{1,\Gamma_-} \bar{c}_{2,\Gamma_+},$$

$$Z_{j,k} = \frac{1}{2}(\kappa_2\lambda_1 c_{2,\Gamma_-}\bar{s}_{1,\Gamma_+} - \kappa_1\lambda_2 c_{1,\Gamma_-}\bar{s}_{2,\Gamma_+})\Lambda^+_{j,k} - \frac{1}{2}(\kappa_2\lambda_1 s_{2,\Gamma_-}\bar{c}_{1,\Gamma_+} - \kappa_1\lambda_2 s_{1,\Gamma_-}\bar{c}_{2,\Gamma_+})\Lambda^-_{j,k},$$

where $\mathcal{O}(f_0)$ denotes terms proportional to f_0. It then follows

$$\frac{X}{Z} = \frac{X_0}{Z_0}\left[1 + \sum_{j,k=1}^{2}\left(\frac{X_{j,k}}{X_0} - \frac{Z_{j,k}}{Z_0}\right) f_{j,k}\right] + \mathcal{O}(f_0),$$

where $X=\{A,B,C,D\}$.

Substituting (C1), we obtain

$$\mathcal{F}^{(1,3)} = \frac{A_0}{Z_0}\mathcal{F}^{(0,1)} - \frac{B_0}{Z_0}\mathcal{F}^{(0,2)} + \omega_1^{(1)}\mathcal{F}^{(0,1)} f_{2,1} + \omega_2^{(1)}\mathcal{F}^{(0,1)} f_{2,2}, \tag{C3}$$

$$\mathcal{F}^{(2,3)} = \frac{C_0}{Z_0}\mathcal{F}^{(0,1)} - \frac{D_0}{Z_0}\mathcal{F}^{(0,2)} + \omega_1^{(2)}\mathcal{F}^{(0,1)} f_{2,1} + \omega_2^{(2)}\mathcal{F}^{(0,1)} f_{2,2}, \tag{C4}$$

where

$$\omega_1^{(1)} = \frac{A_0}{Z_0}\left(\frac{A_{2,1}}{A_0} - \frac{Z_{2,1}}{Z_0}\right) + \frac{B_0}{Z_0}\left(\frac{B_{1,1}}{B_0} - \frac{Z_{1,1}}{Z_0}\right),$$

$$\omega_2^{(1)} = \frac{A_0}{Z_0}\left(\frac{A_{2,2}}{A_0} - \frac{Z_{2,2}}{Z_0}\right) + \frac{B_0}{Z_0}\left(\frac{B_{1,2}}{B_0} - \frac{Z_{1,2}}{Z_0}\right),$$

$$\omega_1^{(2)} = \frac{C_0}{Z_0}\left(\frac{C_{2,1}}{C_0} - \frac{Z_{2,1}}{Z_0}\right) + \frac{D_0}{Z_0}\left(\frac{D_{1,1}}{D_0} - \frac{Z_{1,1}}{Z_0}\right),$$

$$\omega_2^{(2)} = \frac{C_0}{Z_0}\left(\frac{C_{2,2}}{C_0} - \frac{Z_{2,2}}{Z_0}\right) + \frac{D_0}{Z_0}\left(\frac{D_{1,2}}{D_0} - \frac{Z_{1,2}}{Z_0}\right).$$

From Eqs. (28), we get

$$D_+(\phi_3^+ - \phi_0^+) = \frac{8}{\kappa_1}\mathcal{F}^{(0,1)} c^-_{0,1} + \frac{8}{\kappa_2}\mathcal{F}^{(1,3)} c^-_{1,3},$$

$$D_+(\phi_1^+ - \phi_2^+) = \frac{8}{\kappa_1}\mathcal{F}^{(0,1)} c^-_{0,1} - \frac{8}{\kappa_2}\mathcal{F}^{(0,2)} c^-_{0,2}.$$

Introducing solution (32) in the first equation above, we find

$$D_+(\phi_3^+ - \phi_0^+) = D_+(\Gamma_+ + \Delta_+) = \partial_x\Gamma_+ D_+(\phi_1^+ - \phi_2^+) + D_+\Delta_+ = \frac{8}{\kappa_1}\mathcal{F}^{(0,1)} c^-_{0,1} + \frac{8}{\kappa_2}\mathcal{F}^{(1,3)} c^-_{1,3}.$$

Using Eq. (C3) in the above expression, and taking into account that $\mathcal{F}^{(0,1)}$, $\mathcal{F}^{(0,2)}$, $\mathcal{F}^{(0,1)} f_{2,1}$, and $\mathcal{F}^{(0,1)} f_{2,2}$ are independent, we arrive at the following conditions:

$$\frac{c_{0,1}^{-}}{\kappa_1}(\partial_x\Gamma_+-1)-\frac{c_{1,\Gamma_-}}{\kappa_2}\frac{A_0}{Z_0}+\frac{gs_{0,2}^{-}}{4\kappa_2}\Lambda_{1,2}^{+}+\frac{gs_{0,1}^{-}}{4\kappa_1}\Lambda_{1,1}^{+}=0,$$

$$\frac{c_{0,2}^{-}}{\kappa_2}\partial_x\Gamma_+-\frac{c_{1,\Gamma_-}}{\kappa_2}\frac{B_0}{Z_0}-\frac{gs_{0,1}^{-}}{4\kappa_1}\Lambda_{2,1}^{+}-\frac{gs_{0,2}^{-}}{4\kappa_2}\Lambda_{2,2}^{+}=0,$$

$$\frac{c_{0,2}^{-}}{\kappa_2}\partial_x\Lambda_{1,1}^{+}+\frac{c_{0,1}^{-}}{\kappa_1}\partial_x\Lambda_{2,1}^{+}+\frac{gs_{0,2}^{-}}{4\kappa_2}\Lambda_0^{+}-\frac{c_{1,\Gamma_-}}{\kappa_2}\omega_1^{(1)}+\frac{s_{1,\Gamma_-}}{2\kappa_2}\left(\frac{A_0}{Z_0}\Lambda_{2,1}^{-}+\frac{B_0}{Z_0}\Lambda_{1,1}^{-}\right)=0,$$

$$\frac{c_{0,2}^{-}}{\kappa_2}\partial_x\Lambda_{1,2}^{+}+\frac{c_{0,1}^{-}}{\kappa_1}\partial_x\Lambda_{2,2}^{+}-\frac{gs_{0,1}^{-}}{4\kappa_1}\Lambda_0^{+}-\frac{c_{1,\Gamma_-}}{\kappa_2}\omega_2^{(1)}+\frac{s_{1,\Gamma_-}}{2\kappa_2}\left(\frac{A_0}{Z_0}\Lambda_{2,2}^{-}+\frac{B_0}{Z_0}\Lambda_{1,2}^{-}\right)=0. \quad (C5)$$

Moreover, the chirality condition on (32) gives

$$\bar{D}_-(\phi_3^+-\phi_0^+)=\bar{D}_-(\Gamma_++\Delta_+)=0,$$

from where we obtain the following equations:

$$\frac{\overline{c}_{0,1}^{+}}{\lambda_1}\partial_y\Gamma_++\frac{g\overline{s}_{0,1}^{+}}{4\lambda_1}\Lambda_{1,1}^{+}+\frac{g\overline{s}_{0,2}^{+}}{4\lambda_2}\Lambda_{1,2}^{+}=0,$$

$$\frac{\overline{c}_{0,2}^{+}}{\lambda_2}\partial_y\Gamma_+-\frac{g\overline{s}_{0,1}^{+}}{4\lambda_1}\Lambda_{2,1}^{+}-\frac{g\overline{s}_{0,2}^{+}}{4\lambda_2}\Lambda_{2,2}^{+}=0,$$

$$\frac{\overline{c}_{0,2}^{+}}{\lambda_2}\partial_y\Lambda_{1,1}^{+}+\frac{\overline{c}_{0,1}^{+}}{\lambda_1}\partial_y\Lambda_{2,1}^{+}+\frac{g\overline{s}_{0,2}^{+}}{4\lambda_2}\Lambda_0^{+}=0,$$

$$\frac{\overline{c}_{0,2}^{+}}{\lambda_2}\partial_y\Lambda_{1,2}^{+}+\frac{\overline{c}_{0,1}^{+}}{\lambda_1}\partial_y\Lambda_{2,2}^{+}-\frac{g\overline{s}_{0,1}^{+}}{4\lambda_1}\Lambda_0^{+}=0. \quad (C6)$$

The two sets of equations, namely, (C5) and (C6), give the following solutions:

$$\Lambda_{1,1}^{+}=\Lambda_{2,2}^{+}=-\frac{8\mu_-}{g\eta_+\eta_-}\cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_{1,2}^{+}=\frac{8\mu_-}{g\eta_+\eta_-}\left(\frac{\lambda_2}{\lambda_1}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_{2,1}^{+}=\frac{8\mu_-}{g\eta_+\eta_-}\left(\frac{\lambda_1}{\lambda_2}\right)\sin\left(\frac{y}{2}\right),$$

$$\Lambda_0^{+}=-\frac{32\mu_-}{(g\eta_+\eta_-)^2}\sin\left(\frac{x}{2}\right)\left[\cos\left(\frac{y}{2}\right)(a+\cos x-\cos y)-2\mu_+\cos\left(\frac{x}{2}\right)\right],$$

where

$$\mu_\pm=\frac{\gamma_1}{\gamma_2}\pm\frac{\gamma_2}{\gamma_1},$$

$$a=\frac{1}{2}\left(\frac{\gamma_1^2}{\gamma_2^2}+\frac{\gamma_2^2}{\gamma_1^2}\right)+3,$$

$$\eta_\pm=\mu_+-2\cos\left(\frac{x\pm y}{2}\right). \tag{C7}$$

In order to determine the coefficients Λ^- we make use of

$$\bar{D}_-(\phi_3^- - \phi_0^-)=\frac{8}{\lambda_1}\mathcal{F}^{(0,1)}\overline{c}^+_{0,1}-\frac{8}{\lambda_2}\mathcal{F}^{(1,3)}\overline{c}^+_{1,3},$$

$$\bar{D}_-(\phi_1^- - \phi_2^-)=-\frac{8}{\lambda_1}\mathcal{F}^{(0,1)}\overline{c}^+_{0,1}+\frac{8}{\lambda_2}\mathcal{F}^{(0,2)}\overline{c}^+_{0,2},$$

which are obtained from (30). Introducing (32) in the first of these equations, we find

$$\bar{D}_-(\phi_3^- - \phi_0^-)=\bar{D}_-(\Gamma_-+\Delta_-)=\partial_y\Gamma_-\bar{D}_-(\phi_1^- - \phi_2^-)+\bar{D}_-\Delta_-=\frac{8}{\lambda_1}\mathcal{F}^{(0,1)}\overline{c}^+_{0,1}-\frac{8}{\lambda_2}\mathcal{F}^{(1,3)}\overline{c}^+_{1,3}.$$

Using Eq. (C3) in the above expression and taking into account that $\mathcal{F}^{(0,1)}$, $\mathcal{F}^{(0,2)}$, $\mathcal{F}^{(0,1)}f_{2,1}$, and $\mathcal{F}^{(0,1)}f_{2,2}$ are independent, we arrive at the following expressions:

$$\frac{\overline{c}^+_{0,1}}{\lambda_1}(\partial_y\Gamma_-+1)-\frac{\overline{c}_{1,\Gamma_+}}{\lambda_2}\frac{A_0}{Z_0}+\frac{g\overline{s}^+_{0,1}}{4\lambda_1}\Lambda^-_{1,1}+\frac{g\overline{s}^+_{0,2}}{4\lambda_2}\Lambda^-_{1,2}=0,$$

$$\frac{\overline{c}^+_{0,2}}{\lambda_2}\partial_y\Gamma_- -\frac{\overline{c}_{1,\Gamma_+}}{\lambda_2}\frac{B_0}{Z_0}-\frac{g\overline{s}^+_{0,1}}{4\lambda_1}\Lambda^-_{2,1}-\frac{g\overline{s}^+_{0,2}}{4\lambda_2}\Lambda^-_{2,2}=0,$$

$$\frac{\overline{c}^+_{0,2}}{\lambda_2}\partial_y\Lambda^-_{1,1}+\frac{\overline{c}^+_{0,1}}{\lambda_1}\partial_y\Lambda^-_{2,1}+\frac{g\overline{s}^+_{0,2}}{4\lambda_2}\Lambda^-_0-\frac{\overline{c}_{1,\Gamma_+}}{\lambda_2}\omega_1^{(1)}-\frac{\overline{s}_{1,\Gamma_+}}{2\lambda_2}\left(\frac{A_0}{Z_0}\Lambda^+_{2,1}+\frac{B_0}{Z_0}\Lambda^+_{1,1}\right)=0,$$

$$\frac{\overline{c}^+_{0,2}}{\lambda_2}\partial_y\Lambda^-_{1,2}+\frac{\overline{c}^+_{0,1}}{\lambda_1}\partial_y\Lambda^-_{2,2}-\frac{g\overline{s}^+_{0,1}}{4\lambda_1}\Lambda^-_0-\frac{\overline{c}_{1,\Gamma_+}}{\lambda_2}\omega_2^{(1)}-\frac{\overline{s}_{1,\Gamma_+}}{2\lambda_2}\left(\frac{A_0}{Z_0}\Lambda^+_{2,2}+\frac{B_0}{Z_0}\Lambda^+_{1,2}\right)=0. \tag{C8}$$

The chiral condition,

$$D_+(\phi_3^- - \phi_0^-)=D_+(\Gamma_-+\Delta_-)=0,$$

leads us to

$$\frac{c^-_{0,1}}{\kappa_1}\partial_x\Gamma_-+\frac{gs^-_{0,1}}{4\kappa_1}\Lambda^-_{1,1}+\frac{gs^-_{0,2}}{4\kappa_2}\Lambda^-_{1,2}=0,$$

$$\frac{c^-_{0,2}}{\kappa_2}\partial_x\Gamma_- -\frac{gs^-_{0,1}}{4\kappa_1}\Lambda^-_{2,1}-\frac{gs^-_{0,2}}{4\kappa_2}\Lambda^-_{2,2}=0,$$

$$\frac{c^-_{0,2}}{\kappa_2}\partial_x\Lambda^-_{1,1}+\frac{c^-_{0,1}}{\kappa_1}\partial_x\Lambda^-_{2,1}+\frac{gs^-_{0,2}}{4\kappa_2}\Lambda^-_0=0,$$

$$\frac{c_{0,2}^{-}}{\kappa_2}\partial_x\Lambda_{1,2}^{-}+\frac{c_{0,1}^{-}}{\kappa_1}\partial_x\Lambda_{2,2}^{-}-\frac{gs_{0,1}^{-}}{4\kappa_1}\Lambda_0^{-}=0. \tag{C9}$$

Solving (C8) and (C9) for Λ^-, we find

$$\Lambda_{1,1}^{-}=\Lambda_{2,2}^{-}=\frac{8\mu_-}{g\eta_+\eta_-}\cos\left(\frac{y}{2}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_{1,2}^{-}=-\frac{8\mu_-}{g\eta_+\eta_-}\left(\frac{\kappa_2}{\kappa_1}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_{2,1}^{-}=-\frac{8\mu_-}{g\eta_+\eta_-}\left(\frac{\kappa_1}{\kappa_2}\right)\sin\left(\frac{x}{2}\right),$$

$$\Lambda_0^{-}=-\frac{32\mu_-}{(g\eta_+\eta_-)^2}\sin\left(\frac{y}{2}\right)\left[\cos\left(\frac{x}{2}\right)(a-\cos x+\cos y)-2\mu_+\cos\left(\frac{y}{2}\right)\right],$$

where $\mu_\pm$, a, and $\eta_\pm$ are given in (C7).

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