Noise and ultraviolet divergences in the dynamics of the chiral condensate in QCD

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Abstract. The time evolution of the matter produced in high energy heavy-ion collisions seems to be well described by relativistic viscous hydrodynamics. In addition to the hydrodynamic degrees of freedom related to energy-momentum conservation, degrees of freedom associated with order parameters of broken continuous symmetries must be considered because they are all coupled to each other. Of particular interest is the coupling of degrees of freedom associated with the chiral symmetry of QCD. Quantum and thermal fluctuations of the chiral fields act as noise sources in the classical equations of motion, turning them into stochastic differential equations in the form of Ginzburg-Landau-Langevin (GLL) equations. Analytic solutions of GLL equations are attainable only in very special circumstances and extensive numerical simulations are necessary, usually by discretizing the equations on a spatial lattice. However, a not much appreciated issue in the numerical simulations of GLL equations is that ultraviolet divergences in the form of lattice-spacing dependence plague the solutions. The divergences are related to the well-known Rayleigh-Jeans catastrophe in classical field theory. In the present communication we present a systematic lattice renormalization method to control the catastrophe. We discuss the implementation of the method for a GLL equation derived in the context of a model for the QCD chiral phase transition and consider the nonequilibrium evolution of the chiral condensate during the hydrodynamic flow of the quark-gluon plasma.

1. Introduction

Experiments of high energy heavy-ion collisions conducted at the Relativistic Heavy Ion Collider (RHIC) have produced a state of matter that is strongly interacting and evolves as a low-viscosity fluid [1, 2, 3, 4, 5, 6] – the strongly interacting quark-gluon plasma (sQGP). The characterization of the produced matter as strongly interacting came as a great surprise, since earlier expectations based on the property of asymptotic freedom of quantum chromodynamics (QCD) were that RHIC would produce a gas-like system of weakly interacting quarks and gluons. The case for a strongly-interacting system was made on the observation that the space-time evolution of the produced sQGP is well described by relativistic viscous hydrodynamics. In particular, hydrodynamics accounts for the measured momentum anisotropies of the detected hadrons produced in the collision. The momentum anisotropies are the translation to momentum space of the initial spatial eccentricity of the collision; they are encoded in the Fourier moments v_2, v_3, v_4, $\cdots$ of the measured azimuthal distribution of particles [7]. A weakly–interacting, gas–like fluid would have no means to induce momentum anisotropies unless the particles of the fluid are strongly interacting.

Crucial physics input to a hydrodynamic description of the sQGP are the equation of state (EOS) and transport coefficients, as shear (η) and bulk (ζ) viscosities; they reflect the properties of the flowing hadronic matter and should be calculable from QCD. Assuming validity of hydrodynamics for the sQGP, extraction of transport coefficients from measured momentum anisotropy patterns can pinpoint the location of the QCD phase transition or of a crossover from hadronic to quark-gluon matter, as suggested in Ref. [8]. However, it should be noted that in addition to the hydrodynamic degrees of freedom related to energy-momentum conservation, degrees of freedom associated with order parameters of broken continuous symmetries must be considered because they are all coupled to each other. Of particular interest is the coupling of degrees of freedom associated with chiral symmetry, which is an approximate symmetry of QCD that is responsible for the smallness of the pion mass and is supposed to be restored at high temperatures and baryon density. A first work in the direction of coupling chiral degrees of freedom to viscous hydrodynamics was recently conducted in Ref. [9]. That work investigated the influence of the coupling of chiral fields (described by the linear σ model) to viscous hydrodynamic equations on the charged-hadron elliptic flow v_2 and on the ratio $v_4/(v_2)^2$ considering the kinematical conditions of a Au + Au collision at $\sqrt{s} = 200$ GeV per nucleon. The conclusions of Ref. [9] were that v_2 is very sensitive to the temperature dependence of the (shear) viscosity to entropy ratio η/s, while v_2 is not very sensitive to the coupling of the chiral fields. In addition, it was found that the ratio $v_4/(v_2)^2$ is much more sensitive than v_2 to both the coupling of the chiral sources and the temperature dependence of η/s.

One important limitation of the study in Ref. [9] was the neglect of quantum and thermal fluctuations of the chiral fields. Temperature effects in that reference were taken into account via the one-loop effective potential for the chiral fields. Fluctuations, on the other hand, act as noise sources in the classical equations of motion, turning them stochastic differential equations in the form of Ginzburg-Landau-Langevin (GLL) equations. Such equations describe the time evolution of an order parameter under the influence of an environment which is encoded usually in the form of dissipation and noise terms - for a good text book and references on the subject, see Ref. [10]. Typically, when a system consisting of a two-phase mixture in a homogeneous phase is rapidly driven across the critical coexistence temperature into a nonequilibrium state, fluctuations around the initial homogeneous state will develop, the system will break into domains of different phases in space, and ultimately will reach a new equilibrium state.

In the present communication we concentrate on a difficulty that appears in the numerical simulation of GLL equations. Analytic solutions of GLL equations are attainable only in very special circumstances, like when nonlinear couplings are neglected. Linearization of the equations are usually valid in a restricted time interval only [11] and extensive numerical simulations, usually by discretizing the equations on a space lattice, are necessary for obtaining equilibrium solutions. One difficulty is related to the well-known Rayleigh-Jeans ultraviolet catastrophe of classical field theory [12], which manifest themselves via severe lattice-spacing dependence of the solutions of the GLL equations. This is an intrinsic feature of the problem, not related to a particular discretization method of the equations. In order to obtain sensible results in numerical simulations of GLL equations, a renormalization procedure has to be implemented. The implementation of a simple and efficient method [13, 14] to control the lattice-spacing dependence of the solutions of a GLL derived from the linear σ model will be one of the main topics of the present communication.

2. Ginzburg-Landau-Langevin equation - chiral symmetry
Let us start with a brief review of the usage of a GLL equation in statistical physics. A GLL equation for an order parameter field $\varphi(x,t)$ in presence of a noise field $\zeta(x,t)$ is typically

given by [10]

$$\Gamma \frac{\partial \varphi(x,t)}{\partial t} = -\frac{\delta H[\varphi]}{\delta \varphi(x,t)} + \zeta(x,t), \tag{1}$$

where Γ is an Onsager (dissipation) coefficient and $H[\varphi]$ is the Ginzburg-Landau-Wilson (GLW) Hamiltonian, which is typically of the form

$$H[\varphi] = \int dx \left[\frac{1}{2}\kappa(\nabla\varphi)^2 + U_0(\varphi) \right], \tag{2}$$

with κ a positive constant (the range of the interaction in e.g. Ising-like models). A prototype $U(\varphi)$ is

$$U_0(\varphi) = \frac{1}{2} r_0 \varphi^2 + \frac{1}{4} u_0 \varphi^4, \tag{3}$$

with r_0 and $u_0 > 0$ constants that can depend on the temperature. For $r_0 < 0$, $U_0(\varphi)$ is of a double-well shape. For simplicity, let us consider a simple white-noise correlation function for the noise field:

$$\langle \zeta(x,t) \rangle_\zeta = 0, \qquad \langle \zeta(x,t)\zeta(x',t') \rangle_\zeta = 2\Gamma T \, \delta(t-t') \, \delta(x-x'), \tag{4}$$

where $\langle \cdots \rangle_\zeta$ means average over noise realizations and T is the external temperature. From the Fokker-Planck equation associated with the GLL equation of Eq. (1), it is not difficult to show that the equilibrium probability distribution of field configurations $\varphi(x)$ is the Boltzmann factor

$$P_{\text{eq}}[\varphi] = e^{-H[\varphi]/T}. \tag{5}$$

The equilibrium partition function is then given by the functional integral

$$Z[\varphi] = \int \mathcal{D}\varphi \, e^{-H[\varphi]/T}. \tag{6}$$

The meaning of this is that a correlation function, e.g. a two-point correlation function $\langle \varphi(x_1)\varphi(x_2) \rangle$, defined as

$$\langle \varphi(x_1)\varphi(x_2) \rangle = \frac{1}{Z} \int \mathcal{D}\varphi \, \varphi(x_1)\varphi(x_2) \, e^{-H[\varphi]/T}, \tag{7}$$

can also be calculated via the long-time solutions of the GLL equation as

$$\langle \varphi(x_1)\varphi(x_2) \rangle = \lim_{t \to \infty} \frac{1}{N_R} \sum_{r=1}^{N_R} \varphi_r(x_1,t)\varphi_r(x_2,t), \tag{8}$$

where $\varphi_r(x,t)$ is a solution of the GLL equation of Eq. (1), with the index r indicating r-th noise realization, and N_r is the total number of realizations (N_r is supposed to be large). However, despite straightforward, the calculation of the averages in Eq. (8) face the problem that the solutions of the GLL equations suffer from sever lattice spacing dependence, as mentioned in the previous Section. Before discussing this, let us consider the derivation of GLL equations in the context of the quark-gluon plasma.

GLL equations can be derived from a microscopic Hamiltonian describing interparticle interactions by means of the Schwinger-Keldysh (closed-time-path) effective action formalism [15]. In general, the resulting equations can be rather involved, with the order parameter appearing nonlinearly in functionals nonlocal in space and time (memory). Simple forms like the ones shown in

Eqs. (1) - (4) are obtained only after a series of approximations and truncations. In the context of the strongly-interacting quark-gluon plasma (SQGP), Ref. [16] is a recent example where such a simple equation can be derived. Let us review the logic behind such approaches, starting with the hydrodynamic description of the sQGP used in Ref. [9]. Quarks and antiquarks are assumed to constitute a heat bath in local thermal equilibrium, propagated via fluid dynamic equations at the the level of densities, while the chiral fields are propagated via medium-corrected classical equations of motion. Specifically, the energy-momentum tensor $T^{\mu\nu}$ of the combined quark $q = (u, d)$ and chiral $\phi = (\phi^0, \phi^i) = (\sigma, \pi^i), i = 1, 2, 3$ fields is split as

$$T^{\mu\nu} = T_q^{\mu\nu} + T_\phi^{\mu\nu}, \tag{9}$$

with

$$T_q^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu} + \Pi^{\mu\nu}, \tag{10}$$

where the unknowns are the fluid four-velocity u_μ (normalized as $u_\mu u^\mu = 1$), the viscous shear-tensor $\Pi^{\mu\nu}$, the energy density $\epsilon = \epsilon(\phi, T)$, and the pressure $p = p(\phi, T)$ - in the spirit of a hydrodynamics description, they are supposed in local equilibrium at a temperature T. The fluid equations of motion are obtained from the condition of energy-momentum conservation $D_\mu T^{\mu\nu} = 0$, where D_μ is the geometric covariant derivative [9] – the equations of motion for the components of $\Pi^{\mu\nu}$ are those of the dissipative hydrodynamic formalism of Refs. [17, 18, 19]. To close the set of equations, one needs explicit expressions for the temperature dependence of ϵ and p, and the equations of motion for the chiral fields.

Here, like in Ref. [9], we use the linear σ model, so that the contribution from the chiral fields to the energy-momentum tensor in Eq. (9) is given by

$$T_\phi^{\mu\nu} = \sum_{a=0}^{4} \frac{\partial \langle \mathcal{L}_\phi \rangle}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \langle \mathcal{L}_\phi \rangle, \tag{11}$$

where $\langle \cdots \rangle$ means local thermal average and

$$\mathcal{L} = \mathcal{L}_q + \mathcal{L}_\phi, \tag{12}$$

with

$$\mathcal{L}_q = \bar{q} \left[i\gamma^\mu \partial_\mu - g \left(\sigma + \gamma_5 \vec{\tau} \cdot \vec{\pi} \right) \right] q, \tag{13}$$

and

$$\mathcal{L}_\phi = \frac{1}{2} \left(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi} \right) - U(\sigma, \vec{\pi}), \tag{14}$$

where

$$U(\sigma, \vec{\pi}) = \frac{\lambda^2}{4} \left(\phi^2 - v^2 \right)^2 - h_q \sigma - U_0 \,, \tag{15}$$

is the potential which exhibits chiral symmetry breaking, and $\phi^2 = \sigma^2 + \vec{\pi}^2$. The parameters of the model are fixed as follows. The vacuum expectation values of the fields (condensates) are taken $\langle \sigma \rangle = f_\pi$ and $\langle \vec{\pi} \rangle = 0$, with the pion decay constant $f_\pi = 93$ MeV. Also, the partially conserved axial–vector current relation yields $h_q = f_\pi m_\pi^2$ with $m_\pi = 138$ MeV. This leads to $v^2 = f_\pi^2 - m_\pi^2/\lambda^2$. Using $\lambda^2 = 20$, the value of the mass of the σ field is $m_\sigma = \sqrt{2\lambda^2 f_\pi^2 + m_\pi^2} \sim 600$ MeV. The constant U_0 is chosen such that the potential energy vanishes in the ground state. Note that these are fitted to experimental values, because at the level of approximation worked in Ref. [9], they do not become corrected by quantum or thermal effects.

The equations of motion of the chiral fields are given by

$$\Box \sigma + \frac{\delta V_{\text{eq}}(\sigma, \vec{\pi})}{\delta \sigma} = -g\rho_s , \tag{16}$$

$$\Box \vec{\pi} + \frac{\delta V_{\text{eq}}(\sigma, \vec{\pi})}{\delta \vec{\pi}} = -g\vec{\rho}_{ps} , \tag{17}$$

where $\Box$ is the D'Alambertian and ρ_s and $\vec{\rho}_{ps}$ are the equilibrium thermal averages of the scalar and pseudo-scalar chiral condensates :

$$\rho_s = \langle \bar{q}q \rangle = I_\sigma \, \sigma, \qquad \vec{\rho}_{ps} = \langle \bar{q}\gamma_5\vec{\tau}q \rangle = I_\pi \, \vec{\pi}, \tag{18}$$

where

$$I_\sigma = 2gd_q \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \left(1 + e^{E/T} \right)^{-1} , \tag{19}$$

$$I_\pi = 2gd_q \int \frac{d^3p}{(2\pi)^3} \frac{1}{E} \left(1 + e^{E/T} \right)^{-1} . \tag{20}$$

Here, $d_q = 12$ is the color–spin–isospin charge degeneracy of the quarks and $E = (p^2 + m_q^2)^{1/2}$ is the single-quark energy, with the quark mass given by $m_q^2 = g^2(\sigma^2 + \pi^2)$. We consider the case of zero net baryon number (zero baryon chemical potential). In addition, V_{eq} is the one-loop equilibrium effective potential

$$V_{\text{eq}}(\sigma, \vec{\pi}) = U(\sigma, \vec{\pi}) - 2d_qT \int \frac{d^3p}{(2\pi)^3} \ln \left(1 + e^{-E/T} \right) . \tag{21}$$

Here we have changed the definition of d_q as compared to the d_q in Ref. [9] by a factor of two, incorporating this factor in the definition of ρ_s, $\vec{\rho}_{ps}$ and V_{eq}.

In Eqs. (16) and (17), the influence of the quark fluid on the chiral fields is approximated by an equilibrium average over quantum and thermal fluctuations, encoded by the thermal averages of the scalar and pseudo-scalar chiral condensates ρ_s and $\vec{\rho}_{ps}$. A mean field treatment like this neglects physical effects like the decay of mesons into quark-antiquark pairs and the scattering of mesons by the quarks and antiquarks of the fluid. Such microscopic processes lead to dissipation and noise terms in the effective effective equations of motion for the coarse-grained chiral fields, which are of the form of GLL equations. Within the linear sigma model, a GLL equation can be derived using the Schwinger-Keldysh (closed-time-path) effective action formalism[15]. Written in the rest frame of the fluid, the GLL equation for the σ field derived in Ref. [16] is given by

$$\frac{\partial^2 \sigma}{\partial t^2} - \nabla^2 \sigma + \eta \frac{\partial \sigma}{\partial t} + \frac{\delta U}{\delta \sigma} + g\rho_s = \xi, \tag{22}$$

with the noise field satisfying

$$\langle \xi(x,t) \rangle_\xi = 0, \qquad \langle \xi(x,t)\xi(x',t') \rangle_\xi = m_\sigma\eta \coth\left(\frac{m_\sigma}{2T} \right) \delta(t - t')\, \delta^3(x - x'), \tag{23}$$

where the dissipation coefficient is given by

$$\eta = g^2 \frac{d_q}{\pi} \frac{1}{m_\sigma^2} \left(\frac{m_\sigma^2}{4} - m_q^2 \right)^{3/2} \left[1 - 2 \left(1 + e^{m_\sigma/2T} \right)^{-1} \right] . \tag{24}$$

Here the processes $\sigma \to \bar{q}q$ are included; but processes $\sigma \leftrightarrow 2\pi$ are not. The last ones are extremely important when $m_\sigma > 2m_\pi$, which occur at low temperatures. On the other hand, at low temperatures η as given in Eq. (24) is very small and σ decay and recombination processes must be included to take into account of the correct physics. Using the result of a study in Ref. [20] in the context of the σ model, processes $\sigma \leftrightarrow 2\pi$ can be taken into account with a noise correlation of the form

$$\langle \xi(x,t)\xi(x',t')\rangle_\xi = 2\eta_\sigma T\, \delta(t-t')\, \delta^3(x-x'), \tag{25}$$

with $\eta_\sigma = 2.2 \text{ fm}^{-1}$ for a temperature $T = 123$ MeV .

From Eq. (15), one has

$$\frac{\delta U}{\delta \sigma} = -\lambda v^2 \sigma + \lambda \sigma^3, \tag{26}$$

where we used for simplicity $h_q = 0$ in the equation of motion. The GLL equation of Eq. (22) can then be written as

$$\frac{\partial^2 \sigma}{\partial t^2} - \nabla^2 \sigma + \eta \frac{\partial \sigma}{\partial t} - \left(\lambda v^2 - I_\sigma\right)\sigma + \lambda \sigma^3 = \xi. \tag{27}$$

Note that the presence of the second order time derivative turns the GLL equation a hyperbolic partial differential equation, in contrast to the purely diffusive parabolic equation of Eq. (1). The presence of the second order time derivative is a consequence of causality imposed by relativity. In the context of statistical physics, hyperbolic GLL equations appear when memory effects are incorporated in the phase separation and phase ordering processes [21, 22]. One consequence of the hyperbolic nature of the equation is that time-dependent oscillations appear in the evolution of the order parameter toward equilibrium, as we shall see below. Regarding the equilibrium probability distribution of field configurations $\sigma(x)$, one can show that it is given by the Bolztmann factor (the proof is a little more elaborated than in the case of a parabolic equation, see e.g. Ref. [23]):

$$P_{\text{eq}}[\sigma] = e^{-H[\sigma]/T'}, \tag{28}$$

where $T' = (m_\sigma/2)\coth(m_\sigma/2T)$ and

$$H[\sigma] = \int d^3x \left[\frac{1}{2}\left(\nabla \sigma\right)^2 - \frac{1}{2}\left(\lambda v^2 - I_\sigma\right)\sigma^2 + \frac{1}{4}\lambda \sigma^4 \right]. \tag{29}$$

As already mentioned analytic solutions of such nonlinear equations are attainable only in very special circumstances, like when nonlinear terms in $H[\sigma]$ are dropped. Such linear approximations are usually valid in a restricted time interval only [11] and extensive numerical simulations, usually by discretizing the equations on a space lattice, are necessary for obtaining solutions over a large time interval. In the next Section, we will obtain explicit numerical solutions of Eq. (27) and discuss a method do regularize the equation so that results independent of the lattice spacing are obtained.

3. Lattice simulation of the GLL equation
In this Section we discuss the physical situation inwhich the system initially at a high temperature, where the σ field is almost zero, is instantaneously quenched to a low temperature phase; e.g. below the critical (or crossover) temperature.

It is useful to work with dimensionless quantities. We rescale space-time coordinates (t,x) and the field σ: $x = x'/(\lambda v^2)^{1/2}$, $t = t'/(\lambda v^2)^{1/2}$, $\sigma' = \sigma/v$. The GLL equation in Eq. (27) can be rewritten in terms of these dimensionless variables as (henceforth we drop the primes)

$$\frac{\partial^2 \sigma}{\partial t^2} - \nabla^2 \sigma + \bar{\eta}\frac{\partial \sigma}{\partial t} - \epsilon \sigma + \sigma^3 = \zeta, \tag{30}$$

where the dimensionless noise field $\zeta(x,t)$ has zero mean and correlation

$$\langle \zeta(x,t)\zeta(x',t')\rangle_\zeta = 2\gamma\,\delta(t-t')\,\delta^3(x-x'), \tag{31}$$

with

$$\bar{\eta} = \eta/(\lambda v^2)^{1/2}, \qquad \epsilon = -1 + I_\sigma/(\lambda v^2), \qquad \gamma = \eta T/v^2. \tag{32}$$

Here we used Eq. (25) for the noise correlation, appropriate for the physical situation we are considering. In the present work we use γ varying the range of $0.5 \leq \gamma \leq 2$, $\bar{\eta} = 1$, and $\epsilon = -1$ (corresponding to the neglect of $I_\sigma/(\lambda v^2)$, which should not be a bad approximation for low temperatures). Evidently when hydrodynamic equations are coupled to the chiral equations, one is not free to choose the values of $\bar{\eta}$, ϵ and γ, since they are all functions of the local temperature determined by the hydrodynamic equations.

We discretize Eq. (30) on a three-dimensional spatial lattice and use a leapfrog algorithm for the time evolution. We use the notation

$$\sigma(x,t) = \sigma(x,y,z,t) \to \sigma_{ijk}(n), \quad t = n\Delta t, \quad n = 0, 1, \cdots, N_t - 1, \tag{33}$$

$$(x,y,x) = (ia, ja, ka), \quad i, j, k = 0, 1, \cdots, N - 1,, \tag{34}$$

where a is the (dimensionless) lattice spacing. The leapfrog algorithm is defined by the following iteration scheme:

$$\frac{\partial \sigma(x,t)}{\partial t} \to \dot{\sigma}_{ijk}(n) = \frac{1}{2}\left[\dot{\sigma}_{ijk}(n+1/2) + \dot{\sigma}(n-1/2)\right], \tag{35}$$

$$\frac{\partial^2 \sigma(x,t)}{\partial t^2} \to \ddot{\sigma}_{ijk}(n) = \frac{1}{\Delta t}\left[\dot{\sigma}_{ijk}(n+1/2) - \dot{\sigma}_{ijk}(n-1/2)\right], \tag{36}$$

$$\dot{\sigma}_{ijk}(n+1/2) = \frac{1}{\Delta t}\left[\sigma_{ijk}(n+1) - \sigma_{ijk}(n)\right], \tag{37}$$

Now we rewrite Eq. (30) in terms of the discrete field variables $\sigma_{ijk}(n)$:

$$\frac{1}{\Delta t}\left[\dot{\sigma}_{ijk}(n+1/2) - \dot{\sigma}_{ijk}(n-1/2)\right] = A(\sigma,\zeta)_{ijk}(n) - \frac{\bar{\eta}}{2}\left[\dot{\sigma}_{ijk}(n+1/2) + \dot{\sigma}_{ijk}(n-1/2)\right], \tag{38}$$

where

$$A(\sigma,\zeta) = \left(\nabla^2\sigma\right) - \epsilon\,\sigma + \sigma^3 + \zeta. \tag{39}$$

The equation that has to be solved iteratively is then

$$\dot{\sigma}_{ijk}(n+1/2) = \frac{1}{1+(\bar{\eta}/2)\Delta t}\left\{\left[1 - (\bar{\eta}/2)\Delta t\right]\dot{\sigma}_{ijk}(n-1/2) + A(\sigma,\zeta)_{ijk}(n)\,\Delta t\right\}, \tag{40}$$

with the field updated as

$$\sigma_{ijk}(n+1) = \sigma_{ijk}(n) + \dot{\sigma}_{ijk}(n+1/2)\,\Delta t. \tag{41}$$

The noise field is generated at each time and lattice point as

$$\zeta_{ijk}(n) = \sqrt{\frac{2\gamma}{a^3\Delta t}}\,G_{ijk}(n), \tag{42}$$

where $G_{ijk}(n)$ is a zero-mean unit-variance Gaussian random number.

In Fig. 1 we present results for the spatially averaged value of the (noise-averaged) field as a function of time

$$\bar{\sigma}(t) = \frac{1}{N^3} \sum_{i,j,k=1}^{N} \langle \sigma_{ijk}(n) \rangle_\zeta, \tag{43}$$

where $\langle \sigma_{ijk}(n) \rangle_\zeta$ is the average over noise realizations. Results are shown for three values of lattice spacings, $a = 0.5, 0.8$ and 1.0 – in physical units, they correspond roughly to 0.35 fm, 0.57 fm and 0.7 fm, respectively. For the time spacing, we use $\Delta t = 0.001$ – corresponding to 7×10^{-4} fm/c. We show results for $\gamma = 1$ and $\gamma = 2$.
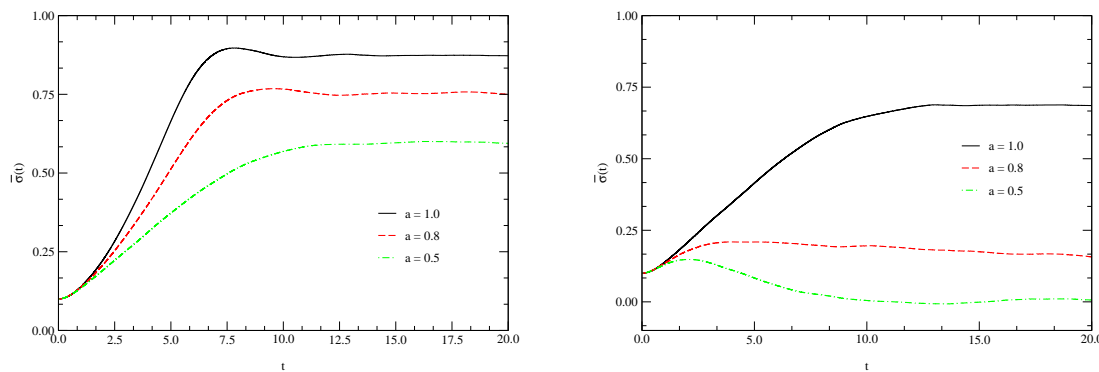
Figure 1. Volume average of the noise averaged σ field as a function of the time, Eq. (43). The left panel is for $\gamma = 1$ and the right panel is for $\gamma = 2$.

The results in Fig. 1 show very clearly the severe lattice spacing dependence of the solutions. The sensitivity is stronger for larger values of γ – note that γ is a measure of the strength of the quantum and thermal fluctuations. As said previously, the sensitivity of the solutions on a is due to the noise sources and is a manifestation of the ultraviolet divergences of classical field theory; when $\zeta = 0$, the sensitivity is absent. Although we have solved the GLL equation for fixed values of $\bar{\eta}$, ϵ and γ, it should be clear that in the coupled hydrochiral evolution, the lattice spacing sensitivity of the GLL equation can introduce large errors in the outcome of the simulation. In the next Section we discuss a method to control the lattice spacing sensitivity.

4. Renormalization of the GLL equation
The equilibrium probability distribution for the field configurations σ that are solutions of Eq. (22) is the Boltzmann factor given in Eq. (28). Equilibrium expectation values and correlation functions of σ with this partition function leads to ultraviolet divergences in spatial dimension larger than one – these divergences are completely unrelated to the usual divergences of the quantum theory. In the dynamics toward equilibrium, the divergences manifest themselves in the wild lattice spacing dependence of the solutions as $a \to 0$, as seen in the previous Section. The lattice dependence can be eliminated by renormalizing the GLL equation through the addition of appropriate counterterms. In three-dimensions, the theory is super-renormalizable, only the term proportional to σ^2 in Eq. (29) needs renormalization.

In the calculation of the macroscopic free energy (or the effective potential, $\Omega = -T \log Z$), there are only two divergent perturbative graphs in three spatial dimensions, the ones shown in Fig. 2: the tadpole diagram (left graph), which is linearly divergent, and the setting-sun diagram (right graph), which is logarithmically divergent. The divergent parts of these graphs

can be calculated as a function of the lattice spacing – for an explicit and detailed derivation, see Ref. [13]. They are given in terms of the dimensionless quantities introduced above as [13, 24]:

$$\Omega_{div}(\sigma) = \frac{1}{2}\epsilon(\mu)\sigma^2, \tag{44}$$
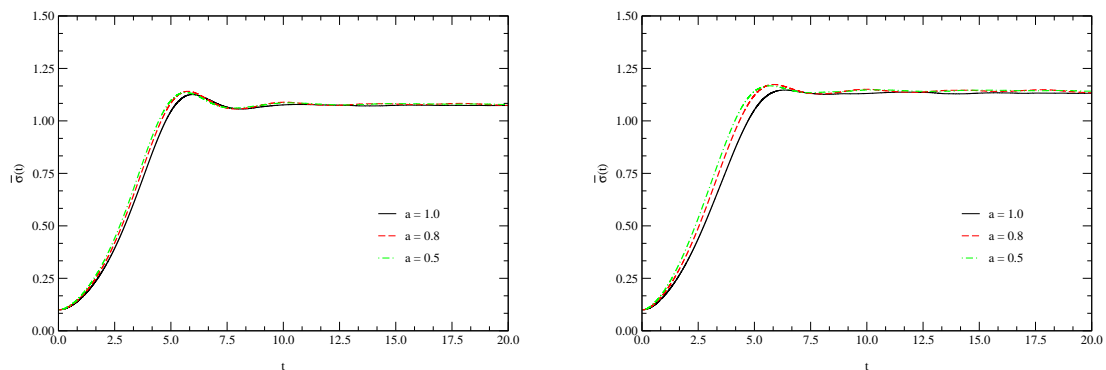
with

$$\epsilon(\mu) = 3\gamma\frac{\Sigma}{4\pi a} - 6\gamma^2 \left\{ \frac{1}{16\pi^2} \left[\log\left(\frac{6}{\mu a}\right) + 0.09 \right] \right\}, \tag{45}$$

where $\Sigma \simeq 3.1759$, and μ is a renormalization scale. Finite results are then obtained by subtracting this term from the original (rescaled) GLL equation; the parameter v^2 then becomes renormalized and μ dependent, it has to be fitted to a measurable quantity. Once it is fitted at some scale μ, its value at another scale μ' is given by renormalization group equations, as usual [10].
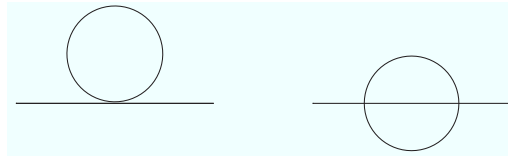
Figure 2. Divergent Feynman graphs that contribute to the effective potential in three spatial dimensions. Left graph is the tadpole and the right graph is the setting-sun.

In Fig. 3 we present the results with the above counter-term added to the GLL equation – we used the value $\mu = 1$, but the results are not much dependent on the value of μ, since the first term in Eq. (45) dominates the divergence. As seen, the subtraction leads to solutions that are independent of the lattice spacing within very good precision. Similar results are obtained for more intense noises, i.e. for larger values of γ.

Figure 3. Volume average of the noise averaged σ field as a function of the time, Eq. (43), obtained with the counter-term added to the GLL equation. The left panel is for $\gamma = 1$ and the right panel is for $\gamma = 2$.

5. Conclusions

In summary, we have discussed a simple and systematic method to renormalize GLL equations on a spatial lattice. Numerical results of simulations were presented for a GLL equation derived in the context of the linear σ model. The usage of the renormalization method when coupling the chiral equations to hydrodynamic fluid equations is straightforward. Work in this direction is in progress.

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