Journal of
Mathematical Physics

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Citation: J. Math. Phys. 53, 102901 (2012); doi: 10.1063/1.4748301
View online: http://dx.doi.org/10.1063/1.4748301
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v53/i10
Published by the AIP Publishing LLC.

JOURNAL OF MATHEMATICAL PHYSICS **53**, 102901 (2012)

Two-dimensional background field gravity: A Hamilton-Jacobi analysis

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(Received 30 May 2012; accepted 13 August 2012; published online 10 September 2012)

We analyse the constraint structure of the background field model for two-dimensional gravity via the Hamilton-Jacobi formalism. This analysis consists in finding the complete set of involutive Hamiltonians that assure the integrability of the system. We then calculate the characteristic equations of the system, also establishing the equivalence between these equations and the field equations. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4748301]

Dedicated to Professor H. G. Valqui on the occasion of his 80th birthday

I. INTRODUCTION

Until now, there is no satisfactory quantum theory for the four-dimensional gravity. In order to understand some properties of gravity at quantum level, it has been found at lower dimensional models of gravity interesting topics of research (see Ref. 1 and references therein). There are some trivialities on those models, for example, the three-dimensional pure gravity has no local degrees of freedom. For the two-dimensional case, it is well known that the Einstein tensor is identically zero, then Einstein's equations are always satisfied for pure gravity.

Because of this trivial behavior of two-dimensional gravity it is customary to make some reformulations in the Einstein-Hilbert action. One of the most used models is due to Jackiw and Teitelboim,[2] in which the action is given by

$$I_{JT} = \int d^2x \sqrt{g}\psi \left(R - k\right),$$

where ψ is a dilaton field used as a lagrangian multiplier, and k is the cosmological constant. The Euler-Lagrange (EL) equations for the Jackiw-Teitelboim (JT) model are given by

$$R - k = 0, \quad \nabla_\mu \nabla_\nu \psi + \frac{1}{2} k g_{\mu\nu} \psi = 0.$$

The first is the constant curvature equation, and the second is the equation of motion for the dilaton field which is determined without making further restrictions on the metric. For an extensive review of two-dimensional dilaton models of gravity, we refer to Ref. 3.

The JT model can be reformulated as an $SO(2, 1)$ gauge theory described by the action

$$I = \int_{\mathcal{M}} Tr(B \wedge F),$$

which is also known as the background field (BF) action[4] (we postponed the conventions for Sec. III). The Hamiltonian analysis of the BF model has been studied in Ref. 5 through Dirac's method. The

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aim of this article is to analyse the constraint structure of the BF model for two-dimensional gravity by means of the Hamilton-Jacobi (HJ) formalism.[6]

The HJ formalism is an alternative method to analyse constrained systems. It was born as a generalization of Carathéodory's work[7] on variational principles and first-order partial differential equations. In this formalism we obtain a set of Hamilton-Jacobi partial differential equations (HJPDEs) also known as Hamiltonian densities, and the integrability of this system is achieved if the Frobenius' integrability conditions (ICs) are satisfied. There are several applications and improvements of this formulation.[8] Concerning two-dimensional gravity, the HJ formalism has been used to study the Polyakov's model in the front-form dynamics,[9] the usual Einstein-Hilbert action,[10] and also the linearized gravity.[11]

The main advantages in using the HJ approach come from the fact that it provides a full theoretical approach, rather than a consistency construction, to the canonical formulation of singular systems. One of these advantages is the absence of the so-called Dirac's conjecture, which states that first-class constraints are generators of the gauge transformations. However, this conjecture is not generally valid as shown in Ref. 12. Moreover, at the classical level, gauge fixing is not required for equivalence between the canonical and lagrangian descriptions. The HJ formalism also provides a very natural description of systems with non-involutive (equivalent to second-class) constraints,[13] that leads to modified generalized brackets (in analogy with Dirac's brackets).

In Sec. II we briefly discuss the HJ formalism. In Sec. III we introduce the BF model. In Sec. IV we make the proper HJ analysis which consists in finding the complete set of involutive Hamiltonians of the theory. In Sec. V we compute the characteristic equations (CEs) and analyse the dynamical evolution along the independent parameters of the theory. Finally we discuss the results.

II. THE HAMILTON-JACOBI FORMALISM

Let us consider a system described by a Lagrangian density $L = L(x^i, \dot{x}^i, t)$, $i = 1, 2, \ldots, N$, with a singular Hessian matrix of rank $P \leq N$. In this case we separate the variables $x^i = (x^a, x^z)$, where $a = 1, 2, \ldots, P$ and $z = 1, 2, \ldots, R$ with $P + R = N$. The variables x^a are related to the inversible part of the Hessian, while x^z are related to its zero-modes. Following Carathéodory's variational approach, the necessary and sufficient condition for extremizing the action is given by $p_0 + p_a\dot{x}^a + p_z\dot{x}^z - L = 0$, where the canonical momenta are defined by $p_a = \partial S/\partial x^a$, $p_z = \partial S/\partial x^z$, and $p_0 = \partial S/\partial t$.

The singularity of the Hessian matrix assures that there are R canonical constraints $H'_z \equiv p_z - \partial L/\partial \dot{x}^z = 0$. This allows us to define the canonical Hamiltonian $H_0 = p_a\dot{x}^a + p_z\dot{x}^z - L$, such that we have a set of $R + 1$ first-order HJPDE

$$H'_\alpha \equiv p_\alpha - \partial L/\partial \dot{t}^\alpha = 0, \qquad \alpha = 0, 1, 2, ..R, \tag{1}$$

where $t^\alpha = (x^0 \equiv t, x^z)$. The Cauchy's method allows one to relate (1) to a set of total differential equations, the characteristic equations

$$dx^a = \frac{\partial H'_\alpha}{\partial p_a}dt^\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial x^a}dt^\alpha, \quad dS = p_a dx^a - H_\alpha dt^\alpha. \tag{2}$$

Solutions of the first two equations are curves of $R + 1$ parameters t^α on the phase space defined by the conjugated variables (x^a, p_a). From (2), and considering t^α as independent parameters, the evolution of any phase space function $F = F(x^a, t^\alpha, p_a, p_\alpha)$ is given by the fundamental differential

$$dF = \{F, H'_\alpha\}dt^\alpha, \tag{3}$$

where the Poisson brackets (PBs) are defined on the extended phase space of the variables $(x^a, t^\alpha, p_a, p_\alpha)$. Therefore, the canonical constraints H'_α play the role of generators of the dynamical evolution of the system: they are considered as the Hamiltonians of the system. Note that for a regular system, the CE (3) become Hamilton's equations.

A complete solution of the set of HJPDE is given by a family of surfaces orthogonal to the characteristic curves. The Frobenius' integrability condition assures the existence of this solution, as well as the independence between the parameters t^α, and it is expressed in terms of vector fields tangent to the family of surfaces. In terms of the Hamiltonians H'_α, which are generators of these vectors, the IC is written as

$$\{H'_\alpha, H'_\beta\} = C_{\alpha\beta}^{\ \ \gamma} H'_\gamma, \tag{4}$$

i.e., the Hamiltonians must close a Lie algebra with the PB. Hamiltonians that satisfy (4) are called involutive Hamiltonians. The presence of Hamiltonians that do not satisfy the above condition, the so called non-involutive Hamiltonians, indicates that the system of HJPDE is not complete, or the parameters are not independent. In this case, the equivalent IC $dH'_\alpha = 0$ should provide new constraints to the system, as well as indicate the dependence between the parameters.[13] The HJ constraint analysis is actually the search for a complete set of involutive Hamiltonians of the system.

It is possible that after a complete set of HJPDE is found, there remains a subset of non-involutive Hamiltonians H'_x. In this case we may proceed with the method outlined in Ref. 13, and build the matrix with elements $M_{xy} \equiv \{H'_x, H'_y\}$. This procedure redefines the dynamics by eliminating the parameters related to these Hamiltonians with the use of the generalized brackets (GBs). If the M matrix is singular of rank $K \leq R$, we have a regular submatrix $M_{\bar{a}\bar{b}}$, with $\bar{a} = 1, 2, \ldots K$, and we define the GB by

$$\{A, B\}^* = \{A, B\} - \{A, H'_{\bar{a}}\}(M_{\bar{a}\bar{b}})^{-1}\{H'_{\bar{b}}, B\}. \tag{5}$$

We can write the fundamental differential as

$$dF = \{F, H'_{\bar{\alpha}}\}^* dt^{\bar{\alpha}}, \qquad \bar{\alpha} = 0, K+1, \ldots, R. \tag{6}$$

After the procedure of finding possible new Hamiltonians and eliminating possible dependence between the parameters, the Hamiltonians $H'_{\bar{\alpha}}$ become the complete set of involutive Hamiltonians of the system, this time with the PB substituted by the GB.

III. THE BF MODEL

Let us consider a gauge group G acting on fields of a two-dimensional manifold $\mathcal{M}$. The BF theory consists on the gauge connection 1-form A and a scalar field B, also called background field (B-field), whose action is given by

$$I_{BF}[B, A] = \int_{\mathcal{M}} Tr(B \wedge F). \tag{7}$$

The trace is taken on the adjoint representation of G, and the field strength F is related to the gauge connection 1-form A by the covariant exterior derivative $F = DA = dA + A \wedge A$. The corresponding EL equations are

$$F = 0, \quad DB = 0. \tag{8}$$

The interpretation of these equations is straightforward: we have a flat connection A, since it has zero field strength, while the B field is parallel to A. With this machinery we may proceed to make a model for the gauge field of the Poincaré group $ISO(1; 1)$, in which we write the connection A as a combination of the 1-form zweibein e^I and the 1-form spin connection ω

$$A = e^I P_I + \omega \Lambda,$$

where $I = 0, 1$, and P_I and Λ are the generators of translations and Lorentz boost, respectively. The zweibein and the spin connection are considered independent, and the Poincaré algebra is given by

$$[\Lambda, P_I] = \epsilon_I^{\ J} P_J, \quad [P_I, P_J] = 0, \tag{9}$$

the Levi-Civita symbol is defined such that $\epsilon_{01} = 1$.

The Killing metric is defined by these generators as

$$g_{ab} \equiv Tr(J_a J_b),$$

with $J_0 = P_0$, $J_1 = P_1$, and $J_2 = \Lambda$. In the two-dimensional case, the Killing metric is degenerated, then it is not possible to build a consistent gauge theory. To solve this problem the presence of a non-vanishing cosmological constant k is needed, so we can deform the Poincaré algebra (9) to the (anti) de Sitter algebra

$$[\Lambda, P_I] = \epsilon_I{}^J P_J, \quad [P_I, P_J] = k\epsilon_{IJ}\Lambda. \tag{10}$$

In this case the Killing metric results to be invariant and non-degenerate:

$$g_{ab} = \begin{pmatrix} k\eta_{IJ} & 0 \\ 0 & 1 \end{pmatrix}.$$

The (anti) de Sitter algebra is expressed in terms of the generators J_a by $[J_a, J_b] = f_{ab}{}^c J_c$, where $f_{ab}{}^c \equiv \varepsilon_{abc} g^{cd}$, and $\varepsilon_{012} = 1$.

In terms of the zweibein and the spin connection we have the field strength

$$F = \left(de^I - \omega\epsilon^I{}_J \wedge e^J\right) P_I + \left(d\omega + \frac{k}{2}\epsilon_{IJ} e^I \wedge e^J\right)\Lambda \equiv T^I P_I + \mathcal{R}\Lambda,$$

where T^I and $\mathcal{R}$ represent the torsion and curvature 2-forms of the zweibein field in the first-order formalism. The flat connection equation $F = 0$ becomes

$$de^I - \omega\epsilon^I{}_J \wedge e^J = 0, \tag{11}$$

$$d\omega + \frac{k}{2}\epsilon_{IJ} e^I \wedge e^J = 0. \tag{12}$$

Equation (11) is the torsion-free equation $T^I = 0$, and along with (12), and considering an inversible zweibein, it is possible to compute the spin connection in terms of e^I. In this case Eq. (12) becomes the equation of constant Ricci's scalar curvature $R = k$. This is the standard procedure to show the equivalence between the two-dimensional BF gravity and the JT model (see Refs. 4 and 14).

IV. THE HAMILTON-JACOBI ANALYSIS

Instead of using the differential forms, it is preferred to use its components $A = A^a_\mu J_a dx^\mu$ and $B = B^a J_a$, so the action can be written as

$$I_{BF}[B, A] = \int dx^2 B_a(\partial_0 A^a_1 - \partial_1 A^a_0 + f_{bc}{}^a A^b_0 A^c_1). \tag{13}$$

In terms of the components, the flat connection equation becomes

$$\partial_0 A^a_1 - \partial_1 A^a_0 + f_{bc}{}^a A^b_0 A^c_1 = 0, \tag{14}$$

while the equation for the B-field becomes

$$\delta^\mu_0 D_1 B_a - \delta^\mu_1 D_0 B_a = 0, \tag{15}$$

where $D_\mu B_a \equiv \partial_\mu B_a + f_{ab}{}^c A^b_\mu B_c$. It is clear that (15) is equivalent to $D_\mu B_a = 0$.

It is well known that the elimination of the divergence terms in the Lagrangian density does not modify the EL equations, but it changes the functional form of the canonical momenta. Therefore, we consider the equivalent action

$$I_{BF}[B, A] = \int d^2x\mathcal{L} = \int d^2x \left(B_a \partial_0 A^a_1 + A^a_0 D_1 B_a\right). \tag{16}$$

Considering A^a_μ and B_a as the variables of the configuration space of the theory, we identify the canonical conjugated momenta

$$\pi^\mu_a \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 A^a_\mu)} = \delta^\mu_1 B_a, \qquad \Pi^a \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 B_a)} = 0,$$

as their respective canonical constraints. The canonical Hamiltonian density $\mathcal{H}_0 \equiv \pi^\mu_a \partial_0 A^a_\mu + \Pi^a \partial_0 B_a - \mathcal{L}$ is then given by

$$\mathcal{H}_0 = -A^a_0 D_1 B_a. \tag{17}$$

We have then the set of HJPDE

$$\mathcal{H}' \equiv \pi + \mathcal{H}_0 = 0, \quad \to x^0 \equiv t, \tag{18}$$

$$\mathcal{H}_a'^0 \equiv \pi_a^0 = 0 \quad \to A_0^a \equiv \lambda_0^a, \tag{19}$$

$$\mathcal{H}_a'^1 \equiv \pi_a^1 - B_a = 0 \quad \to A_1^a \equiv \lambda_1^a, \tag{20}$$

$$\mathcal{H}'^a \equiv \Pi^a = 0 \quad \to B_a \equiv \epsilon_a. \tag{21}$$

The left hand sides of these equations are the Hamiltonian densities for which we have related the set of parameters $(x^0, \lambda_\mu^a, \epsilon_a)$. The fundamental PBs of the theory are given by

$$\{A_\mu^a(x), \pi_b^\nu(y)\} = \delta_b^a \delta_\nu^\mu \delta(x-y), \qquad \{B_a(x), \Pi^b(y)\} = \delta_b^a \delta(x-y), \tag{22}$$

and the fundamental differential,

$$dF = \int dy \left[\{F, \mathcal{H}'(y)\} dt + \{F, \mathcal{H}_a'^\mu(y)\} d\lambda_\mu^a(y) + \{F, \mathcal{H}'^a(y)\} d\epsilon_a(y) \right]. \tag{23}$$

The fundamental differential is used to test the integrability of the system in order to find other possible Hamiltonian densities to complete the system (18). We may proceed in another way: we analyse the algebra of the previous Hamiltonian densities, build the GB, and then find the remaining constraints. For this particular case, the this method simplifies calculations and clarifies certain aspects of the formalism.

Notice that the subset of Hamiltonian densities $h^0 \equiv \mathcal{H}'^a$ and $h^1 \equiv \mathcal{H}_a'^1$ is not involutive, then we build the matrix $M(x, y)$ with components

$$M^{rs}(x, y) \equiv \{h^r(x), h^s(y)\} = \begin{pmatrix} 0_{3\times 3} & 1_{3\times 3} \\ -1_{3\times 3} & 0_{3\times 3} \end{pmatrix} \delta(x-y),$$

where $r, s = 0, 1$. The inverse matrix $M^{-1}(x, y)$ is given by

$$M_{rs}^{-1}(x, y) = \begin{pmatrix} 0_{3\times 3} & -1_{3\times 3} \\ 1_{3\times 3} & 0_{3\times 3} \end{pmatrix} \delta(x-y).$$

With this matrix, we eliminate the parameters $\left(\lambda_1^a, \epsilon_a\right)$ related to the non-involutive Hamiltonian densities by building the GB

$$\{F(x), G(y)\}^* = \{F(x), G(y)\} - \int dz dw \{F(x), h^r(z)\} M_{rs}^{-1}(z, w) \{h^s(w), G(y)\}. \tag{24}$$

The only non-zero GBs are given by

$$\{A_\mu^a, \pi_b^\nu\}^* = \delta_b^a \delta_\nu^\mu \delta(x-y), \qquad \{A_\mu^a(x), B_b(y)\}^* = \delta_b^a \delta_\mu^1 \delta(x-y). \tag{25}$$

The reduction of the phase space is now evident, since B_b and Π^b are not canonical conjugated anymore. In fact, now B_b plays the role of π_b^1. It is still necessary to test the integrability of the Hamiltonian $\mathcal{H}_a'^0$. For this purpose we notice that after the construction of the GB the fundamental differential is now given by

$$dF = \int dy \left[\{F, \mathcal{H}'\}^* dt + \{F, \mathcal{H}_a'^0\}^* d\lambda^a \right],$$

where we have renamed $\lambda^a \equiv \lambda_0^a$.

Since $\{\mathcal{H}_a'^0, \mathcal{H}'\}^* = -D_1 B_a$, to achieve integrability we have to consider the new Hamiltonian density

$$\mathcal{C}_a' \equiv D_1 B_a = 0. \tag{26}$$

It is straightforward to see that the IC for $\mathcal{C}'_a$ is satisfied: they are in involution with the other Hamiltonians under the GB. In particular, we have the algebra

$$\{\mathcal{C}'_a(x), \mathcal{C}'_b(y)\}^* = f_{ab}{}^{c}\mathcal{C}'_c(x)\delta(x-y), \tag{27}$$

and the IC programme is closed for this system.

V. THE CHARACTERISTIC EQUATIONS

In the HJ description, we consider the complete set of involutive Hamiltonians $\mathcal{H}'$, $\mathcal{H}'^0_a$, and $\mathcal{C}'_a$ as generators of the dynamical evolution of the system. Note that the functions $\mathcal{H}'$ and $\mathcal{H}'^0_a$ are related to the parameters t and λ^a, respectively. However, $\mathcal{C}'_a$ is not related to any variable of the theory. To consider this Hamiltonian as a generator we expand the phase-space with a new set of variables ω^a. The fundamental differential of the system is then given by

$$dF = \int dy\left[\{F, \mathcal{H}'(y)\}^* dt + \{F, \mathcal{H}'^0_a(y)\}^* d\lambda^a(y) + \{F, \mathcal{C}'_a(y)\}^* d\omega^a(y)\right]. \tag{28}$$

From (28) we obtain the CE

$$dB_a = -f_{ab}{}^{c}B_c\left[A^b_0 dt - d\omega^b\right], \tag{29}$$

$$d\Pi^a = 0, \tag{30}$$

and

$$dA^a_\mu = \delta^0_\mu d\lambda^a + \delta^1_\mu\left[D_1 A^a_0 dt - D_1 d\omega^a\right], \tag{31}$$

$$d\pi^\mu_a = \delta^0_\mu D_1 B_a dt - \delta^1_\mu f_{ab}{}^{c}B_c\left[A^b_0 dt - d\omega^b\right]. \tag{32}$$

These equations should be equivalent to the EL equations (14) and (15). Because of the presence of the parameters ω^a, we need to clarify under what conditions the equivalence of these equations are valid.

The integrability assures independence between the parameters (t, λ^a, ω^a), so the dynamics in the direction of any parameter is independent of the others. In this case, we have that temporal evolution alone gives

$$D_0 B_a = 0, \tag{33}$$

$$\partial_0 \Pi^a = 0, \tag{34}$$

and

$$\partial_0 A^a_\mu = \delta^1_\mu D_1 A^a_0, \tag{35}$$

$$\partial_0 \pi^\mu_a = \delta^0_\mu D_1 B_a - \delta^1_\mu f_{ab}{}^{c}B_c A^b_0. \tag{36}$$

We see that (33) is one of the EL equations for the B field (15). Equation (34) represents the IC for the Hamiltonian (21), note that this is related to the fact that the B_a fields serve as Lagrange multipliers in the action. The component $\mu = 0$ of (35) states that A^a_0 has no dynamics, while the component $\mu = 1$ is the EL equation (14). For $\mu = 0$, (36) yields $\partial_0\pi^0_a = D_1 B_a = \mathcal{C}'_a$, which is zero by imposition of integrability. Therefore, this equation reproduces the EL equation $D_1 B_a = 0$. At last, for $\mu = 1$ (36) gives

$$\partial_0 \pi^1_a + f_{ab}{}^{c}B_c A^b_0 = 0.$$

Using (33) it becomes $\partial_0 \pi^1_a = \partial_0 B_a$, which corroborates the interpretation of B_a as a conjugate momenta related to A^a_1. Therefore, we showed that the time evolution of the characteristic equations (33)–(35) is completely equivalent to the EL equations (14) and (15).

The evolution in the direction of the parameters (λ^a, ω^a), on the other hand, gives the following set of infinitesimal transformations:

$$\delta A^a_\mu = \{A^a_\mu, \mathcal{H}'^0_b\}^* \delta\lambda^b + \{A^a_\mu, \mathcal{C}'_b\}^* \delta\omega^b = \delta^0_\mu \delta\lambda^a - \delta^1_\mu D_1 \delta\omega^a, \tag{37}$$

$$\delta B_a = \{B_a, \mathcal{C}'_b\}^* \delta\omega^b = f_{ab}{}^c B_c \delta\omega^b. \tag{38}$$

These variations left invariant the action (13) if we choose

$$\delta\omega^a = -\xi^a, \quad \delta\lambda^a = D_0 \xi^a, \tag{39}$$

in other words, the BF theory is invariant under the gauge transformations

$$\delta A^a_\mu = D_\mu \xi^a, \tag{40}$$

$$\delta B_a = f_{ab}{}^c B_c \xi^b. \tag{41}$$

Since the set of involutive Hamiltonians $\left(\mathcal{H}'^0_b, \mathcal{C}'_b\right)$ generates the transformation (37), we have that the generator of the gauge transformation is given by

$$G = \int dy \left[\mathcal{H}'^0_a(y) D_0 \xi^a(y) - \mathcal{C}'_a(y) \xi^a(y)\right], \tag{42}$$

and it can be directly verified that

$$\begin{aligned} \{B_a, G\}^* &= -f_{ab}{}^c B_c \delta\omega^b = f_{ab}{}^c B_c \xi^b, \\ \left\{A^a_\mu, G\right\}^* &= \delta^0_\mu \delta\dot{\omega}^a - \delta^1_\mu D_1 \delta\omega^a = D_\mu \xi^a. \end{aligned}$$

VI. FINAL REMARKS

In this work we have dealt with the Hamilton-Jacobi constraint structure of the two dimensional BF gravity. The Frobenius' integrability conditions are a cornerstone of the mathematical structure of this formalism. We saw in Sec. IV that the Poisson algebra of the canonical constraints of the BF model allowed the construction of the GB, due to the presence of the two non-involutive Hamiltonian densities $\left(\mathcal{H}'^1_a, \mathcal{H}'^a\right)$. We built the GB, and by proceeding with the integrability, we obtained the set $\left(\mathcal{H}', \mathcal{H}'^0_a, \mathcal{C}'_a\right)$ as the complete set of involutive Hamiltonian densities of the theory.

These results can be compared with the ones obtained in Ref. 5. The set of involutive Hamiltonians match with their set of first-class constraints. In particular, the generators $\mathcal{C}'_a$ are the first-class secondary constraints found by Constantinidis *et al.* This is an expected feature, since the gauge character of the theory cannot be changed under a distinct formalism. However, Ref. 5 does not present the equivalent non-involutive (second-class) Hamiltonians $\left(\mathcal{H}'^1_a, \mathcal{H}'^a\right)$, and therefore the use of the GB (24) has no analogy in their analysis in terms of Dirac's brackets. This is so because Ref. 5 does not consider the B-field as a variable of the formalism, and then they had to ignore the constraint $\mathcal{H}'^1_a = 0$ as a proper primary constraint for the sake of consistency.

We built the fundamental differential (28), from where we obtained the characteristic equations (29)–(31) of the system. These equations depend explicitly of the parameter ω^a related to the Hamiltonian $\mathcal{C}'_a$, and this dependence does not spoil the equivalence between the CE and the EL equations of the system. This is so because integrability implies that the parameters must be independent, therefore time evolution is independent of the dynamics of the other parameters.

We notice that all the involutive Hamiltonians contribute to the construction of the gauge generator (42), and now we can count the degrees of freedom of this theory. The dimension of the phase-space of the BF model is 18, 6 of these dimensions are related to the (B_a, Π^a) canonical variables, and 12 are related to $\left(A_{\mu}^{a}, \pi_{a}^{\nu}\right)$. The construction of the GB reduces the phase-space since we identify the variable B_a with the momenta π_a^1. The momenta Π^a are eliminated by the GB. Our reduced phase-space has now the dimension of 12. On the other hand, we have six generators of gauge transformation, three for each involutive Hamiltonian. These generators reduce the number of dynamical variables in 12. As a result we have zero degrees of freedom for BF gravity.

ACKNOWLEDGMENTS

The authors would like to thank the referee for useful comments. M.C.B. was supported by FAPESP. B.M.P. was partially supported by CNPq and CAPES. C.E.V. was supported by CAPES.

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