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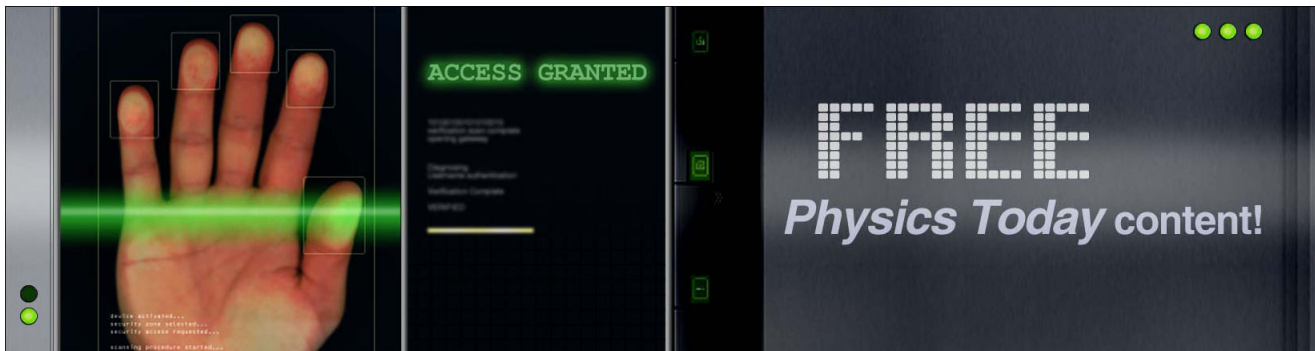
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## ADVERTISEMENT



## Shrimp-shape domains in a dissipative kicked rotator

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Some dynamical properties for a dissipative kicked rotator are studied. Our results show that when dissipation is taken into account a drastic change happens in the structure of the phase space in the sense that the mixed structure is modified and attracting fixed points and chaotic attractors are observed. A detailed numerical investigation in a two-dimensional parameter space based on the behavior of the Lyapunov exponent is considered. Our results show the existence of infinite self-similar shrimp-shaped structures corresponding to periodic attractors, embedded in a large region corresponding to the chaotic regime. © 2011 American Institute of Physics.

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**A strongly dissipative kicked rotator is studied. The investigation is based mainly on the asymptotic behavior of the Lyapunov exponents. Our results show that by reducing the two-dimensional map to a one-dimensional map in the limit of infinite kicks, the Feigenbaum's  $\delta$  is recovered. An investigation in the parameter space (the dissipation parameter and the kicking parameter) reveals the existence of infinite families of self-similar structures of shrimp-shape embedded in a large region corresponding to the chaotic regime. The organization of stability shrimps reported here for the discrete-time kicked rotator agrees well with the parameter space organization reported recently in the literature for several flows (continuous-time systems).**

### I. INTRODUCTION

Studies in dissipative systems have attracted much attention during the last decades, and it has been used in various fields of science including optics,<sup>1,2</sup> turbulence and fluid dynamics,<sup>3,4</sup> nanotechnology,<sup>5,6</sup> atomic and molecular physics,<sup>7,8</sup> and quantum and relativistic systems.<sup>9,10</sup> One system of special importance for its large applicability, which has been the subject of extensive research, is the standard map. Proposed originally by Chirikov<sup>11,12</sup> in 1969, the standard map is a dynamical system which describes the motion of a kicked rotator. Since the pioneering paper of 1969, the standard map has been applied in many different fields of science including solid state physics,<sup>13</sup> statistical mechanics,<sup>14</sup> accelerator physics,<sup>15</sup> problems of quantum mechanics and quantum chaos,<sup>16,17</sup> plasma physics,<sup>18</sup> ratchet transport,<sup>19</sup> and many others.

The dynamics of the kicked rotator is controlled by the kicking parameter  $K$  and in the absence of dissipative forces; if  $K$  is small enough, the structure of the phase space is mixed<sup>20–34</sup> in the sense that Kolmogorov-Arnold-Moser (KAM) invariant tori and regular islands are observed coexisting with chaotic seas. As the parameter  $K$  increases and becomes larger than  $K_c \approx 0.971635\dots$ , the last invariant spanning curve disappears and the system presents a globally chaotic component in the sense that a chaotic orbit can spread over the phase space. However, the introduction of dissipation in the model changes completely the mixed structure and the system exhibits attractors.<sup>35–37</sup> In the regime of strong dissipation, the model exhibits a period doubling bifurcation cascade, and the so called Feigenbaum's  $\delta$ , which is the rate of the bifurcations, can be obtained numerically. On the other hand, when weak dissipation is taken into account, a drastic change occurs in the behavior of the average energy. The unlimited energy growth present in the Hamiltonian case<sup>38</sup> for the case  $K \gg K_c$  is no longer observed. The average action exhibits a characteristic saturation value which can be described using scaling arguments. Additionally, such a behavior can be described remarkably well by an empirical universal function of the type  $f(x) = x^\beta/(1+x)^\beta$ , where  $\beta$  is the acceleration exponent. Such a function can also be applied to many dissipative systems.<sup>39–42</sup>

In this paper, we will explore some properties of a dissipative standard map seeking to understand and describe its parameter space. We specifically take into account its two-dimensional parameter space, namely, the dissipation parameter  $\gamma$  and the amplitude of the kicks  $K$ . We revisit the parameter space of the dissipative standard map where the existence of self-similar structures called *shrimps* was shown.<sup>44</sup> According to Ref. 43, “*Shrimps are formed by a regular set of adjacent windows centered around the main pair of intersecting superstable parabolic arcs. A shrimp is a doubly infinite mosaic of stability domains composed by*

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an innermost main domain plus all the adjacent stability domains arising from two period-doubling cascades together with their corresponding domains of chaos. Shrimps should not be confused with their innermost main domain of periodicity.” Results by Gaspard *et al.*<sup>45</sup> in 1984, Rössler *et al.*<sup>46</sup> in 1989, and Komuro *et al.*<sup>47</sup> in 1991 already demonstrated the existence of self-similar periodic structures in a 2D-mapping of the Chua’s system, the logistic map, and the mapping of the double scroll circuit, respectively. However, it was after the pioneering paper of Gallas<sup>48</sup> in 1993 studying the parameter space of the Hénon map that the parameter space attracted much attention, and since then, it has already been shown that such shrimp-shaped domains can be found in many theoretical models.<sup>49–58</sup> Very recently, they were also observed experimentally in a circuit of the Nishio-Inaba family.<sup>59</sup> A recent result from one of the pioneers of chaos theory,<sup>60</sup> was also devoted to this intriguing and rich parameter space structures. Here, in order to classify regions in the parameter space with regular or chaotic behavior, we use as a tool the Lyapunov exponent. We adopt the following procedure: starting with a fixed initial condition, after a long transient, the Lyapunov exponent is obtained and, for each combination of  $(K, \gamma)$ , a color is attributed. After that we give an increment in the parameters. We use the last value obtained for the dynamical variables  $(I, \theta)$  before the increment, as the new initial condition after the increment. This ensures that we are always in the basin of the same attractor. These self-similar very well organized structures of shrimp-shape are shown embedded into a large region corresponding to chaotic attractors.

The paper is organized as follows. In Sec. II, we describe all the necessary details to obtain the two-dimensional map that describes the dynamics of the system and also we present and discuss our numerical results. Conclusions are drawn in Sec. III.

## II. THE MODEL AND THE NUMERICAL RESULTS

The Hamiltonian that describes the dynamics of the kicked rotator has the following form:<sup>61,62</sup>

$$H(I, \theta, t) = \frac{I^2}{2} + K \cos(\theta) \sum_{n=-\infty}^{\infty} \delta(t - n), \quad (1)$$

where  $I$  and  $\theta$  are the action and angle variables, respectively.  $K$  is the amplitude of the delta-function pulses (kicks), the kicking parameter. The equations of motion can be easily found and are given by

$$\dot{I} = K \sin(\theta) \sum_{n=-\infty}^{\infty} \delta(t - n), \quad \dot{\theta} = I. \quad (2)$$

Assuming that  $(I_n, \theta_n)$  are the values of the variables just before the  $(n + 1)^{\text{th}}$  kick,  $(I_{n+1}, \theta_{n+1})$  represent their values just before the  $(n + 2)^{\text{th}}$  kick, and introducing a dissipative parameter  $\gamma$ , the dynamics of a dissipative kicked rotator is described by the following two-dimensional nonlinear map:

$$S : \begin{cases} I_{n+1} = (1 - \gamma)I_n + K \sin(\theta_n) \\ \theta_{n+1} = [\theta_n + I_{n+1}] \text{mod}(2\pi) \end{cases}, \quad (3)$$

where  $\gamma \in [0, 1]$  is the dissipation parameter. When dissipation is taken into account the structure of the phase space is changed. Then, an elliptical fixed point (generally surrounded by KAM islands) turns into a sink. Regions of the chaotic sea might be replaced by chaotic attractors. Figure 1(a) shows the structure of the phase space for the conservative dynamics ( $\gamma = 0$ ) with  $K = 1$ . As it is well known for such a value of the kicking parameter  $K$ , the last invariant torus is destroyed and the phase space has one large chaotic sea, the nested structures of thin chaotic layers, and KAM islands. Figure 1(b) shows the basin of attraction where the main fixed points are of period 1 (red and black), 2 (cyan), 3 (maroon), and 4 (green). The dissipation parameter considered is  $\gamma = 10^{-2}$ . The procedure used to construct the basin of attraction is to divide both  $I \in [0, 2\pi]$  and  $\theta \in [0, 2\pi]$  into grids of 1000 parts each, thus leading to a total of  $10^6$  different initial conditions. Each initial condition is iterated up to  $n = 5 \times 10^5$ . One can see that many periodic attractors emerge for such a choice of control parameters. It is important to stress that other attractors could in principle exist. If they exist, however, their basins of attraction are too small to be observed.

We now consider the regime of strong dissipation. It corresponds to the case where the action  $I$  loses more than 70% of its value upon a kick. We considered the case of

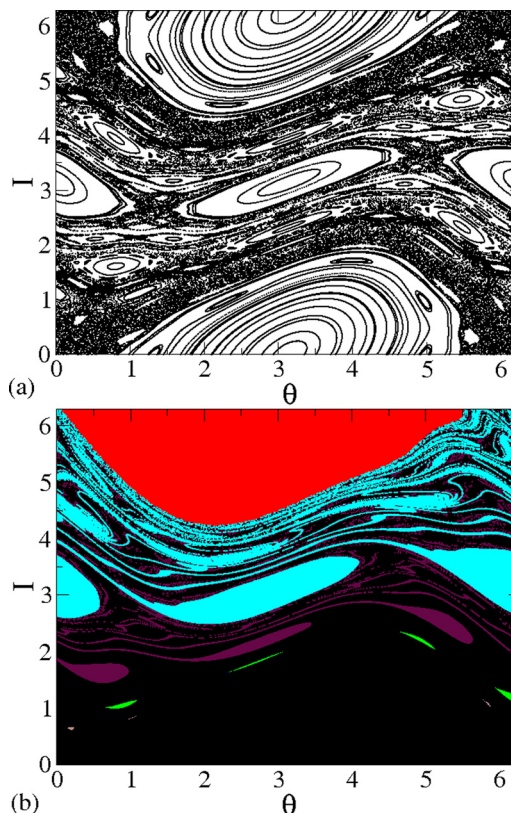


FIG. 1. (Color online) (a) Phase space for the conservative standard map ( $\gamma = 0$ ) with  $K = 1$ . (b) Basin of attraction for the attracting fixed points (sinks) of period 1 (red and black), 2 (cyan), 3 (maroon), and 4 (green). The control parameters used to construct the basin of attraction were  $K = 1$  and  $\gamma = 10^{-2}$ .



$\gamma = 0.80$ . To explore some typical behavior, we have used the initial conditions  $(\theta_0, I_0) = (3, 5.53)$  and investigated its attraction to periodic orbits, and looked at the bifurcations as  $K$  varies in the range where global chaos is observed in the non-dissipative regime, i.e.,  $\gamma = 0$ . Figure 2(a) shows the behavior of the asymptotic action plotted against the control parameter  $K$ , where a sequence of period doubling bifurcations is evident.<sup>65-67</sup> The bifurcations observed in (a) are marked by the vanishing Lyapunov exponent at the same control parameter  $K$  as shown in Fig. 2(b). As discussed by Eckmann and Ruelle,<sup>64</sup> the Lyapunov exponents are defined as

$$\lambda_j = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |\Lambda_j|, \quad j = 1, 2, \quad (4)$$

where  $\Lambda_j$  are the eigenvalues of  $M = \prod_{i=1}^n J_i(\theta_i, I_i)$  and  $J_i$  is the Jacobian matrix evaluated over the orbit  $(\theta_i, I_i)$ . However, a direct implementation of a computational algorithm to evaluate Eq. (4) has a severe limitation to obtain  $M$ . For the limit of short  $n$ , the components of  $M$  can assume different orders of magnitude for chaotic orbits and periodic attractors, making the implementation of the algorithm impracticable. To avoid such a problem,  $J$  can be written as  $J = \Theta T$ , where  $\Theta$  is an orthogonal matrix and  $T$  is a right up triangular matrix.  $M$  is rewritten as  $M = J_n J_{n-1} \dots J_2 \Theta_1 \Theta_1^{-1} J_1$ , where  $T_1 = \Theta_1^{-1} J_1$ . A product of  $J_2 \Theta_1$  defines a new  $J'_2$ . In a next step, one can show that  $M = J_n J_{n-1} \dots J_3 \Theta_2 \Theta_2^{-1} J'_2 T_1$ . The same procedure can be used to obtain  $T_2 = \Theta_2^{-1} J'_2$  and so on. Using this procedure, the problem is reduced to evaluate the

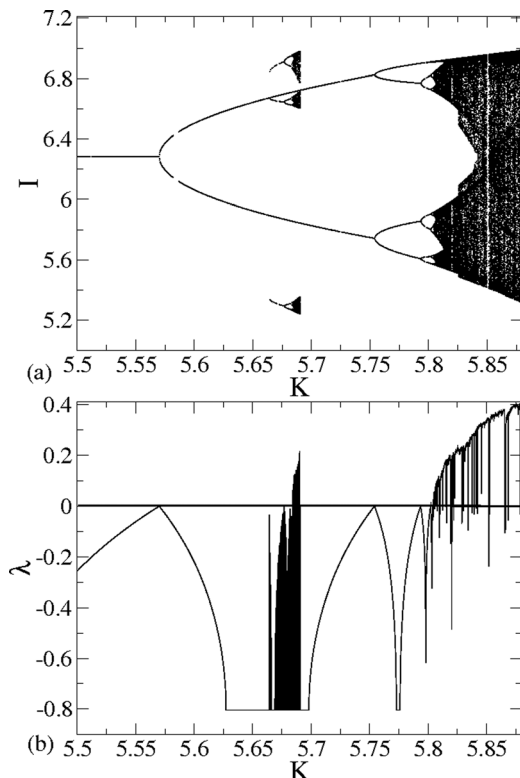


FIG. 2. Bifurcation cascade for (a)  $I \times K$ ; (b) the Lyapunov exponent associated to (a). The damping coefficient used was  $\gamma = 0.80$ .

TABLE I. The value of  $n$ , the period of the bifurcation, the values of the parameter  $K$  where the bifurcation happen, and the convergence of the Feigenbaum's  $\delta$  considering bifurcations up to the eleventh order.

$n$	Period	$K$	$\delta$
1	2	5.57011554475050	—
2	4	5.75421890196820	4.66723703234012
3	8	5.79366480024635	4.60078306721265
4	16	5.80223853600000	4.65331150449313
5	32	5.80408103800000	4.66432491229193
6	64	5.80447605808000	4.66821746028430
7	128	5.80456067712000	4.66893856195158
8	256	5.80457880094812	4.66910749234449
9	512	5.80458268259534	4.66917075858837
10	1024	5.80458351393080	4.66920050635855
11	2048	5.80458369197744	—

diagonal elements of  $T_i : T_{11}^i, T_{22}^i$ . Finally, the Lyapunov exponents are given by

$$\lambda_j = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln |T_{jj}^i|, \quad j = 1, 2. \quad (5)$$

If at least one of the  $\lambda_j$  is positive then the orbit is said to be chaotic. Figure 2(b) shows the behavior of the Lyapunov exponents corresponding to Fig. 2(a). One can see also that

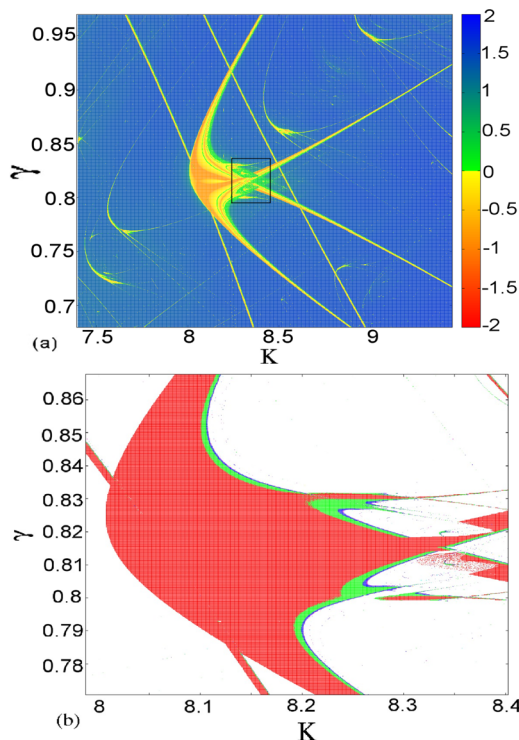


FIG. 3. (Color online) (a) Phase diagram of  $K$  vs.  $\gamma$  where the regular structure of shrimp-shape is shown. The color scale corresponds to the Lyapunov exponent for a given combination of  $(K, \gamma)$ . Regular regions are shown in a red-yellow scale, while chaotic behavior is shown in a green-blue scale. (b) Magnification of the main structure in (a) where white indicates chaos (positive Lyapunov exponent) and periodic solution (negative Lyapunov exponent) is shown in colors, each color indicates a given period, namely, red correspond to period 4, green period 8, and blue period 16 and larger periods are no longer visible. The color coding runs between the maximum and minimum values of the entire plot and its numerical value is indicated.



when the bifurcations occur, the exponent  $\lambda$  vanishes. The Lyapunov exponents between  $5.66 < K < 5.69$  correspond to the small sequence of bifurcations observed in Fig. 2(a) for the same range of the control parameter  $K$ . Feigenbaum<sup>68,69</sup> observed that there is an universal feature along the bifurcations. The period doubling bifurcations converge geometrically to the chaos border at a constant rate  $\delta$ . The procedure used to obtain the Feigenbaum constant  $\delta$  is as follows: let  $K_1$  represent the control parameter value at which period-1 gives birth to a period-2 orbit,  $K_2$  is the value where period-2 changes to period-4, and so on. In general, the parameter  $K_n$  corresponds to the control parameter value at which a period- $2^n$  orbit is born. Thus, the Feigenbaum's  $\delta$  is written as

$$\delta = \lim_{n \rightarrow \infty} \frac{K_n - K_{n-1}}{K_{n+1} - K_n}. \quad (6)$$

The theoretical value for the Feigenbaum constant  $\delta$  is  $\delta = 4.669201609\dots$ . Considering the numerical data obtained through the Lyapunov exponents calculation, the Feigenbaum's  $\delta$  obtained for the dissipative standard map is

$\delta = 4.66920050635855\dots$  considering bifurcations up to 11th order (see Table I). Here, it is important to emphasize that such a constant is obtained for dissipative systems where the two-dimensional model is reduced to a one-dimensional as  $n \rightarrow \infty$ . Such a universal constant has been found numerically in other systems, namely, the logistic map<sup>70</sup> and the dissipative Fermi-Ulam model<sup>71</sup> just to mention two of them. On the other hand, if this "two-dimensional character is preserved," the universal constant may be different.<sup>72,73</sup>

To investigate the parameter space, we may change both the dissipation parameter  $\gamma$  and the intensity of the nonlinearity  $K$  (the kicking parameter) of the system (3). For each combination of them and after a long transient, the Lyapunov exponent is computed. Based on its value, a color is attributed to each combination of  $(K, \gamma)$ . Figure 3 shows the structure of the parameter space for the dissipative standard map where a shrimp shaped structure is evident. Such a region of the parameter space indeed agrees with Ref. 44. The procedure used to construct the figure was to divide both  $K \in [7.4, 9.45]$  and  $\gamma \in [0.68, 0.97]$  into windows of 2000 parts each, thus leading to a total of  $4 \times 10^6$  different initial

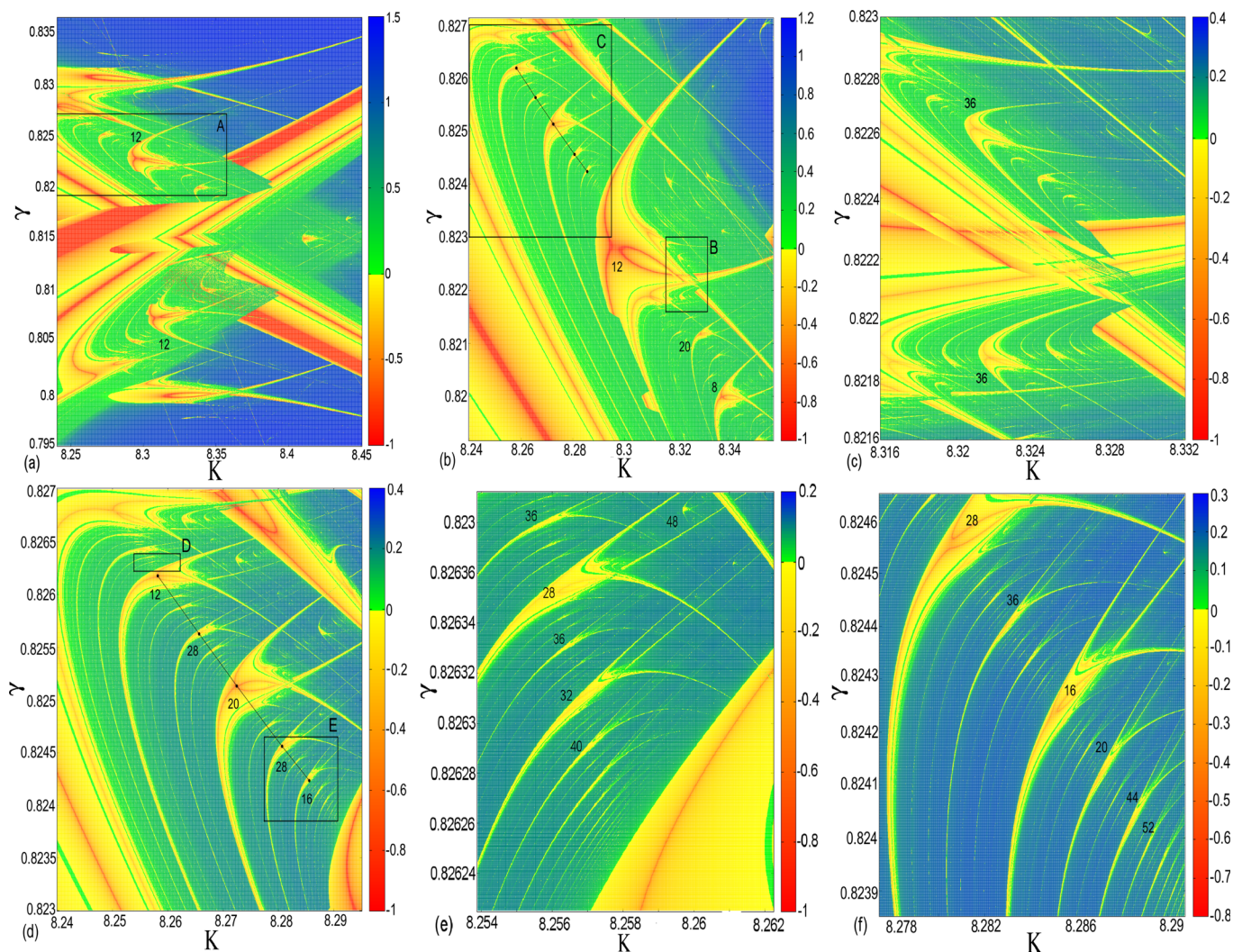


FIG. 4. (Color online) (a) Magnification of the box in Fig. 3(a); (b) magnification of the box A in Fig. 4(a); (c) and (d) are magnification of the boxes B and C in Fig. 4(b); and (e) and (f) are magnification of the boxes D and E in Fig. 4(d). Numbers represent the period of the main structure of each shrimp. The color coding runs between the maximum and minimum value of the entire plot, and its numerical value is indicated, which changes from plot to plot.

conditions. Starting with  $I_0 = 5.53$  and  $\theta_0 = 3$  as initial condition, for each increment in  $K$  and  $\gamma$ , we follow the attractor. This means that we have used the last value obtained for  $(\theta, I)$  before the increment, as the new initial condition after the increment. Using this approach, we lose the information about many other attractors, because it can happen that the chosen initial condition belongs to the basin of attraction of another attractor. In our simulations, we considered a transient of  $n = 1 \times 10^7$  iterations, and the Lyapunov exponent was computed for the next  $n = 1 \times 10^6$  iterations. The exponents are coded with a continuous color scale ranging from green-blue (positive exponents) to red-yellow (negative exponents). It is important to emphasize that the color scaling was changed from plot to plot. Figure 3(b) shows a magnification of the main structure in Fig. 3(a), and colors indicate the period. Each shrimp consists of a main body followed by an infinite sequence of bifurcations following the rule  $k \times 2^n$ , where  $k$  is the period of the main body. In the case of Fig. 3(b),  $k = 4$  which is shown in red, green corresponds to period 8, blue denotes period 16, and so on.

Figure 4 shows several figures with magnification of regions in the parameter space. Figure 4(a) shows an enlargement of the region inside the box of Fig. 3(a). As one can see, there are two small shrimps of period 12. Another magnification (inside the box A in Fig. 4(a)) reveals the existence of many other shrimps in Fig. 4(b). An enlargement of the region inside box B in Fig. 4(b) reveals the same structure observed in Fig. 4(a), which indicates that if infinite enlargements are made, such structures will have essentially the same shape, however, with different periods. Additionally, these structures seem to be organized in a very specific direction in the parameter space [see Fig. 4(d)], which can be approximated by the equation

$$K = 19.660 - 13.801\gamma. \quad (7)$$

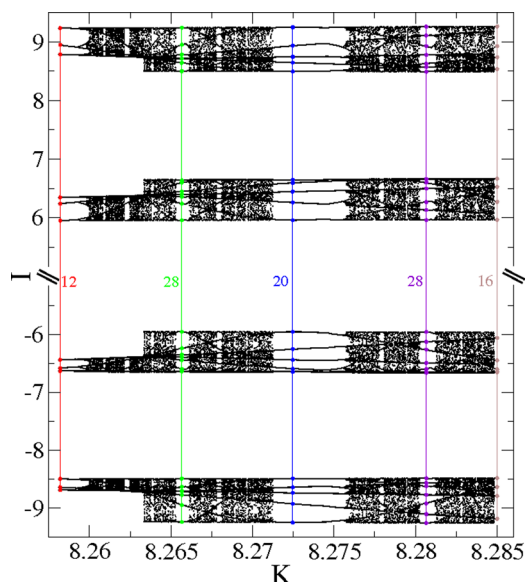


FIG. 5. (Color online) Bifurcation diagram changing simultaneously  $K$  and  $\gamma$  according to Eq. (7). Numbers represent the period of the shrimps shown in Fig. 4(d).

Figure 5 shows a bifurcation diagram obtained by turning  $K$  and  $\gamma$  simultaneously according to Eq. (7). As one can see, the diagram reflects very well the information obtained from Fig. 4(d) (alternation between chaotic behavior and regular regions) where Eq. (7) is fitted. Numbers represent the period ( $12 \rightarrow 28 \rightarrow 20 \rightarrow 28 \rightarrow 16$ ). However, an infinite family of shrimps of higher order does exist as can be seen in Figs. 4(e) and 4(f). The results presented here agree well with the results which have been recently presented in the literature for flows, i.e., for systems described by systems of differential equations.<sup>52–58</sup>

### III. CONCLUSIONS

Some results for a dissipative standard map have been addressed. When dissipation is taken into account, the mixed structure of the phase space is changed, the elliptic fixed points are replaced by attracting fixed points. For the regime of strong dissipation, the model exhibits a period doubling bifurcation cascade, where the Feigenbaum  $\delta = 4.66920050635855\dots$  was numerically obtained. Based on the Lyapunov exponent, we show that the parameter space  $(K, \gamma)$ , where  $\gamma$  is the dissipation parameter and  $K$  the kicking parameter, is very rich and exhibits infinite families of self-similar shrimp-shape structures, corresponding to periodic attractors, embedded in a large region corresponding to chaotic attractors. We observe that they are organized in a particular way described along the straight line according to  $K = 19.660 - 13.801\gamma$ , where the alternation between regular behavior and chaos has been shown in period-doubling bifurcation cascades.

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<sup>1</sup>I. R. Senitzky, *Phys. Rev. Lett.* **15**, 233 (1965).

<sup>2</sup>M. N. Shneider and P. F. Barker, *Phys. Rev. A* **71**, 053403 (2005).

<sup>3</sup>V. L'vov, A. Pomyalov, I. Procaccia, and V. Tiberkevich, *Phys. Rev. Lett.* **92**, 244503 (2004).

<sup>4</sup>P. Parmananda, M. Hildebrand, and M. Eiswirth, *Phys. Rev. E* **56**, 239 (1997).

<sup>5</sup>M. Steiner, M. Freitag, V. Perebeinos, J. C. Tsang, J. P. Small, M. Kinoshita, D. Yuan, J. Liu, and P. Avouris, *Nat. Nanotechnol.* **4**, 320 (2009).

<sup>6</sup>Y. Zhao, C. Ma, G. Chen, and Q. Jiang, *Phys. Rev. Lett.* **91**, 175504 (2003).

<sup>7</sup>G. Katz, M. A. Ratner, and R. Kosloff, *Phys. Rev. Lett.* **98**, 203006 (2007).

<sup>8</sup>D. J. Tannor and A. Bartana, *J. Phys. Chem. A* **103**, 10359 (1999).

<sup>9</sup>W. V. Liu and W. C. Schieve, *Phys. Rev. Lett.* **78**, 3278 (1997).

<sup>10</sup>H.-P. Breuer and F. Petruccione, *Phys. Rev. A* **55**, 3101 (1997).

<sup>11</sup>B. V. Chirikov, *Research Concerning the Theory of Nonlinear Resonance and Stochasticity*, Preprint No. 267, Institute of Nuclear Physics, Novosibirsk, 1969.

<sup>12</sup>B. V. Chirikov, *Phys. Rep.* **52**, 263 (1979).

<sup>13</sup>F. M. Izraelev, *Physica D* **1**, 243 (1980).



- <sup>14</sup>T. H. Stix, *Phys. Rev. Lett.* **36**, 10 (1976).
- <sup>15</sup>H. L. Cycon, R. Froese, W. Kirsch, and B. Simon, *Schrödinger Operators* (Springer, Berlin, 1987).
- <sup>16</sup>G. Casati, I. Guarneri, J. Ford, and F. Vivaldi, *Phys. Rev. A* **34**, 1413 (1986).
- <sup>17</sup>F. L. Moore, J. C. Robinson, C. F. Bharucha, B. Sudaram, and M. G. Raizen, *Phys. Rev. Lett.* **75**, 4598 (1995).
- <sup>18</sup>S. Aubry, *Physica D* **7**, 240 (1983).
- <sup>19</sup>A. Celestino, C. Manchein, H. A. Albuquerque, and M. V. Beims, *Phys. Rev. Lett.* **106**, 234101 (2011).
- <sup>20</sup>S. O. Kamphorst and S. P. Carvalho, *Nonlinearity* **12**, 1363 (1999).
- <sup>21</sup>V. Lopac, I. Mrkonjić, N. Pavin, and D. Radić, *Physica D* **217**, 88 (2006).
- <sup>22</sup>D. F. M. Oliveira and E. D. Leonel, *Commun. Nonlinear Sci. Numer. Simul.* **15**, 1092 (2010).
- <sup>23</sup>M. V. Berry, *Eur. J. Phys.* **2**, 91 (1981).
- <sup>24</sup>M. Robnik, *J. Phys. A* **16**, 3971 (1983).
- <sup>25</sup>M. V. Berry and M. Robnik, *J. Phys. A* **17**, 2413 (1984).
- <sup>26</sup>M. Robnik and M. V. Berry, *J. Phys. A* **18**, 1361 (1985).
- <sup>27</sup>T. Prosen and M. Robnik, *J. Phys. A* **26**, 2371 (1993).
- <sup>28</sup>T. Prosen and M. Robnik, *J. Phys. A* **27**, 8059 (1994).
- <sup>29</sup>M. Robnik, *Nonlinear Phenom. in Complex Syst. (Minsk)* **1**, 1 (1998).
- <sup>30</sup>T. Prosen and M. Robnik, *J. Phys. A* **32**, 1863 (1999).
- <sup>31</sup>R. Markarian, S. O. Kamphorst, and S. P. de Carvalho, *Commun. Math. Phys.* **174**, 661 (1996).
- <sup>32</sup>G. Veble, M. Robnik, and J. Liu, *J. Phys. A* **32**, 6423 (1999).
- <sup>33</sup>V. Lopac, I. Mrkonjić, and D. Radić, *Phys. Rev. E* **66**, 036202 (2001).
- <sup>34</sup>E. D. Leonel and P. V. E. McClintock, *J. Phys. A* **38**, 823 (2005).
- <sup>35</sup>E. D. Leonel and A. L. P. Livorati, *Physica A* **387**, 1155 (2008).
- <sup>36</sup>A. L. P. Livorati, D. G. Ladeira, and E. D. Leonel, *Phys. Rev. E* **78**, 056205 (2008).
- <sup>37</sup>D. F. M. Oliveira and E. D. Leonel, *Phys. Lett. A* **374**, 3016 (2010).
- <sup>38</sup>D. G. Ladeira and J. K. L. da Silva, *J. Phys. A* **40**, 11467 (2007).
- <sup>39</sup>D. F. M. Oliveira, M. Robnik, and E. D. Leonel, arXiv:1102.2266.
- <sup>40</sup>D. F. M. Oliveira and M. Robnik, *Phys. Rev. E* **83**, 026202 (2011).
- <sup>41</sup>D. F. M. Oliveira, J. Vollmer, and E. D. Leonel, *Physica D* **240**, 389 (2011).
- <sup>42</sup>D. F. M. Oliveira and E. D. Leonel, *Physica A* **389**, 1009 (2010).
- <sup>43</sup>R. Vitolo, P. Glendinning, and J. A. C. Gallas, *Phys. Rev. E* **84**, 016216 (2011).
- <sup>44</sup>L. Martins and J. A. C. Gallas, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **18**, 1705 (2008).
- <sup>45</sup>P. Gaspard, R. Kapral, and G. Nicolis, *J. Stat. Phys.* **35**, 697 (1984).
- <sup>46</sup>J. Rössler, B. Hess, and M. Markus, *Phys. Rev. A* **39**, 5954 (1989).
- <sup>47</sup>M. Komuro, R. Tokunaga, T. Matsumoto, and A. Hotta, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **1**, 139 (1991).
- <sup>48</sup>J. A. C. Gallas, *Phys. Rev. Lett.* **70**, 2714 (1993).
- <sup>49</sup>J. A. C. Gallas, *Appl. Phys. B* **60**, S203 (1994), [http://www.if.ufrgs.br/~jgallas/OFICINA/BNK/jg\\_festschrift01.pdf](http://www.if.ufrgs.br/~jgallas/OFICINA/BNK/jg_festschrift01.pdf).
- <sup>50</sup>J. A. C. Gallas, *Physica B* **202**, 196 (1994).
- <sup>51</sup>B. R. Hunt, J. A. C. Gallas, C. Grebogi, J. A. Yorke, and H. Koçak, *Physica D* **129**, 35 (1999).
- <sup>52</sup>C. Bonatto, J. C. Garreau, and J. A. C. Gallas, *Phys. Rev. Lett.* **95**, 143905 (2005).
- <sup>53</sup>C. Bonatto and J. A. C. Gallas, *Phys. Rev. Lett.* **107**, 054101 (2008).
- <sup>54</sup>H. A. Albuquerque, R. M. Rubinger, and P. C. Rech, *Phys. Lett. A* **372**, 4793 (2008).
- <sup>55</sup>J. C. D. Cardoso, H. A. Albuquerque, and R. M. Rubinger, *Phys. Lett. A* **373**, 2050 (2009).
- <sup>56</sup>E. V. Viana, R. M. Rubinger, H. A. Albuquerque, A. G. de Oliveira, and G. M. Ribeiro, *Chaos* **20**, 023110 (2010).
- <sup>57</sup>C. Stegemann, H. A. Albuquerque, and P. C. Rech, *Chaos* **20**, 023103 (2010).
- <sup>58</sup>C. Stegemann, H. A. Albuquerque, R. M. Rubinger, and P. C. Rech, *Chaos* **21**, 033105 (2011).
- <sup>59</sup>R. Stoop, P. Benner, and Y. Uwate, *Phys. Rev. Lett.* **105**, 074102 (2010).
- <sup>60</sup>E. N. Lorenz, *Physica D* **237**, 1689 (2008).
- <sup>61</sup>G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics* (Oxford University Press, Oxford, 2006).
- <sup>62</sup>G. M. Zaslavsky, *The Physics of Chaos in Hamiltonian Systems* (Imperial College, London, 2007).
- <sup>63</sup>G. M. Zaslavsky, *Phys. Lett. A* **69A**, 145 (1978).
- <sup>64</sup>J. P. Eckmann and D. Ruelle, *Rev. Mod. Phys.* **57**, 617 (1985).
- <sup>65</sup>R. M. May, *Nature* **261**, 459 (1976).
- <sup>66</sup>S. Grossmann and S. Thomae, *Z. Naturforsch.* **32a**, 1353 (1977).
- <sup>67</sup>M. W. Beims and J. A. C. Gallas, *Physica A* **238**, 225 (1997).
- <sup>68</sup>M. Feigenbaum, *J. Stat. Phys.* **19**, 25 (1978).
- <sup>69</sup>M. Feigenbaum, *J. Stat. Phys.* **21**, 669 (1979).
- <sup>70</sup>J. A. C. Gallas, *Int. J. Mod. Phys. C* **3**, 553 (1992).
- <sup>71</sup>D. F. M. Oliveira and E. D. Leonel, *Braz. J. Phys.* **38**, 62 (2008).
- <sup>72</sup>T. C. Bountis, *Physics D* **3**, 577 (1981).
- <sup>73</sup>M. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, *Physica D* **5**, 370 (1982).