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On the Betti number of the union of two generic map images

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Abstract

Let $f : M \to N$ and $g : K \to N$ be generic differentiable maps of compact manifolds without boundary into a manifold such that their intersection satisfies a certain transversality condition. We show, under a certain cohomological condition, that if the images $f(M)$ and $g(K)$ intersect, then the $(v+1)$th Betti number of their union is strictly greater than the sum of their $(v+1)$th Betti numbers, where $v = \dim M + \dim K - \dim N$. This result is applied to the study of coincidence sets and fixed point sets. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $f : M \to N$ and $g : K \to N$ be differentiable maps, where M and K are smooth closed [2] manifolds of dimensions m and k, respectively and N is an n-dimensional smooth manifold. In this paper, we consider the following problems: *Is it possible to separate the images of f and g by homotopies*? *More precisely, is it possible to find f' and g' homotopic to f and g, respectively such that $f'(M) \cap g'(K) = \emptyset$? If this is the case, what type of condition guarantees that $f(M) \cap g(K) = \emptyset$?*

For the first problem, we define primary obstructions $f^*(U_g) \in H^{n-k}(M; \mathbb{Z}_2)$ and $g^*(U_f) \in H^{n-m}(K; \mathbb{Z}_2)$ to the existence of such homotopies (Section 2). These cohomology classes are Poincaré dual to the homology classes represented by the images of

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1 The third author has been partly supported by CNPq, Brazil, and by the Anglo-Japanese Scientific Exchange Programme, run by the Japan Society for the Promotion of Science and the Royal Society.
2 A manifold is said to be *closed* if it is compact and has no boundary.

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the so-called intersection manifold in case f and g are transverse, and it is easily seen that they are homotopy invariants. Thus, if f and g are homotopic to maps whose images do not intersect, then $f^*(U_g)$ and $g^*(U_f)$ necessarily vanish. (Nevertheless, we warn the reader that the vanishing of these primary obstructions does not necessarily imply the existence of such maps.)

For the second problem, we assume that the primary obstructions vanish and want to find a condition which guarantees that f and g have disjoint images. This recognition problem is difficult to solve in general. Thus, in this paper, we assume that f and g are generic in the sense of Ronga [21] and that their intersection satisfies a certain transversality condition. The main result of this paper is Theorem 2.4 which states that such generic maps f and g have disjoint images if and only if the $(m+k-n+1)$th Betti number of $f(M) \cup g(K)$ is equal to the sum of the Betti numbers of $f(M)$ and $g(K)$. In other words, if the images intersect, then the $(m+k-n+1)$th Betti number of $f(M) \cup g(K)$ is strictly greater than the sum of the $(m+k-n+1)$th Betti numbers of $f(M)$ and $g(K)$. The spaces $f(M)$, $g(K)$ and $f(M) \cup g(K)$ can be very complicated in general (see, for example, [4]) and, for the Betti numbers, we use Čech homology instead of the usual singular homology. Furthermore, we assume that the Betti numbers of $f(M)$ and $g(K)$ are finite.

We note that a characterization of embeddings among generic maps in terms of a primary obstruction and Betti numbers has been obtained in [1,3,4] and that our technique is similar to those developed there.

The paper is organized as follows. In Section 2, we define the primary obstructions and state our main theorem in a precise manner. In Section 3, we prepare a lemma on the intersection map of two transverse maps, which will be used in the proof of our main theorem. In Section 4, we prove our main theorem. Some applications to the study of coincidence sets and fixed points sets are given in Section 5. More precisely, we characterize those pairs of maps whose coincidence sets are empty, using a cohomology class similar to the modulo 2 coincidence number and the Betti numbers of the graphs of the maps.

Throughout the paper, all homology and cohomology groups have $\mathbb{Z}_2$ coefficients unless otherwise indicated.

2. Statement of the main result

Let M and N be manifolds of dimensions m and n, respectively and let $f : M \to N$ be a proper continuous map. Then we define $U_f \in H^{n-m}(N)$ to be the image of the fundamental class $[M] \in H_m^c(M)$ by the composite

$$H_m^c(M) \xrightarrow{f_*} H_m^c(N) \xrightarrow{D_N^{-1}} H^{n-m}(N),$$

where H_*^c denotes the (singular) homology of the compatible family with respect to the compact subsets [23, Chapter 6, Section 3], and D_N denotes the Poincaré duality isomorphism. By the definition, it is easy to see that U_f is a homotopy invariant (when M is not compact, the homotopy should be through proper maps).

Suppose that $f:M\to N$ and $g:K\to N$ are proper continuous maps between manifolds. Then we see easily that if $f(M)\cap g(K)=\emptyset$, then $f^*(U_g)=0$ in $H^{n-k}(M)$ and $g^*(U_f)=0$ in $H^{n-m}(K)$. Thus, if there exist maps $f':M\to N$ and $g':K\to N$ homotopic to f and g, respectively such that $f'(M)\cap g'(K)=\emptyset$, then $f^*(U_g)$ and $g^*(U_f)$ vanish. Thus these cohomology classes can be regarded as primary obstructions to the existence of such maps. In the next section we will see that these cohomology classes are Poincaré dual to the homology classes represented by the images of the so-called intersection manifold.

Example 2.1. Consider the embeddings $f=\mathrm{id}\times\eta:S^1\times S^1\to S^1\times S^2$ and $g:S^2\to S^1\times S^2$, where $\eta:S^1\to S^2$ is an embedding and g is defined by $g(x)=(c,x)$ for some fixed $c\in S^1$. Then we see that $g^*(U_f)$ vanishes, while $f^*(U_g)\in H^1(S^1\times S^1)$ does not vanish. Thus we cannot find maps f' and g' homotopic to f and g, respectively such that their images do not intersect.

In some cases, the obstructions $f^*(U_g)$ and $g^*(U_f)$ are invariant under bordism. For example, suppose that M is a closed manifold and that $H^{m+k-n}(M)$ is generated by the elements of the form $w_{i_1}(M)\smile w_{i_2}(M)\smile\cdots\smile w_{i_r}(M)$ with $i_1+i_2+\cdots+i_r=m+k-n$, where w_i denotes the ith Stiefel–Whitney class and $\smile$ denotes the cup product. We suppose that K also satisfies the same property. Then if f and $f':M\to N$ are bordant and if g and $g':K\to N$ are bordant, then we have $f^*(U_g)=(f')^*(U_{g'})$ and $g^*(U_f)=(g')^*(U_{f'})$ (for a definition of bordism, see [8,2]). We can prove this fact using an argument similar to [2]. For example, when M and K are even dimensional real, complex or quarternionic projective spaces, this sufficient condition is satisfied.

Next we define the class of differentiable maps which we are going to treat in this paper. We suppose that the manifolds M and N are smooth.

Definition 2.2. Let $f:M\to N$ be a proper differentiable map of class C^2 with $\dim M<\dim N$. We say that f is *generic for the double points*, if it is so in the sense of Ronga [21, Définition, p. 228]; in other words, if the 1-jet extension $j^1f:M\to J^1(M,N)$ of f is transverse to the submanifolds $\Sigma^i=\{\alpha\in J^1(M,N)\colon\ \dim\ker\alpha=i\}$ for all i and if the l-fold product map $f^l:M^l\to N^l$ is transverse to the diagonal δ_N^l of N^l off the super diagonal

$$\Delta_M^l=\bigl\{(x_1,\dots,x_l)\in M^l\colon\ x_i=x_j \text{ for some } i\neq j\bigr\}$$

of M^l for all $l=2,3,4,\dots$. Note that the latter condition is equivalent to that, for every $q\in f(M)$, $f^{-1}(q)$ consists of finitely many points, say $q_1,q_2,\dots,q_s$, and the subspaces $\mathrm{d}f_{q_1}(T_{q_1}M)$, $\mathrm{d}f_{q_2}(T_{q_2}M),\dots,$ $\mathrm{d}f_{q_s}(T_{q_s}M)$ are in general position in T_qN. Note that if $f:M\to N$ is generic for the double points, then f is finite-to-one and that $\dim M<\dim N$ by definition [3].

[3] Even when $\dim M=\dim N$, one can define maps which are generic for the double points as in this definition; however, since we are interested only in differentiable maps $f:M\to N$ with $\dim M<\dim N$ in this paper, we have included this condition in our definition.

Note that the set of the proper maps of class C^r $(2 \leqslant r \leqslant \infty)$ which are generic for the double points is dense in the space $C^r_{\mathrm{pr}}(M, N)$ of all proper maps of class C^r of M into N with the Whitney C^r-topology (or the fine C^r-topology). This fact is easily proved by using Thom's transversality theorem [13, Theorem 4.9, p. 54] and the multijet transversality theorem [13, Theorem 4.13, p. 57] together with the fact that the set of proper maps of class C^r is open in the space $C^r(M, N)$ of all C^r maps of M into N (for example, see [16, Chapter 2, Section 1]).

Let K be a smooth manifold and suppose that $f : M \to N$ and $g : K \to N$ are proper differentiable maps which are generic for the double points.

Definition 2.3. We say that f and g are *transverse with respect to double points* if for every $q \in f(M) \cap g(K)$ with $f^{-1}(q) = \{q_1, q_2, \ldots, q_s\}$ and $g^{-1}(q) = \{q'_1, q'_2, \ldots, q'_t\}$, the $s + t$ subspaces $\mathrm{d}f_{q_1}(T_{q_1}M)$, $\mathrm{d}f_{q_2}(T_{q_2}M), \ldots, \mathrm{d}f_{q_s}(T_{q_s}M)$, $\mathrm{d}g_{q'_1}(T_{q'_1}K)$, $\mathrm{d}g_{q'_2}(T_{q'_2}K)$, $\ldots$, $\mathrm{d}g_{q'_t}(T_{q'_t}K)$ are in general position in T_qN. Note that if f and g are transverse with respect to double points, then they are transverse in the usual sense.

Suppose that $f : M \to N$ and $g : K \to N$ are differentiable maps. Then f and g can be approximated by differentiable maps f_1 and g_1 which are generic for the double points and which are transverse with respect to double points. This fact can be proved by using the above mentioned transversality theorems together with the techniques used in the proof of the multijet transversality theorem (see [13, Chapter II, Section 4]). Note also that if both f and g are embeddings, then f and g are transverse with respect to double points if and only if they are transverse in the usual sense.

In the following, $\check{H}_*$ will denote the Čech (or Alexander–Čech) homology (see [11,23], [24,15,9], for example). For a topological space X, $\check{\beta}_i(X)$ will denote the dimension of the vector space $\check{H}_i(X)$ over $\mathbb{Z}_2$, and $\beta_i(X)$ the dimension of the singular homology $H_i(X)$. Here we note that $\check{H}_*$ is naturally isomorphic to the singular homology H_* for an ANR (absolute neighborhood retract). In particular, this is valid for manifolds.

The main result of this paper is the following.

Theorem 2.4. *Let $f : M \to N$ and $g : K \to N$ be differentiable maps which are generic for the double points, where M and K are smooth closed manifolds of dimensions m and k, respectively and N is an n-dimensional smooth manifold with $m < n$ and $k < n$. Suppose that f and g are transverse with respect to double points and that $\check{\beta}_{v+1}(f(M))$ and $\check{\beta}_{v+1}(g(K))$ are finite, where $v = m + k - n$. Then $f(M) \cap g(K) = \emptyset$ if and only if $f^*(U_g) = 0$ in $H^{n-k}(M)$, $g^*(U_f) = 0$ in $H^{n-m}(K)$ and $\check{\beta}_{v+1}(f(M)) + \check{\beta}_{v+1}(g(K)) = \check{\beta}_{v+1}(f(M) \cup g(K))$.*

Note that when $f(M)$ is an ANR, $\check{\beta}_i(f(M))$ are finite. In fact, the set of all $f \in C^r_{\mathrm{pr}}(M, N)$ which are generic for the double points such that $f(M)$ is an ANR is dense in $C^r_{\mathrm{pr}}(M, N)$ (see Remark 4.9). Note also that there exists a differentiable map which is generic for the double points and whose image is *not* an ANR (see [4]).

3. A lemma on the intersection map

Let $f : M \to N$ and $g : K \to N$ be proper differentiable maps of class C^2 which are mutually transverse; i.e., for every pair $(x, y) \in M \times K$ with $f(x) = g(y)$, we have $\mathrm{d}f_x(T_xM) + \mathrm{d}g_y(T_yK) = T_aN$, where $a = f(x) = g(y)$. (Recall that if f and g are generic for the double points and are transverse with respect to double points, then they are mutually transverse.) We set $\dim M = m$, $\dim K = k$ and $\dim N = n$. It is easy to see that f and g are mutually transverse if and only if the differentiable map $f \times g : M \times K \to N \times N$ is transverse to the diagonal $\delta_N^2 = \{(z, z) \in N \times N\}$. We set $V = (f \times g)^{-1}(\delta_N^2)$ and define the *intersection map* $h : V \to N$ by the composite

$$V \xrightarrow{i} M \times K \xrightarrow{f \times g} N \times N \xrightarrow{\pi} N,$$

where i is the inclusion map and π is the projection to the first factor. Here V is a differentiable manifold of dimension $v = m + k - n$ and we call it the *intersection manifold*. Furthermore, we define $\pi_M : V \to M$ and $\pi_K : V \to K$ by $\pi_M = \pi_1 \circ i$ and $\pi_K = \pi_2 \circ i$, where $\pi_1 : M \times K \to M$ and $\pi_2 : M \times K \to K$ are the projections to the first factor and to the second factor, respectively. We call the following commutative diagram the *pull-back diagram of f and g*:

$$\begin{array}{ccc} V & \xrightarrow{\pi_M} & M \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{g} & N \end{array}$$

Lemma 3.1. *We have $f^*(U_g) = U_{\pi_M}$ in $H^{n-k}(M)$ and $g^*(U_f) = U_{\pi_K}$ in $H^{n-m}(K)$.*

For the proof of Lemma 3.1, we need the following.

Proposition 3.2. *We have $f^* \circ g_! = (\pi_M)_! \circ (\pi_K)^* : H^j(K) \to H^{n-k+j}(M)$ for all j, where $g_!$ and $(\pi_M)_!$ are the Gysin homomorphisms.*

For a definition of Gysin homomorphisms, see [10, p. 53].

Proof. First consider the case where g is an embedding. We identify K with the image $g(K)$ of g. Then we can identify the pull-back diagram of f and g with the diagram

$$\begin{array}{ccc} V = f^{-1}(K) & \xrightarrow{i_1} & M \\ {\scriptstyle \pi_K = f|_{f^{-1}(K)}}\downarrow & & \downarrow{\scriptstyle f} \\ K & \xrightarrow{i_2} & N \end{array}$$

where i_1 and i_2 are the inclusion maps. Then $f : M \to N$ induces a natural map α of the normal bundle ν_1 of $f^{-1}(K)$ in M into the normal bundle ν_2 of K in N. We may assume that a metric is given on each fiber of ν_1 and ν_2 so that α is isometric on each fiber. Then α induces a natural map $F : V^{\nu_1} \to K^{\nu_2}$ between the Thom spaces of ν_1 and ν_2, where

$V^{\nu_1} = E_1/\partial E_1$, $K^{\nu_2} = E_2/\partial E_2$, E_j is the unit disk bundle of ν_j, and ∂E_j is the unit sphere bundle of ν_j $(j = 1, 2)$. We may assume that E_1 and E_2 are embedded in M and N, respectively as tubular neighborhoods of V and K, respectively. Then the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\hat{i}_1} & V^{\nu_1} & \xleftarrow{i'_1} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \pi_K} \\ N & \xrightarrow{\hat{i}_2} & K^{\nu_2} & \xleftarrow{i'_2} & K \end{array}$$

is commutative, where i'_j are the zero sections $(j = 1, 2)$ and $\hat{i}_j$ is the map which coincides with the identity map on $\operatorname{Int} E_j$ $(j = 1, 2)$ and $\hat{i}_1(M - \operatorname{Int} E_1) = \partial E_1/\partial E_1 = *_1$ for $j = 1$ and $\hat{i}_2(N - \operatorname{Int} E_2) = \partial E_2/\partial E_2 = *_2$ for $j = 2$. Then we have the commutative diagram

$$\begin{array}{ccccc} (i_2)_! : H^j(K) & \xrightarrow{\cong} & H^{j+n-k}(K^{\nu_2}, *_2) & \xrightarrow{\hat{i}_2^*} & H^{j+n-k}(N) \\ {\scriptstyle (\pi_K)^*}\downarrow & & {\scriptstyle F^*}\downarrow & & {\scriptstyle f^*}\downarrow \\ (i_1)_! : H^j(V) & \xrightarrow{\cong} & H^{j+n-k}(V^{\nu_1}, *_1) & \xrightarrow{\hat{i}_1^*} & H^{j+n-k}(M) \end{array}$$

which implies that $f^* \circ (i_2)_! = (i_1)_! \circ (\pi_K)^*$, where the two horizontal maps on the left are the Thom isomorphisms. This shows that, in this case, we have $f^* \circ g_! = (\pi_M)_! \circ (\pi_K)^*$. We note that this has already been known for the case $j = 0$ (see [5, Proposition 2.15]).

Now we consider the general case. Let $\varphi : K \to D^d$ be an embedding into the interior $\operatorname{Int} D^d$ of the d-dimensional disk, where d is sufficiently large. Then consider the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\pi}_M} & M \times D^d \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{id}} \\ K & \xrightarrow{\tilde{g}} & N \times D^d \end{array}$$

where $V \subset M \times K$, $\tilde{\pi}_M(x, y) = (x, \varphi(y))$, and $\tilde{g}(y) = (g(y), \varphi(y))$. Then $f \times \mathrm{id}$ is transverse to $\tilde{g}$ and the above diagram can be identified with the pull-back diagram of $f \times \mathrm{id}$ and $\tilde{g}$. Then, since $\tilde{g}$ is an embedding, we can use the argument in the preceding paragraph (see also [10, p. 53]). This completes the proof. □

Note that the above proof works for an arbitrary coefficient module R, if everything is R-oriented.

Proof of Lemma 3.1. By Proposition 3.2, we have

$$f^* \circ g_!(1) = (\pi_M)_! \circ (\pi_K)^*(1) : H^0(K) \to H^{n-k}(M),$$

where $1 \in H^0(K)$ is the Poincaré dual of the fundamental class $[K] \in H_k^c(K)$ of K. It is easy to verify that $f^* \circ g_!(1) = f^*(U_g)$ and $(\pi_M)_! \circ (\pi_K)^*(1) = U_{\pi_M}$, and hence we have $f^*(U_g) = U_{\pi_M}$. The other equality can be proved similarly. This completes the proof. □

The above proof of Lemma 3.1 uses Gysin homomorphisms. Here we give an alternative proof for the case where M, K and N are compact. Let $B = B_0 \cup B_1 \cup \cdots \cup B_n$ be a basis of

$H^*(N)$ over $\mathbb{Z}_2$, where $B_j = \{b_j^1, b_j^2, \ldots, b_j^{\beta_j}\}$ is a basis of $H^j(N)$. Here we may assume that $U_g = ab_{n-k}^1$ for some $a \in \mathbb{Z}_2$. Let $B' = B'_0 \cup B'_1 \cup \cdots \cup B'_n$ be the dual basis of B, where $B'_j = \{b_j'^1, b_j'^2, \ldots, b_j'^{\beta_j}\}$ is a basis of $H^{n-j}(N)$ and $\langle b_j^r \smile b_j'^s,\ [N]\rangle = \delta_{r,s}$, where $\delta_{r,s} = 1$ if $r = s$ and $\delta_{r,s} = 0$, otherwise. Let $d : N \to N \times N$ be the diagonal map. Then it is known that

$$U_d = \sum_j \sum_{r=1}^{\beta_j} b_j^r \times b_j'^r$$

(for example, see [9, 8.21, Exercise 2]). Consider the pull-back diagram

$$\begin{array}{ccc} V & \xrightarrow{\ i\ } & M \times K \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f\times g} \\ N & \xrightarrow{\ d\ } & N \times N \end{array}$$

where h is the intersection map defined at the beginning of this section. Then, since $d : N \to N \times N$ is an embedding, we have $U_i = (f \times g)^*(U_d)$ (see [5]). Thus, for all $\gamma \in H^{m+k-n}(M)$, we have

$$\begin{aligned} \big\langle \gamma, (\pi_M)_*([V])\big\rangle &= \big\langle \pi_1^*(\gamma), U_i \cap [M \times K]\big\rangle \\ &= \big\langle \pi_1^*(\gamma), (f \times g)^*(U_d) \cap [M \times K]\big\rangle \\ &= \sum_{r=1}^{\beta_{n-k}} \big\langle \gamma, f^*(b_{n-k}^r) \cap [M]\big\rangle\big\langle g^*(b_{n-k}'^r), [K]\big\rangle \\ &= \sum_{r=1}^{\beta_{n-k}} \big\langle \gamma, f^*(b_{n-k}^r) \cap [M]\big\rangle\big\langle b_{n-k}'^r, U_g \cap [N]\big\rangle \\ &= \big\langle \gamma, f^*(b_{n-k}^1) \cap [M]\big\rangle\big\langle b_{n-k}'^1 \smile (ab_{n-k}^1), [N]\big\rangle \\ &= \big\langle \gamma, f^*(ab_{n-k}^1) \cap [M]\big\rangle \\ &= \big\langle \gamma, f^*(U_g) \cap [M]\big\rangle. \end{aligned}$$

Therefore we have $(\pi_M)_*([V]) = f^*(U_g) \cap [M]$ in $H_{m+k-n}(M)$, which implies that $f^*(U_g) = U_{\pi_M}$. This completes the alternative proof. □

4. Proof of Theorem 2.4

Proof of Theorem 2.4. First suppose that $f^*(U_g) = 0$, $g^*(U_f) = 0$, $\check{\beta}_{v+1}(f(M)) + \check{\beta}_{v+1}(g(K)) = \check{\beta}_{v+1}(f(M) \cup g(K))$ and $f(M) \cap g(K) \neq \emptyset$, where $v = m + k - n$. Set $A = f(M) \cap g(K)$. Let A_M and A_K be copies of A, and $i_M : A_M \to A$ and $i_K : A_K \to A$ the respective identifications. In the following, we consider A_M and A_K to be subsets of

$f(M)$ and $g(K)$, respectively, and we use the symbol "$\amalg$" for a disjoint union. Consider the following commutative diagram of Čech homology with exact rows (see [18,11]):

$$\begin{array}{ccccc}
\cdots \longrightarrow \check{H}_j(A_M \amalg A_K) & \longrightarrow & \check{H}_j(f(M) \amalg g(K)) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
\cdots \longrightarrow \check{H}_j(A) & \longrightarrow & \check{H}_j(f(M) \cup g(K)) & \longrightarrow & \\
\check{H}_j(f(M) \amalg g(K), A_M \amalg A_K) & \longrightarrow & \check{H}_{j-1}(A_M \amalg A_K) \longrightarrow \cdots & & \\
\downarrow & & \downarrow & & \\
\check{H}_j(f(M) \cup g(K), A) & \longrightarrow & \check{H}_{j-1}(A) \longrightarrow \cdots & &
\end{array}$$

where the vertical homomorphisms are induced by the natural identifications. Since the map $(f(M) \amalg g(K), A_M \amalg A_K) \to (f(M) \cup g(K), A)$ is a relative homeomorphism between compact pairs, the homomorphism $\check{H}_j(f(M) \amalg g(K), A_M \amalg A_K) \to \check{H}_j(f(M) \cup g(K), A)$ is an isomorphism (see [11, Chapter X, Section 5]). Then we obtain the following exact sequence of Čech homology (see [10, Lemma, p. 2]):

$$\begin{aligned}
\check{H}_{v+1}(A_M) \oplus \check{H}_{v+1}(A_K) &\to \check{H}_{v+1}(A) \oplus \check{H}_{v+1}\big(f(M)\big) \oplus \check{H}_{v+1}\big(g(K)\big) \\
&\to \check{H}_{v+1}\big(f(M) \cup g(K)\big) \to \check{H}_v(A_M) \oplus \check{H}_v(A_K) \\
&\overset{\alpha}{\to} \check{H}_v(A) \oplus \check{H}_v\big(f(M)\big) \oplus \check{H}_v\big(g(K)\big),
\end{aligned} \tag{4.1}$$

where $v = m + k - n$ and α is defined by using $i_M \cup i_K : A_M \amalg A_K \to A$ and the inclusions $A_M \to f(M)$ and $A_K \to g(K)$.

Lemma 4.2. *We have $\check{H}_{v+1}(A) = 0$, where $v = m + k - n$.*

Proof. Let $h : V \to N$ be the intersection map defined in the preceding section. We see easily that $A = h(V)$. Since $\dim V = m + k - n = v$, we see that the topological dimension of A is at most v (see [22,7]). Hence we have the conclusion (see [17, p. 152]). □

By the above lemma together with our assumption about the Čech–Betti numbers $\check{\beta}_{v+1}$, we see that α in (4.1) must be injective.

Set $[A] = (f \circ \pi_M)_*[V] = (g \circ \pi_K)_*[V] \in \check{H}_v(A)$.

Lemma 4.3. *The homology class $[A] \in \check{H}_v(A)$ is nonzero if $A \neq \emptyset$.*

For the proof of the above lemma, set

$$V_0 = \big\{(x, y) \in V\colon\ \mathrm{d}f_x \text{ and } \mathrm{d}g_y \text{ are injective}\big\}, \quad \text{and}$$
$$V_1 = \big\{(x, y) \in V_0\colon\ f^{-1}(f(x)) = \{x\},\ g^{-1}(g(y)) = \{y\}\big\}.$$

Lemma 4.4. *The set V_0 is open and dense in V.*

Proof. Set $\Sigma(f) = \{x \in M\colon\ df_x$ is not injective$\}$ and $\Sigma(g) = \{y \in K\colon\ dg_y$ is not injective$\}$. Note that $\Sigma(f)$ and $\Sigma(g)$ are closed subsets of M and K, respectively. Since $V_0 = V - ((\Sigma(f) \times K) \cup (M \times \Sigma(g)))$, it follows that V_0 is open. Set $V_f = V \cap (\Sigma(f) \times K)$ and $V_g = V \cap (M \times \Sigma(g))$.

Suppose that V_0 is not dense in V. Then $V_f \cup V_g$ contains a nonempty open set U of V. Since V_f and V_g are closed subsets of V, we may assume that U is contained in V_f.

For the proof of Lemma 4.4, we need the following.

Lemma 4.5. *Suppose that f and g are transverse in the usual sense. Then a point $(x, y) \in V$ is a singular point of $\pi_K : V \to K$ if and only if $(x, y) \in V_f$.*

Proof. Suppose that $(x, y) \in V$ is a singular point of π_K. Then there exists a nonzero vector $(u_1, u_2) \in T_xM \times T_yK$ such that $(u_1, u_2) \in T_{(x,y)}V$ and $d(\pi_K)_{(x,y)}(u_1, u_2) = u_2 = 0$. Since f and g are transverse, it is easy to see that $(u_1, u_2) \in T_xM \times T_yK$ belongs to $T_{(x,y)}V$ if and only if $df_x(u_1) = dg_y(u_2)$. Thus we see that there exists a nonzero vector $u_1 \in T_xM$ such that $df_x(u_1) = 0$. Hence $(x, y) \in V_f$. We can prove the converse in a similar manner. This completes the proof of Lemma 4.5. □

By Lemma 4.5, the open set U of V consists of singular points of the differentiable map $\pi_K : V \to K$. By taking U smaller if necessary, we may assume that the rank of π_K is constant on U. Then, at each point of U, π_K is locally equivalent to the composition of a submersion and an embedding by the rank theorem (for example, see [6, Section 5]). Thus there exists a point $y_0 \in K$ such that $(\pi_K)^{-1}(y_0) \cap U$ contains an infinite number of points. This implies that the set $\{x \in M\colon\ f(x) = g(y_0)\}$ contains an infinite number of points. However, since f is generic for the double points, f is finite-to-one. This is a contradiction. Thus V_0 is dense in V. This completes the proof of Lemma 4.4. □

Lemma 4.6. *The set V_1 is open and dense in V.*

Proof. Consider the following map:

$$\left(M \times M - \Delta_M^2\right) \times K \overset{f\times f\times g}{\longrightarrow} N \times N \times N.$$

Since f and g are transverse with respect to double points, the above map is transverse to δ_N^3 (see Definitions 2.2 and 2.3). Set $W_f = (f \times f \times g)^{-1}(\delta_N^3)$, which is a submanifold of $(M \times M - \Delta_M^2) \times K$ of dimension $2m + k - 2n = (m + k - n) + (m - n)$. Similarly we define $W_g = (f \times g \times g)^{-1}(\delta_N^3)$, which is a submanifold of $M \times (K \times K - \Delta_K^2)$ of dimension $m + 2k - 2n = (m + k - n) + (k - n)$. Note that $\dim W_f, \dim W_g < m + k - n$. Let $p_f : (M \times M - \Delta_M^2) \times K \to M \times K$ denote the projection to the first and the third factors and $p_g : M \times (K \times K - \Delta_K^2) \to M \times K$ the projection to the first and the second factors. Note that $p_f(W_f), p_g(W_g) \subset V$ and that $V_1 = V_0 - (p_f(W_f) \cup p_g(W_g))$. Since $\dim W_f$ and $\dim W_g$ are strictly smaller than $\dim V = m + k - n$, we see that $V - p_f(W_f)$ and $V - p_g(W_g)$ are dense in V by the Sard theorem.

On the other hand, since W_f is closed in $(M \times M - \Delta_M^2) \times K$, we see that

$$\overline{W}_f = W_f \cup \left(\overline{W}_f \cap (\Delta_M^2 \times K)\right),$$

where $\overline{W}_f$ denotes the closure of W_f in $M \times M \times K$. Since f is generic for the double points, the closure of the selfintersection set $M(f) = \{x \in M\colon\ f^{-1}(f(x)) \neq \{x\}\}$ of f in M coincides with $M(f) \cup \Sigma(f)$ (see [21]). Thus we see that

$$\overline{W}_f \subset W_f \cup \bigl\{(x, x, y) \in M \times M \times K\colon\ x \in \Sigma(f),\ f(x) = g(y)\bigr\}.$$

Thus we have

$$\begin{aligned} p_f(\overline{W}_f) &\subset p_f(W_f) \cup \bigl\{(x, y) \in M \times K\colon\ x \in \Sigma(f),\ f(x) = g(y)\bigr\} \\ &= p_f(W_f) \cup V_f. \end{aligned}$$

By a similar argument, we also have $p_g(\overline{W}_g) \subset p_f(W_g) \cup V_g$, where $\overline{W}_g$ is the closure of W_g in $M \times K \times K$. Note that $p_f(\overline{W}_f)$ and $p_g(\overline{W}_g)$ are closed subsets of $M \times K$, since $\overline{W}_f$ and $\overline{W}_g$ are compact.

Then we have

$$V - p_f(\overline{W}_f) \supset \bigl(V - p_f(W_f)\bigr) \cap (V - V_f).$$

As we have seen above, $V - p_f(W_f)$ is dense in V, and by the proof of Lemma 4.4, $V - V_f$ is open and dense in V. Thus $(V - p_f(W_f)) \cap (V - V_f)$ is dense in V. Thus $V - p_f(\overline{W}_f)$ is an open set containing a dense set, and hence it is open and dense in V. Similarly $V - p_g(\overline{W}_g)$ is also open and dense in V.

Since

$$\begin{aligned} V_1 &= V - \bigl(p_f(W_f) \cup V_f \cup p_g(W_g) \cup V_g\bigr) \\ &= V - \bigl(p_f(\overline{W}_f) \cup V_f \cup p_g(\overline{W}_g) \cup V_g\bigr) \\ &= \bigl(V - p_f(\overline{W}_f)\bigr) \cap \bigl(V - p_g(\overline{W}_g)\bigr) \cap \bigl(V - (V_f \cup V_g)\bigr) \\ &= \bigl(V - p_f(\overline{W}_f)\bigr) \cap \bigl(V - p_g(\overline{W}_g)\bigr) \cap V_0 \end{aligned}$$

and V_0 is open and dense in V by Lemma 4.4, V_1 is also open and dense in V. This completes the proof of Lemma 4.6. □

Proof of Lemma 4.3. Recall that $A = h(V)$. By the definition of V_1 together with Lemma 4.5, we see that $h|_{V_1}$ is an embedding onto an open set of A. Taking a point $x \in V_1$, we have the commutative diagram:

$$\begin{array}{ccccc} H_v(V) & \xrightarrow{h_*} & H_v(A) & \xrightarrow{\alpha_1} & \check{H}_v(A) \\ \downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \iota_3} \\ H_v(V, V - x) & \xrightarrow{h_*} & H_v(A, A - h(x)) & \xrightarrow{\alpha_2} & \check{H}_v(A, A - h(x)) \end{array}$$

where $v = \dim V = m + k - n$, ι_1, ι_2 and ι_3 are induced by the inclusions, and α_1 and α_2 are the natural homomorphisms; in other words, α_1 and α_2 are induced by the inclusions $A \to Z$, where Z runs over all open sets of N containing A (recall that $\check{H}_v(A)$ is nothing but the inverse limit of the singular homology groups $H_v(Z)$. For example, see [9, Chapter VIII, Section 13]. See also [11, Chapter X, Section 2]). Then we see that α_2 is an isomorphism by excision. Since $h_*\colon H_v(V, V - x) \to H_v(A, A - h(x))$ is an isomorphism and the image of $[V] \in H_v(V)$ in $H_v(V, V - x)$ by ι_1 is nonzero, the image of $[A] = h_*[V] \in H_v(A)$ by ι_2 is nonzero. Hence $[A]$ is nonzero in $\check{H}_v(A)$. This completes the proof of Lemma 4.3. □

Let $[A_M] \in \check{H}_v(A_M)$ and $[A_K] \in \check{H}_v(A_K)$ be the homology classes corresponding to $[A] \in \check{H}_v(A)$ by the identifications i_M and i_K, respectively.

Lemma 4.7. *We have that* $(i_M \cup i_K)_*([A_M] \oplus [A_K]) = 0$ *in* $\check{H}_v(A)$.

Proof. This is easily seen, since $(i_M \cup i_K)_*([A_M] \oplus [A_K]) = h_*([V]) + h_*([V]) = 0$. □

Now consider the nonzero element $[A_M] \oplus [A_K]$ of $\check{H}_v(A_M) \oplus \check{H}_v(A_K)$. Then by Lemma 4.7, we have $(i_M \cup i_K)_*([A_M] \oplus [A_K]) = 0$ in $\check{H}_v(A)$. Furthermore, we see that the image of $[A_M]$ (or $[A_K]$) by the homomorphism induced by the inclusion into $f(M)$ (respectively $g(K)$) is equal to $f_*((\pi_M)_*[V]) \in \check{H}_v(f(M))$ (respectively $g_*((\pi_K)_*[V]) \in \check{H}_v(g(K))$). Since, $(\pi_M)_*[V] \in H_v(M) \cong \check{H}_v(M)$ and $(\pi_K)_*[V] \in H_v(K) \cong \check{H}_v(K)$ coincide with the Poincaré duals of $f^*(U_g)$ and $g^*(U_f)$, respectively by Lemma 3.1 and these are zero by our hypothesis, we see that $\alpha([A_M] \oplus [A_K]) = 0$ in (4.1). In other words, α is not injective. This is a contradiction. Hence we have $f(M) \cap g(K) = \emptyset$.

Conversely suppose that $f(M) \cap g(K) = \emptyset$. Then we see easily that $f^*(U_g) = 0$, $g^*(U_f) = 0$ and $\check{\beta}_{v+1}(f(M)) + \check{\beta}_{v+1}(g(K)) = \check{\beta}_{v+1}(f(M) \cup g(K))$. This completes the proof of Theorem 2.4. □

Remark 4.8. The condition that f and g should be transverse with respect to double points is essential. For example, consider the immersions with normal crossings $f : K^2 \to S^3$ and $g : K^2 \to S^3$ as in Fig. 1, where K^2 is the Klein bottle. Then we see easily that they are transverse in the usual sense and that $f^*(U_g)$ and $g^*(U_f)$ vanish. However, we have $\check{\beta}_2(f(K^2)) + \check{\beta}_2(g(K^2)) = \check{\beta}_2(f(K^2) \cup g(K^2))$. In fact, in this case, Lemma 4.3 does not hold.

Consider another example of differentiable maps $f : M \to S^3$ and $g : K \to S^3$ as in Fig. 2, where M and K are the 2-sphere S^2, f is generic for the double points, and g

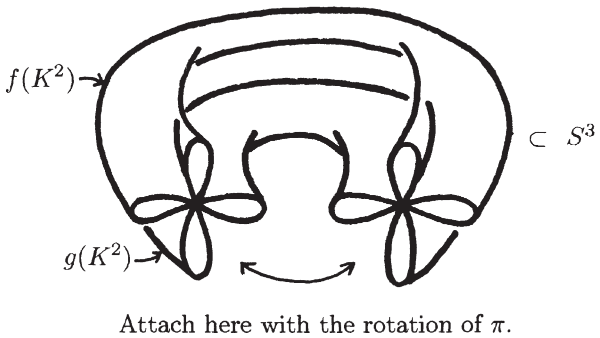

Fig. 1.

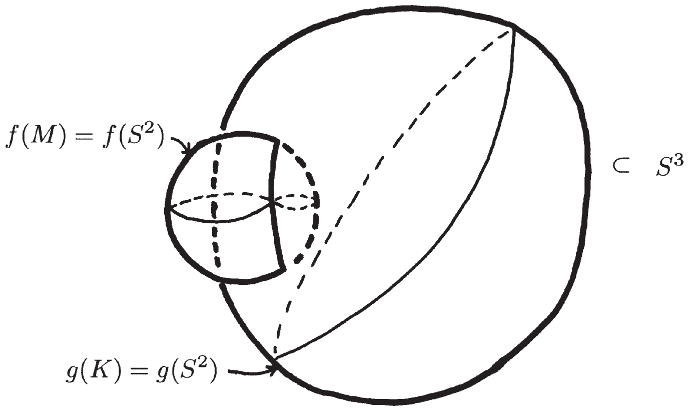

Fig. 2.

is an embedding whose image contains the image of the selfintersection set of f. They are transverse in the usual sense and $f^*(U_g)$ and $g^*(U_f)$ vanish. However, we have $\check{\beta}_2(f(S^2)) + \check{\beta}_2(g(S^2)) = \check{\beta}_2(f(S^2) \cup g(S^2))$. In this case A is contractible and does not carry any fundamental class. Consequently Lemma 4.3 does not hold in this case, either.

Remark 4.9. Suppose that $f: M \to N$ and $g: K \to N$ are smooth maps which are generic in the sense of [12]; in other words, they are C^0-stable. Then f and g are Thom stratifiable. We say that f and g are *strongly transverse* if all the strata of f and g intersect in general position. Note that, in this case, f and g are generic for the double points and are transverse with respect to double points (see [3]). Then $f(M)$, $g(K)$ and $f(M) \cup g(K)$ are all triangulable [14] and the Betti numbers with respect to Čech homology in Theorem 2.4 can be replaced by those with respect to singular homology. In particular, $\check{\beta}_{v+1}(f(M)) = \beta_{v+1}(f(M))$ and $\check{\beta}_{v+1}(g(K)) = \beta_{v+1}(g(K))$ are always finite.

5. Application to coincidence sets and fixed point sets

Let f and $g: M \to N$ be continuous maps between manifolds. We set

$$C = \{x \in M\colon\ f(x) = g(x)\}$$

and call it the *coincidence set* of f and g (see [19,20]). For f, define the graph $G_f: M \to M \times N$ by $G_f(x) = (x, f(x))$. Then it is easy to see that the points in C are in one to one correspondence with the intersection points of the images of G_f and G_g.

Lemma 5.1. *Suppose that f and $g: M \to N$ are differentiable of class C^1 and that $f(x) = g(x)$ $(= a)$ for some $x \in M$. Then G_f and G_g are transverse at x if and only if $\mathrm{d}f_x - \mathrm{d}g_x: T_xM \to T_aN$ is surjective.*

Proof. First note that G_f and G_g are transverse at x if and only if $\operatorname{codim}(\operatorname{Im} d(G_f)_x \cap \operatorname{Im} d(G_g)_x) = \operatorname{codim}(\operatorname{Im} d(G_f)_x) + \operatorname{codim}(\operatorname{Im} d(G_g)_x)$ in $T_{(x,a)}(M \times N)$, which is equivalent to $\dim(\operatorname{Im} d(G_f)_x \cap \operatorname{Im} d(G_g)_x) = m - n$, where $\dim M = m$ and $\dim N = n$. Note that $\operatorname{Im} d(G_f)_x \cap \operatorname{Im} d(G_g)_x = \{(u, v) \in T_x M \times T_a N\colon\ v = df_x(u) = dg_x(u)\}$. Hence we have $\dim(\operatorname{Im} d(G_f)_x \cap \operatorname{Im} d(G_g)_x) = \dim\{u \in T_x M\colon\ df_x(u) = dg_x(u)\} = \dim\ker(df_x - dg_x)$. Finally, this is equal to $m - n$ if and only if $df_x - dg_x$ is surjective. This completes the proof. □

Definition 5.2. Let f and $g : M \to N$ be continuous maps of an m-dimensional manifold into an n-dimensional manifold $(m \geqslant n)$. We define

$$\Lambda(f, g) = (G_f)^*(U_{G_g}) \in H^n(M).$$

Note that $\Lambda(f, g) = \Lambda(g, f)$ and that it is invariant under homotopies of f and g.

Note that if the coincidence set C is empty, then $\Lambda(f, g) = 0$. Note also that when $m = n$ and M is a closed manifold, the number $\Lambda_{f,g} = \langle \Lambda(f, g), [M]\rangle \in \mathbb{Z}_2$ is nothing but the modulo 2 *coincidence number* of f and g [19, p. 247].

Proposition 5.3. *Let f and $g : M \to N$ be differentiable maps of class C^2 of a closed m-dimensional manifold into an n-dimensional manifold $(m \geqslant n)$. We suppose that, for every x in the coincidence set C, $df_x - dg_x$ is surjective. Then $C = \emptyset$ if and only if $\Lambda(f, g) = 0$ and $\beta_{m-n+1}(G_f(M) \cup G_g(M)) = \beta_{m-n+1}(G_f(M)) + \beta_{m-n+1}(G_g(M))$ $(= 2\beta_{m-n+1}(M))$.*

The above proposition is easily obtained by applying Theorem 2.4 to G_f and G_g. Note that, in our case, $G_f(M) \cup G_g(M)$, $G_f(M)$ and $G_g(M)$ are compact polyhedrons and the Betti numbers with respect to Čech homology coincide with those with respect to the usual singular homology.

When $m = n$, we also have the following.

Proposition 5.4. *Let f and $g : M \to N$ be differentiable maps of class C^2 of a closed m-dimensional manifold into an m-dimensional manifold, where M and N are connected. We suppose that, for every x in the coincidence set C, $df_x - dg_x$ is surjective. Then $C = \emptyset$ if and only if $\Lambda_{f,g} = 0 \in \mathbb{Z}_2$ and $\beta_1(G_f(M) \cup G_g(M)) \equiv 0 \bmod 2$.*

Proof. First suppose that $\Lambda_{f,g} = 0$, $\beta_1(G_f(M) \cup G_g(M)) \equiv 0 \bmod 2$ and that $C \neq \emptyset$. Since $\Lambda_{f,g} = 0$, $G_f(M) \cap G_g(M)$ consists of an even number of points; i.e., $\beta_0(G_f(M) \cap G_g(M))$ is even. Furthermore, since $H_1(G_f(M) \cap G_g(M)) = 0$, we have the following Mayer–Vietoris exact sequence:

$$\begin{aligned} 0 \ &\to\ H_1\big(G_f(M)\big) \oplus H_1\big(G_g(M)\big) \to H_1\big(G_f(M) \cup G_g(M)\big) \\ &\to\ H_0\big(G_f(M) \cap G_g(M)\big) \xrightarrow{\alpha_1} H_0\big(G_f(M)\big) \oplus H_0\big(G_g(M)\big) \\ &\xrightarrow{\alpha_2} H_0\big(G_f(M) \cup G_g(M)\big). \end{aligned}$$

Hence we have

$$\begin{aligned}
&\beta_1\bigl(G_f(M)\cup G_g(M)\bigr)-2\beta_1(M)\\
&\quad=\beta_1\bigl(G_f(M)\cup G_g(M)\bigr)-\bigl(\beta_1\bigl(G_f(M)\bigr)+\beta_1\bigl(G_g(M)\bigr)\bigr)=\dim\ker\alpha_1\\
&\quad=\beta_0\bigl(G_f(M)\cap G_g(M)\bigr)-\dim\ker\alpha_2\\
&\quad=\beta_0\bigl(G_f(M)\cap G_g(M)\bigr)-1.
\end{aligned}$$

This implies that $\beta_1(G_f(M)\cup G_g(M))\equiv\beta_0(G_f(M)\cap G_g(M))-1 \bmod 2$, which is a contradiction. Thus the coincidence set C must be empty.

Conversely, if $C=\emptyset$, then we have $\Lambda_{f,g}=0$ and $\beta_1(G_f(M)\cup G_g(M))=2\beta_1(M)\equiv 0 \bmod 2$. This completes the proof. □

As an application of Proposition 5.4, we have the following.

Corollary 5.5. *Let M be an m-dimensional $\mathbb{Z}_2$-homology sphere (i.e., $H_*(M)$ is isomorphic to $H_*(S^m)$ over the coefficient $\mathbb{Z}_2$) and $h:M\to M$ a smooth map of degree one over $\mathbb{Z}_2$. Furthermore, let $f:M\to N$ be a smooth map of M into an m-dimensional connected manifold N and set $g=f\circ h$. Assume that for every x in the coincidence set C of f and g, $\mathrm{d}f_x-\mathrm{d}g_x=\mathrm{d}f_x-(\mathrm{d}f_{h(x)}\circ\mathrm{d}h_x)$ is surjective. Then $C=\emptyset$ if and only if $\beta_1(G_f(M)\cup G_g(M))\equiv 0 \bmod 2$.*

Proof. First, note that $\Lambda(f,g)=(f,g)^*(U_d)$, where $(f,g):M\to N\times N$ is defined by $(f,g)(x)=(f(x),g(x))$ for $x\in M$ and $d:N\to N\times N$ is the diagonal map (for example, see [15, Section 30]). Then, since M is a $\mathbb{Z}_2$-homology sphere, we have $\Lambda_{f,g}=\deg(f)+\deg(g)$, where deg denotes the degree over $\mathbb{Z}_2$ (to see this, use the argument as in [15, pp. 222–223] together with [9, Exercise 8.21]). Thus we have

$$\Lambda_{f,g}=\deg(f)+\deg(f)\cdot\deg(h)=\deg(f)\cdot\bigl(1+\deg(h)\bigr)=0$$

by our hypothesis. Then the result follows from Proposition 5.4. □

As an immediate corollary, we have the following.

Corollary 5.6. *Let $f:S^m\to N$ be a smooth map of the m-dimensional sphere S^m into an m-dimensional connected manifold N such that $\mathrm{d}f_x+\mathrm{d}f_{-x}$ is surjective for every $x\in S^m$ with $f(x)=f(-x)$, where we identify S^m with the unit sphere in $\mathbb{R}^{m+1}$ and we identify the tangent spaces T_xS^m and $T_{-x}S^m$ with the hyperplane in $\mathbb{R}^{m+1}$ perpendicular to x (or to $-x$). Then there exists a point $x_0\in S^m$ with $f(x_0)=f(-x_0)$ if and only if $\beta_1(G_f(S^m)\cup G_g(S^m))\equiv 1 \bmod 2$, where $g:S^m\to N$ is the smooth map defined by $g(x)=f(-x)$ for $x\in S^m$.*

Remark 5.7. In Corollary 5.6, if f has degree zero over $\mathbb{Z}_2$, then there always exists a point $x_0\in S^m$ with $f(x_0)=f(-x_0)$, which is known as a generalization of the classical Borsuk–Ulam theorem (see [8, Section 33]). Thus, in this case, we always have

$\beta_1(G_f(S^m) \cup G_g(S^m)) \equiv 1 \bmod 2$, provided that $\mathrm{d}f_x + \mathrm{d}f_{-x}$ is surjective for every $x \in S^m$ with $f(x) = f(-x)$.

In Propositions 5.3 and 5.4, when $M = N$ and g is the identity map, the coincidence set is called the *fixed point set* of f. In this case, $\Lambda_{f,\mathrm{id}} = \Lambda_f$ is the modulo 2 *Lefschetz number* [19, p. 247]. Setting $\Delta_M = \{(x,x) \in M \times M\}$, we obtain direct corollaries to Propositions 5.3 and 5.4 as follows.

Corollary 5.8. *Let $f : M \to M$ be a differentiable map of class C^2 of a closed m-dimensional manifold. We suppose that, for every x in the fixed point set F of f, $\mathrm{d}f_x : T_xM \to T_xM$ has no eigenvalue* 1. *Then $F = \emptyset$ if and only if $\Lambda_f = 0 \in \mathbb{Z}_2$ and* $\beta_1(G_f(M) \cup \Delta_M) = \beta_1(G_f(M)) + \beta_1(M)$ $(= 2\beta_1(M))$.

Corollary 5.9. *Let $f : M \to M$ be a differentiable map of class C^2 of a closed connected m-dimensional manifold. We suppose that, for every x in the fixed point set F of f, $\mathrm{d}f_x : T_xM \to T_xM$ has no eigenvalue* 1. *Then $F = \emptyset$ if and only if $\Lambda_f = 0 \in \mathbb{Z}_2$ and* $\beta_1(G_f(M) \cup \Delta_M) \equiv 0 \bmod 2$.

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