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Force-versus-time curves during collisions between two identical steel balls

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We describe a numerical procedure for plotting the force-versus-time curves in elastic collisions between identical conducting balls. A system of parametric equations relating the force and the time to a dimensionless parameter is derived from the assumption of a force compatible with Hertz's theory of collision. A simple experimental arrangement consisting of a mechanical system of colliding balls and an electrical circuit containing a crystal oscillator and an electronic counter is used to measure the collision time as a function of the energy of impact. From the data we can determine the relevant parameters. The calculated results agree very well with the expected values and are consistent with the assumption that the collisions are elastic.

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I. INTRODUCTION

Several techniques have been used to obtain the interaction between two objects in a collision as a function of time. The choice of the technique depends primarily on the interaction time between the colliding bodies. When this time is long enough, the position of the two bodies as a function of time can be recorded by using strobe photos[1] or a motion detector coupled to a data acquisition system.[2] The force-versus-time curve is then obtained from the position data set. Usually, the assumption of a sufficiently long interaction time is valid for two bodies that do not touch each other during the collision but that interact by action-at-a-distance forces. An example is the elastic collision between two gliders carrying appropriately positioned magnets that mutually repel as they travel toward each other on a linear air track. If there is physical contact between the colliding bodies, the duration of the collision is usually shorter. In this case the usual technique that allows us to study the evolution of the contact force uses a piezoelectric film placed between the colliding bodies.[3–7] This technique can be applied to both elastic and inelastic collisions and the force-time curve is obtained in real time; its disadvantage is that it is intrusive because the film may significantly change the coefficient of restitution.

The initial motivation of the work reported here was to study the evolution of the contact force in a head-on collision between two identical steel balls by means of a nonintrusive technique. Our approach is based on the assumptions that the collision between the balls is elastic and that the contact force acting on each ball during the collision has the form $F=-kx^{n-1}$, where x is the compression of the balls and k and n are parameters that must be determined empirically. The assumption of elastic collisions is reasonable unless the colliding balls are very large and/or the impact speed is so high as to produce permanent deformation of the surface atom layers of the balls.[8–10] The second assumption expresses a force law similar to Hooke's law. Indeed, the collision can be described as one between mass points interacting by springs.[7,11,12] If these springs behave like Hooke's-law springs, the collision time would be independent of the impact speed. However, experiment shows that the collision time depends on the impact speed and the assumed form of F takes this fact into account. In Sec. III we will show that the experimental results are consistent with this assumption and with Hertz's elastic theory of impact,[12–15] which predicts the exponent $n-1=3/2$.

II. THEORY

Assume the collision between two identical steel balls is as shown in Fig. 1. When at rest, the balls are in contact and the line joining their centers is horizontal (ξ-axis direction). Ball 1 is released from a known height h and strikes ball 2 at its center with speed u_1 with respect to the laboratory system. It is frequently more convenient to solve collision problems in the center-of-mass system instead of the laboratory system. At the initial instant of collision ($t=0$), the speed u^* of the center of mass of the balls in the laboratory system is $u^*=(u_1+u_2)/2=u_1/2$,[16] because the balls have equal mass ($m_1=m_2=m$) and $u_2=0$. The speed of ball 1 with respect to the center of mass is then $u_1-u^*=(1/2)u_1$. Consequently, the kinetic energy E^* of ball 1 in the center-of-mass system just before the impact is $1/4$ of its kinetic energy in the laboratory system,

$$E^* = mgh/4, \tag{1}$$

where g is the acceleration due to gravity.

When the incident ball comes into contact with the other ball, the contact extends over a small region of circular shape around point O [see Fig. 1(b)]. This point and the origin of the $\xi\eta$ axes (in the center-of-mass system) remain together during the collision. Due to local compression, ball 1 approaches point O. If x is the distance of approach, the approach speed of the incident ball with respect to point O is dx/dt.

The time during which the balls remain in contact, that is, the collision time τ, can be derived from energy conservation expressed in the center-of-mass system for ball 1. During the collision,

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E^*, \tag{2}$$

where $V(x)$ is the potential energy of deformation. For a contact force of the form $F(x)=-kx^{n-1}$, the potential energy is given by $V(x)=(k/n)x^n$. From this form and Eq. (2), it follows that

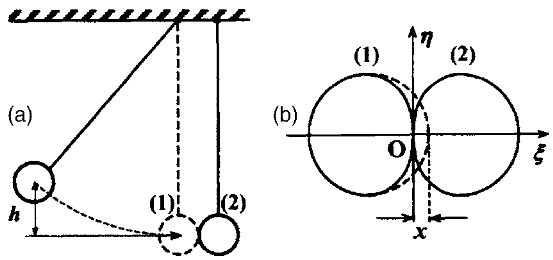

Fig. 1. Collision between two identical steel balls. (a) Ball 1 is released from a known height h and strikes ball 2, which is initially at rest. (b) Balls during the collision, where x is the amount of compression of each ball.

$$t(x) = \int_0^t dt = \sqrt{\frac{m}{2}} \int_0^x \frac{dw}{\sqrt{E^* - (k/n)w^n}}. \tag{3}$$

We write $w = x_0 z$, with x_0 the maximum distance of compression and z a dimensionless parameter $(0 \leqslant z \leqslant 1)$. The collision begins at $z=0$ and maximum compression is achieved at $z=1$. When $z=1$, the derivative in Eq. (2) vanishes $(dx/dt = 0)$, so that the energy $E^* = V(x_0) = (k/n)x_0^n$. Thus, the force and the time are coupled through the parametric equations,

$$F(z) = -kx_0^{n-1}z^{n-1},$$

$$t(z) = x_0 \sqrt{\frac{2}{gh}} \int_0^z \frac{dv}{\sqrt{1-v^n}}. \tag{4}$$

Equation (4) expresses the coordinates of the points on the $F(t)$ curve in the range $0 \leqslant t \leqslant \tau/2$ (compression stage) in terms of the parameter z. For an elastic collision it follows that the time $t(z)$ at $z=1$ is $1/2$ of the collision time τ, so that the amplitude x_0 can be written as

$$x_0 = \frac{\tau}{\gamma} \sqrt{\frac{gh}{8}}, \tag{5}$$

where $\gamma = \int_0^1 (dv/\sqrt{1-v^n})$.

Given that $x_0 = (nE^*/k)^{1/n}$ and $E^* = mgh/4$, we can also write

$$\tau = ph^q, \tag{6}$$

where $p = \gamma(nmg/4k)^{1/n}(8/g)^{1/2}$ and $q = (2-n)/2n$.

In our experiments we measure the quantities τ and h. The mass m can be easily determined and the acceleration g is known. The parameters q and p (and thus n and k) can be obtained by fitting the data for τ versus h using Eq. (6). Equation (6) shows that the force $F \propto x^{n-1}$ leads to a collision time $\tau \propto h^{(2-n)/2n}$. Note that τ increases with h (and E^*) for $n<2$, is independent of h for $n=2$, and decreases with increasing h for $n>2$.

III. EXPERIMENT

The experiment consists of two identical steel balls (ball bearings) suspended by copper wires from an insulated support (see Fig. 2). To fix the wires to the steel balls, we made a small cylindrical cavity (6 mm depth by 4 mm diameter) in each of the balls using the electron-erosion technique. A slightly larger brass cylinder with a wire pair previously welded to it was introduced under pressure in each cavity. The other ends of the wires were fixed to a carrier that slides

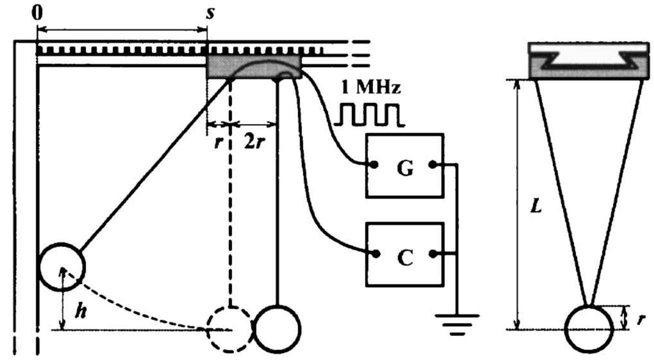

Fig. 2. Experimental setup. (a) Copper wires fixed to an insulated carrier suspend the steel balls. The height h is calculated from the position s of the carrier on the graduated rail and the vertical suspension distance L. The balls connect the oscillator G to the counter C at the beginning of the collision and disconnect it at the end. (b) Two wires were used to assure that the movement of the balls is restricted to one plane.

on a rail. A thumb-adjust lock knob securely clamps the carrier to the rail, which has an engraved scale in millimeters. The balls are in contact when at rest and the line joining their centers is horizontal. When ball 1 is pulled back until it touches the vertical pole of the setup, the height h is $h = L - (L^2 - s^2)^{1/2}$; the vertical suspension distance L is known and the length s is read directly on the rail scale. Ball 1 is then carefully released to strike ball 2 (initially at rest) at its center. The collision time τ is measured by connecting the output of a stable crystal oscillator[17] (operating at 1 MHz) to an electronic counter[18] through the switch formed by the two balls (see Fig. 2).[19]

The experiments were done using two different pairs of steel balls, (a) and (b). The balls of pair (a) have a diameter $d_a = 38.1$ mm and mass $m_a = 225.8$ g; for pair (b) $d_b = 27.0$ mm and $m_b = 80.2$ g. The vertical suspension distance $L = 50.0$ cm in both cases.

Figure 3 shows the collision time τ (averaged over 10 measurements) as a function of the height h. The dots represent the data and the solid lines correspond to the fits using Eq. (6). The fits were performed using a least-squares routine and the uncertainties were estimated by calculating the variances.[20] The best values for q and p and their uncertainties are in Fig. 3. These values were used to calculate n and

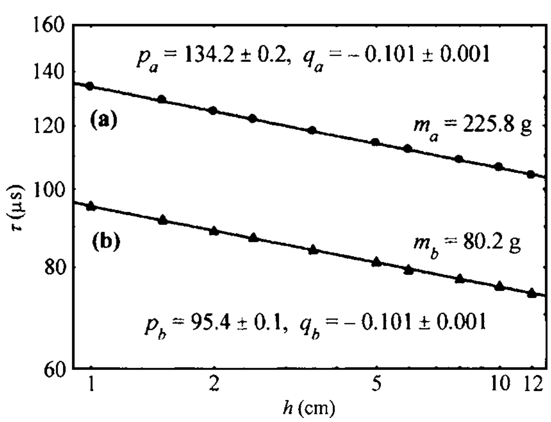

Fig. 3. The collision time τ as a function of the drop height h plotted on a log-log scale. The dots represent the experimental data and the solid lines correspond to the fits to Eq. (6).

Table I. Computed values obtained from the curve fits in Fig. 3. The definitions of the parameters are given in the text.

Balls	m (g)	n	k (cgs units)
(a)	225.8	2.51 ± 0.01	$(4.55 \pm 0.13) \times 10^{12}$
(b)	80.2	2.51 ± 0.01	$(3.67 \pm 0.11) \times 10^{12}$

k, which are shown in Table I. The value for n obtained in this way is in good agreement with the value $n = 5/2$ predicted by Hertz's theory of impact. If we compare Eq. (6) with the equivalent expression obtained by Hertz,[13] we find that $p = 1.47(2g)^{-1/10}[5\sqrt{2}\pi\rho(1-\nu^2)/4Y]^{2/5}d$, where ρ, Y, and ν are the specific mass, Young's modulus and Poisson's ratio of the material, respectively.

The proportionality between the parameter p and the diameter d of the balls is confirmed by the measurements, $p_a/p_b = 1.407$ and $d_a/d_b = 1.411$. By using the values $g = 976\ \text{cm/s}^2$, $\rho = 7.8\ \text{g/cm}^3$, $p_b = 95.4 \times 10^{-6}\ \text{cm}^{1/10}\ \text{s}$, $d_b = 2.70\ \text{cm}$, and $\nu = 0.29$,[21] we obtain $Y = 2.1 \times 10^{12}\ \text{dynes/cm}^2$, a reasonable value for the Young's modulus considering that for most steel balls, Y is between 2.0×10^{12} and $2.2 \times 10^{12}\ \text{dynes/cm}^2$.[21]

Although the experimental results show that the collision time obeys the relation $\tau \propto h^q$ over the range of h we studied, this result alone does not imply a force law of the form $F \propto x^{n-1}$. It was shown in Sec. II that a force of the form $F \propto x^{n-1}$ leads to the relation $\tau \propto h^q$, but not necessarily the other way around. The assumption that the opposite is also valid is supported by the good agreement between the experimental data and the results predicted by Hertz's theory.

Figure 4 shows examples of $F(t)$ for the two pairs of balls. The plotted curves correspond to the minimum and maximum height h used in the experiments. The coordinates $[t(z), F(z)]$ associated with the compression stage $(0 \leqslant t \leqslant \tau/2)$ are numerically calculated by substituting the values of k, n, g, x_0, and h into Eq. (4) and varying the parameter z from 0 to 1 in steps of 0.005 (200-point resolution).[22] The coordinates associated with the decompression stage in the time interval $\tau/2 \leqslant t \leqslant \tau$ are obtained by transforming the time $t(z) \to \tau - t(z)$, with z varying from 1 to 0. The Gaussian aspect of the curves is due to the fact that n is larger than 2 in the present case, because for $n = 2$, the curves would coincide with the first half cycle of a sine wave.

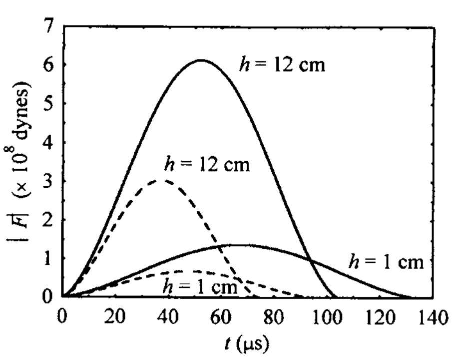

Fig. 4. Force-versus-time curves. The solid lines refer to the pair of balls (a), whereas the dashed ones to pair (b). Two curves were plotted for each pair, corresponding to the minimum and maximum height h used in the experiments.

The accuracy of the results can be verified by comparing the momentum change of the impinging ball with the impulse $J = \int_0^\tau F dt$ evaluated by a numerical integration of the force-versus-time curves. For an elastic collision, the impinging ball comes to rest after impact so that the magnitude of the momentum change is $P = m\sqrt{2gh}$. In the example for the pair of balls (a) at $h = 12$ cm, the impulse $J_a = 34\,630$ dynes s in comparison to $P_a = 34\,558$ dynes s, with a discrepancy of only 0.2%. For the lighter pair of balls (b) at $h = 12$ cm, the impulse $J_b = 12\,198$ dynes s in comparison to $P_b = 12\,274$ dynes s, with a 0.6% discrepancy.

IV. CONCLUSIONS

We have described a method to accurately determine $F(t)$ in an elastic collision by measuring the collision time for different impact energies. Our theoretical expressions apply to collisions between two identical steel balls, but the theory could be extended to more general cases such as the impact between balls with different masses. Our phenomenological approach allows us to illustrate several important concepts of mechanics and to verify the predictions of Hertz's theory of impact without recourse to complicated mathematics. The experiment is easy to perform and is appropriate for undergraduate laboratories.

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[16] See Ref. 1, Chaps. 8 and 11.
[17] Part ACO9923, Abracon Corporation, 29 Journey, Aliso Viejo, CA 92656, www.abracon.com
[18] Part CD4026BE, Texas Instruments, 12500 TI Boulevard, Dallas, TX 75243-4136, www.ti.com
[19] The transient switching occurring at times very near the initial (or final) contact is not considered in our model because the switch resistance goes to zero in an interval of time that is typically much shorter than the collision time. See, for example, Fig. 6(b) in I. Stensgaard and E. Laegsgaard, "Listening to the coefficient of restitution—revisited," Am. J. Phys.69, 301–305 (2001).
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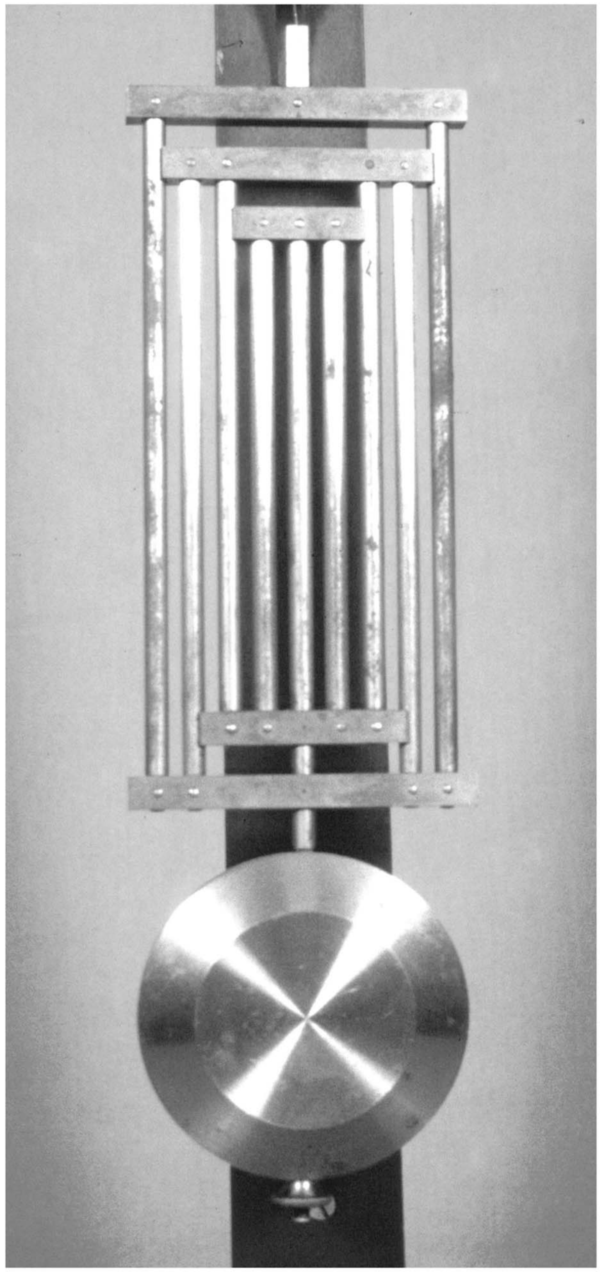
[22] The curves were plotted using the ParametricPlot command of Mathematica 4.1, www.wolfram.com



Gridiron Pendulum. The period of a pendulum depends on its length, but the length of a metal pendulum rod changes with the temperature; as the temperature goes up, the length increases and the clock controlled by the pendulum runs slow. In 1725 the Englishman John Harrison invented the gridiron pendulum to solve the problem of temperature compensation. Brass and iron rods, with different temperature coefficients of expansion, were fastened alternately top and bottom in a gridiron pattern to keep the center of mass at the same point. This gridiron pendulum is in the apparatus collection of Amherst College, and is post 1880. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)