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On the quantization of Poincaré and de Sitter gauge models

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The gauge model based on the Yang–Mills equations for the Poincaré group cannot be consistently quantized, at least in a perturbative approach. The regulated theory, obtained by adding the counterterms required by consistency and renormalizability, is just the gauge theory for a de Sitter group.

I. INTRODUCTION

Gauge theories for the Poincaré and de Sitter groups have been extensively studied as alternative theories for gravitation.[1] In this paper, "gauge theories" are to be considered as synonymous for models in which the field equations are the Yang–Mills equations for the group. That the gauge model for the Poincaré group could describe gravitation has already been shown elsewhere.[2] On the other hand, the quantization of such a model is expected from the start to face difficulties because of two peculiarities of the group: it is nonsemisimple and it acts on space-time itself. As a consequence of the first peculiarity, the Yang–Mills equations are not derivable from a Lagrangian.[2] As a result of the second peculiarity, all source fields belong, besides some tensor or spinor representation, to a "kinematic" representation whose generators are derivative fields on space-time. The number of derivatives appearing in currents and invariants is thereby augmented, representing a great threat to renormalizability. It will be shown here that such a model presents an inconsistency in the gauge field vertices, a problem that seems to stem from the absence of a Lagrangian. In order to illustrate what happens let us consider an unrealistic but instructive model. Suppose we did not know the Yukawa coupling Lagrangian $\mathscr{L}_I = g\varphi\Psi\overline{\Psi}$, but we had somehow arrived at the field equations in the form

$$\overrightarrow{D}\Psi = g\varphi\Psi, \tag{1.1}$$

$$\overline{\Psi}\overleftarrow{D} = -g\varphi\overline{\Psi}, \tag{1.2}$$

$$(\Box^2 + m^2)\varphi = g'\overline{\Psi}\Psi, \tag{1.3}$$

where $\overrightarrow{D} = i\gamma^\mu \partial_\mu - m$. Suppose further that we had some evidence (say, "experimental") that $g' \neq g$. This is a baffling situation from an intuitive point of view, but the problem can be made more definite if, ignoring the Lagrangian, we try to quantize the system by the Källén–Yang–Feldman (KYF) formalism.[3] The trouble is clear: as seen from the channels of Ψ and $\overline{\Psi}$, the coupling constant is g; as seen from the φ channel, it would be g'. The $\varphi\overline{\Psi}\Psi$ vertex obtained from Eqs. (1.1) and (1.2) would be different from that obtained from (1.3). This trivial remark points to a fundamental inconsistency of those equations, which are coherent only when $g = g'$. On the other hand, if we examine them in the light of Vainberg's theorem,[4] which gives necessary and sufficient conditions for the existence of a Lagrangian for a given set of equations, we find that $g = g'$ is necessary for (1.1)–(1.3) to be derivable from a Lagrangian.

We show in Sec. II, by using the KYF formalism, that this kind of inconsistency is present in the Yang–Mills equations for the Poincaré group.

The fact that the Poincaré group comes out as an Inönü–Wigner contraction limit of the de Sitter groups is exploited in Sec. III to provide more insight on the problem. The de Sitter groups being semisimple, a Lagrangian model can be built up, the path integral formalism may be used to supply the Feynman rules, and the Poincaré model is then seen as a limit case. The comparison of the de Sitter and Poincaré cases sheds some light on the way the inconsistencies, absent in the former, emerge in the latter. Geometrical considerations suggest that the de Sitter models can be viewed as smoothed versions of the Poincaré model.

Inconsistencies in field theories appear mainly when renormalization is involved, and sometimes find remedy in the addition of counterterms to the Lagrangian, with consequent modifications in the field equations. A notorious example is the electrodynamics of scalar mesons, which only becomes renormalizable if a self-interaction term $\lambda\varphi^4$ is added to the purely electromagnetic Lagrangian. As here no Lagrangian is at hand, we may think of changing the equations directly. A study of the possibilities arising in this line of thought is given in Sec. IV, where, by combining requirements of vertex consistency and renormalizability, successive counterterms are introduced in the Yang–Mills equations. Curiously enough, the final well-behaved resulting theory is just a de Sitter gauge model, which in this way appears as a "functionally corrected" Poincaré model.

II. VERTEX INCONSISTENCY

The Poincaré Lie algebra is the semidirect product of the Lorentz algebra and the algebra of the translations in space-time. It is convenient to use the double index notation $J_{\alpha\beta}$ ($\alpha,\beta = 1,2,3,4$, with $\alpha < \beta$), for the Lorentz generators and to take J_α for the translation generators. Individual indices can be raised and lowered by the Minkowski metric $\eta_{\alpha\beta}$.

Taking $A^\alpha{}_{\beta\mu}$ and $B^\gamma{}_\mu$ as the gauge potentials related, respectively, to the Lorentz sector (which constitutes a gauge subtheory) and the translation sector, the corresponding field tensors turn out to be[2]

$$F^{\alpha\beta}{}_{\mu\nu} = \partial_\mu A^{\alpha\beta}{}_\nu - \partial_\nu A^{\alpha\beta}{}_\mu - gA^\alpha{}_{\gamma\mu}A^{\gamma\beta}{}_\nu + gA^\alpha{}_{\gamma\nu}A^{\gamma\beta}{}_\mu, \tag{2.1}$$

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$$\tau^{\alpha}{}_{\mu\nu} = \partial_\mu B^{\alpha}_{\nu} - \partial_\nu B^{\alpha}{}_{\mu} - gA^{\alpha}{}_{\gamma\mu} B^{\gamma}_{\nu} + gA^{\alpha}{}_{\gamma\nu} B^{\gamma}{}_{\mu}. \qquad (2.2)$$

The Yang–Mills equations for the Poincaré group are

$$\partial_\mu F^{\alpha\beta\mu\nu} - gA^{\alpha}{}_{\gamma\mu} F^{\gamma\beta\mu\nu} + gF^{\alpha}{}_{\gamma}{}^{\mu\nu} A^{\gamma\beta}{}_{\mu} = gS^{\alpha\beta\nu}, \qquad (2.3)$$

$$\partial_\mu \tau^{\alpha\mu\nu} - gA^{\alpha}{}_{\gamma\mu} \tau^{\gamma\mu\nu} + gF^{\alpha}{}_{\gamma}{}^{\mu\nu} B^{\gamma}{}_{\mu} = gl^2 \theta^{\alpha\nu}, \qquad (2.4)$$

where $S^{\alpha\beta\nu}$ is the source spin density, $\theta^{\alpha\nu}$ is a source energy-momentum including coupling to the gauge fields, and l is the Planck length.

There is no Lagrangian density from which the above field equations can be derived.[2] It will be seen in Sec. IV that some pieces of this system of coupled equations can have Lagrangians, but the fact is that the whole system cannot. Attempts to redefine the fields so as to make the theory more tractable either disfigure its character by changing the meaning of the fundamental fields or make it trivial. For example, if the treatment used for the Korteweg–de Vries equation is applied here, the fields $B^{\alpha}{}_{\mu}$ must be some derivative $\partial_\mu \varphi^{\alpha}$, corresponding to the vacuum of the model.

In the absence of a Lagrangian, the natural way possibly open to quantization is the KYF formalism. It is convenient to use (2.1) and (2.2) in (2.3) and (2.4) so that equations acquire the form

$$\Box A^{\alpha\beta}{}_{\nu} - \partial_\nu (\partial^\mu A^{\alpha\beta}{}_{\mu})$$
$$= gV^{\alpha\beta}{}_{\nu}[A] - g^2 W^{\alpha\beta}{}_{\nu}[A] + S^{\alpha\beta}{}_{\nu}, \qquad (2.5)$$

$$\Box B^{\alpha\nu} - \partial^\nu (\partial^\mu B^{\alpha}{}_{\mu})$$
$$= gU^{\alpha\nu}_{\beta\lambda}[A] B^{\beta\lambda} - g^2 Z^{\alpha\nu}_{\beta\lambda}[A] B^{\beta\lambda} + l^2 \theta^{\alpha\nu}, \qquad (2.6)$$

where

$$V^{\alpha\beta}{}_{\nu}[A] = (A_{\gamma}{}^{\beta}{}_{\rho} \delta^{\alpha}{}_{\epsilon} - A^{\alpha}{}_{\epsilon\rho} \delta_{\gamma}{}^{\beta})$$
$$\times (\delta_\nu{}^{\rho} \partial^{\sigma} - \eta^{\rho\sigma} \partial_\nu) A^{\epsilon\gamma}{}_{\sigma}, \qquad (2.7)$$

$$W^{\alpha\beta}{}_{\nu}[A] = (\delta_{\epsilon}{}^{\alpha} \eta^{\beta\gamma} - \delta_{\epsilon}{}^{\beta} \eta^{\alpha\gamma})$$
$$\times (\delta_\nu{}^{\sigma} \eta^{\lambda\rho} - \delta_\nu{}^{\lambda} \eta^{\sigma\rho}) A^{\epsilon}{}_{\varphi\rho} A^{\varphi}{}_{\delta\lambda} A^{\delta}{}_{\gamma\sigma}, \qquad (2.8)$$

$$U^{\alpha\nu}_{\beta\lambda}[A] = \delta_\lambda{}^{\nu} (\partial_\mu A^{\alpha}{}_{\beta}{}^{\mu} + 2A^{\alpha}{}_{\beta\mu} \partial^\mu)$$
$$- A^{\alpha}{}_{\beta}{}^{\nu} \partial_\lambda - A^{\alpha}{}_{\beta\lambda} \partial^\nu$$
$$+ \partial^\nu A^{\alpha}{}_{\beta\lambda} - 2\partial_\lambda A^{\alpha}{}_{\beta}{}^{\nu}, \qquad (2.9)$$

$$Z^{\alpha\nu}_{\beta\lambda}[A] = A^{\alpha}{}_{\gamma\mu} (A^{\gamma}{}_{\beta}{}^{\mu} \delta_\lambda{}^{\nu} - 2A^{\gamma}{}_{\beta}{}^{\nu} \delta_\lambda{}^{\mu})$$
$$+ A^{\alpha}{}_{\gamma}{}^{\nu} A^{\gamma}{}_{\beta\lambda}. \qquad (2.10)$$

Gauge-fixing terms should be added to the left-hand side but they will not be important for the argument that follows.

Let us consider the sourceless case. To simplify the discussion, we shall rewrite (2.5) and (2.6) symbolically as

$$KA = gV[A] - g^2 W[A], \qquad (2.11)$$

$$KB = gU[A]B - g^2 Z[A]B. \qquad (2.12)$$

In the KYF formalism, we look for a perturbative solution in the form

$$A = \mathring{A} + gK^{-1}V[A] - g^2 K^{-1} W[A], \qquad (2.13)$$

$$B = \mathring{B} + gK^{-1}U[A]B - g^2 K^{-1} Z[A]B. \qquad (2.14)$$

Iteration to the desired order is then performed by replacing the A's and B's successively in terms of the free solutions $\mathring{A}$ and $\mathring{B}$. The operator K^{-1} represents a convolution with the Green's function of the differential operator in the lhs of (2.5) and (2.6) with Feynman boundary conditions.[5] We shall refer to K^{-1} simply as the Feynman propagator in some supposedly fixed gauge. The Feynman rules are obtained by projecting each one of these perturbative solutions on outgoing fields of the same kind. Each time they "hit" the free propagator, these outgoing fields produce free-fields of the same kind, so that the first contributions give precisely the basic vertices for the Feynman rules. In the case (2.11), such vertices are of the form $gAV[A]$ and $g^2 AW[A]$ and from them just the expected three-leg and four-leg vertices for a gauge model for the Lorentz group are obtained. Equation (2.11) is, of course, a set of coupled equations, one for each potential $A^{\alpha}{}_{\beta\mu}$. Take, for instance, the component $A^{1}{}_{2\mu}$. The projection is to be made on an outgoing field, $\mathring{A}^{1}{}_{2\mu}$, of exactly the same kind. Other potentials $\mathring{A}^{2}{}_{3\mu}$, $\mathring{A}^{3}{}_{0\mu}$, etc. appear in the vertices. In the equations for $A^{2}{}_{3\mu}$ and $A^{3}{}_{0\mu}$, the projections are made on outgoing fields $\mathring{A}^{2}{}_{3\mu}$ and $\mathring{A}^{3}{}_{0\mu}$, respectively. The important point is that the three-leg vertex involving $A^{1}{}_{2\mu}$, $A^{2}{}_{3\mu}$, and $A^{3}{}_{0\mu}$ will appear the same when obtained from each one of their respective equations. In other words, the expression for a vertex can be obtained from the equation related to any of its legs, and the result is independent of the choice of the leg. This general fact of perturbative field theory is easily found for (2.11), which are in reality the Yang–Mills equations for a Lorentz gauge model. Ghost fields could be introduced in principle through the old laborious Feynman method,[6,7] but (2.11) alone has a Lagrangian and in fact it would be simpler to pursue the whole treatment for the Lorentz sector by the path integration methods.

Now we come to the main point. The same considerations above, when applied to the whole set (2.11) and (2.12), lead to an insurmountable difficulty: vertices like $gB(\partial A)B$, $gBA(\partial A)$, and $g^2 BAAB$ do come out from (2.12) but not from (2.11). There are AB couplings in (2.12) but no field B appears in (2.11). Thus the expression for a vertex is no longer obtained from the equation for any of its legs, it is now dependent on the choice of the leg. With some freedom of language, we might say that the B's are able to "feel" the A's, but not the other way round. Or still, that vertices involving B's and A's are present for outgoing B's but not for outgoing A's. The same kind of inconsistency would appear in our defective Yukawa model [(1.1)–(1.3)] with $g = 0$ and $g' \neq 0$.

From this fundamental vertex inconsistency we conclude that, at least from the point of view of the KYF formalism, a model with (2.3) and (2.4) as field equations is not amenable to quantization.

III. RELATION TO DE SITTER MODELS

For usual gauge models, it is simpler to obtain the whole set of Feynman rules by the path integral approach and it will be instructive to examine our special case in the light of this standard procedure. It requires a Lagrangian, which is missing, but we can resort to the well-known fact that the Poincaré group P is an Inonü-Wigner contraction[8] of the two de Sitter (dS) groups.[9] As dS is semisimple, we can easily write down both the Yang–Mills equations and the corresponding Lagrangian for a dS gauge model. The comparison of the two cases will allow us to see why and where the procedure breaks down in the Poincaré model.

The relations between classical gauge models for P and dS have been studied in detail[2,10] and here we shall only recall the main points. In order to see what happens to gauge fields in the contraction process it is convenient to look at the contraction as acting on the group parameters ω^{ab} ($a,b = 1,...,5$, $a<b$). The parameters $\omega^{\alpha\beta}$ ($\alpha,\beta = 1,...,4$), related to the Lorentz subgroup, remain untouched. The parameters $\omega^{\alpha 5}$ represent "rotation" angles, compact or not, depending on the relative sign of $\eta_{\alpha\alpha}$ and η_{55}, where η_{ab} is the diagonalized dS invariant metric. Contraction requires redefining such angles as $L\omega^{\alpha 5} = a^{\alpha}$, where the a^{α} are the translation parameters and L is a length parameter taken to infinity in the contraction limit. A translation is thereby viewed as the limit of some infinitesimal rotation with an infinite radius. The dS generators J_{ab} obey

$$[J_{cd}, J_{ef}] = -if^{ab}{}_{cd,ef}\, J_{ab}, \tag{3.1}$$

where

$$f^{ab}{}_{cd,ef} = \eta_{de}\delta_c^{[a}\delta_f^{b]} - \eta_{df}\delta_c^{[a}\delta_e^{b]} - \eta_{ce}\delta_d^{[a}\delta_f^{b]} + \eta_{cf}\delta_d^{[a}\delta_e^{b]}, \tag{3.2}$$

with $[ab]$ meaning antisymmetrization in the indices. If $A^{ab}{}_{\mu}$ are the gauge potentials for the dS gauge model, then $A^{\alpha\beta}{}_{\mu}$ remain the same through the contraction process, but $A^{\alpha 5}{}_{\mu}$ must be redefined so that $A^{\alpha 5}{}_{\mu} = L^{-1}B^{\alpha}{}_{\mu}$, where $B^{\alpha}{}_{\mu}$ is the translation gauge potential of the previous section. This can be checked, for example, by comparing the vacuum potentials $A^{\alpha 5}{}_{\mu} = \partial_{\mu}\omega^{\alpha 5}$ and $B^{\alpha}{}_{\mu} = \partial_{\mu}a^{\alpha}$. By the same process, if $F^{ab}{}_{\mu\nu}$ are the dS field strengths, the $F^{\alpha\beta}{}_{\mu\nu}$ become the field strengths (2.1) related to the Lorentz subgroup, while $\tau^{\alpha}{}_{\mu\nu} = LF^{\alpha 5}{}_{\mu\nu}$ become the translation field strengths (2.2). The Yang–Mills equations for the dS model,

$$\partial_{\mu}F^{ab\mu\nu} - gA^{a}{}_{c\mu}F^{cb\mu\nu} + gF^{a}{}_{c}{}^{\mu\nu}A^{cb}{}_{\mu} = 0, \tag{3.3}$$

reduce exactly to the sourceless versions of (2.3) and (2.4) in the contraction limit $L \to \infty$, and the same happens to the corresponding Bianchi identities.

The contraction procedure has been frequently used to approach questions involving P,[11] mainly because it allows a point to point comparison to the better behaved dS group. It has been so in the demonstration of the nonexistence of a Lagrangian for the set of equations (2.3) and (2.4).[2] Equation (3.3) comes from the typical Lagrangian

$$\mathscr{L} = -\tfrac{1}{4}F^{ab}{}_{\mu\nu}F_{ab}{}^{\mu\nu}, \tag{3.4}$$

in which the algebra double indices are lowered and raised by the Cartan–Killing metric of dS. In the contraction limit, such a metric becomes degenerate and the field equations lose some terms. In particular, the cubic term in B, present in (3.3) and related to the four-leg vertex typical of gauge theories, is suppressed (as discussed below).

Path integral quantization can be performed without too much ado and Feynman rules of the usual kind are obtained for the dS model. For the P model, we start by making the substitution $A^{\alpha 5}{}_{\mu} = L^{-1}B^{\alpha}{}_{\mu}$ and follow the same procedure while keeping in mind what happens at the limit. Also, the ghost fields with ($\alpha 5$) indices must be substituted in an analogous way, but as they will not be important for our central problem we shall not discuss them. In reality we shall concentrate on the inconsistency of the P model, leaving aside all the details having no bearing upon it. Once the substitution is made, (3.4) becomes

$$\mathscr{L} = -\tfrac{1}{4}[(F^{\alpha\beta}{}_{\mu\nu} + gL^{-2}B^{\alpha}{}_{\mu}B^{\beta}{}_{\nu})^2 + L^{-2}(\tau^{\alpha}{}_{\mu\nu})^2]. \tag{3.5}$$

It is clear that the limit cannot be taken immediately: only the part

$$\mathscr{L}_{\mathscr{L}} = -\tfrac{1}{4}(F^{\alpha\beta}{}_{\mu\nu})^2, \tag{3.6}$$

corresponding to a Lorentz group gauge model, with only (2.3) for the field equations, would remain. As is frequently the case in the contraction formalism, we should first perform all the calculations and take the limit at the last step, although here we shall keep an eye on the relations to the field equations. Because it will be enough to make our point, we shall only examine in detail the three-field vertex: (3.5) is written as

$$\begin{aligned}\mathscr{L} = \mathscr{L}_{\mathscr{L}} - (L^{-2}/4)\,[\,&(\partial_{[\mu}B^{\alpha}{}_{\nu]})^2 \\ &+ 2gf_{\alpha\beta,\gamma 5,\epsilon 5}\,(\partial_{[\mu}A^{\alpha\beta}{}_{\nu]}B^{\gamma}{}_{\mu}B^{\gamma}{}_{\nu}B^{\epsilon}{}_{\nu} \\ &+ \partial_{[\mu}B^{\epsilon}{}_{\nu]}A^{\alpha\beta}{}_{[\mu}B^{\gamma}{}_{\nu]}) + o(g^2)\,],\end{aligned} \tag{3.7}$$

where we have kept the dS structure constants (3.2).

We can obtain (2.3) and (2.4) from (3.5) simply by taking variations with respect to $A_{\alpha\beta\nu}$ and $B_{\alpha\nu}$, respectively, and then taking $L \to \infty$. An important point is that (2.4) is obtained with an overall factor L^{-2}, which cancels out. A consequence is that the contributions coming from the three-field terms in (3.7), proportional to L^{-2}, will remain in (2.4) but will be suppressed in (2.3). We see in this way how it happens that the BA coupling, present in (2.4), vanishes in (2.3), and find the same inconsistency of the previous section. The same happens to the terms A^2B^2 omitted in (3.7). The terms in B^4 have a L^{-4} factor and are totally suppressed.

Another consequence of (3.5) is that, once the $B^{\alpha}{}_{\mu}$ become (beside the $A^{\alpha\beta}{}_{\nu}$) the fundamental fields in substitution to the $A^{\alpha 5}{}_{\mu}$, the conjugate momenta become ill-defined. The vanishing of their time components is usual in a gauge theory, but here also the space components vanish: the momenta conjugate to $B^{\alpha}{}_{j}$ is $\pi^{\alpha}{}_{j} = L^{-2}\tau^{\alpha}{}_{j4}$, so that in the limit the canonical quantization is jeopardized.

In the Feynman rules for gauge models, the group dependence rests basically in the structure constants,[12] whose cyclic symmetry is used precisely to make the vertices symmetric in the external legs.[13] The cyclic symmetry is absent for nonsemisimple groups, which suggests that the inconsistency found here might be a common illness of all models involving such groups.

We have seen that, as long as we take the Yang–Mills equations as the very foundations of the theory, the Poincaré model is inevitably inconsistent. Let us forget the equations for a moment and use the contraction procedure to obtain a quantized theory. This amounts to taking (3.5) seriously and obtaining the resulting Feynman rules. The task is rather lengthy albeit standard. The results are simple and, once found, easily understood. Here we shall only describe the main points of the resulting theory, trying to justify them by general arguments.

(i) The Lorentz sector constitutes a gauge subtheory, with the usual rules.

(ii) As seen in (3.5), the propagator of the B fields will be just the usual one, in some fixed gauge, times a factor L^2; the same applies to the corresponding ghosts.

(iii) Vertices are as usual, with the difference that each B leg (or corresponding ghost) gains a factor L^{-1} (an obvious consequence of the $A^{\alpha 5}{}_{\mu} \to L^{-1} B^{\alpha}{}_{\mu}$ substitution).

Note that no final factor of L comes out from internal B lines in a diagram, since the L^2 factor in the propagator is just compensated by the L^{-1} factors in the two vertices connected. Graphs with external B legs will retain L^{-1} factors. However, if we calculate an S matrix element with N external B legs, the same L^{-N} factor will appear in each term in the perturbative series and, consequently, cancel out. Only when graphs with different numbers of external B legs are compared will the L^{-1} factors play a role.

The geometric setting for a P gauge model is best seen as an associated bundle, with Minkowski space as the base manifold and the fibers being tangent (also Minkowski) spaces on which the group acts. In the analogous setting for a dS model,[2] each tangent space is replaced by a dS space characterized by a length parameter L. When $L \to \infty$, each dS space approaches a tangent Minkowski space. If we use conformal coordinates[8] for each dS space, its points will be projected on a Minkowski space. In such coordinates, the natural dS group parameters are precisely $\omega^{\alpha\beta}$ and a^{α}, and the gauge fields become naturally $A^{\alpha\beta}{}_{\mu}$ and $B^{\alpha}{}_{\mu}$. The quantized theory sketched above is in reality a dS model, viewed in conformal coordinates. To use an analogy, a dS model stands to a P model like a parabola to its asymptote, which is approached more and more when L becomes larger and larger, but it is never really attained. The dS model appears as a "smoothing" of the incongruous P model and seems to be its nearest quantizable theory. In Sec. IV we shall arrive again at a dS model from a rather different approach.

IV. CONSISTENCY AND LAGRANGIAN CHARACTER

Lagrangian theories do not exhibit the inconsistency described above. We could ask whether or not vertex consistency implies the presence of a Lagrangian or, in other words, whether only Lagrangian theories are quantizable in a coherent way. We shall not consider this very general question here. We shall restrict ourselves to Eqs. (2.3) and (2.4) in the sourceless case and proceed to a kind of naive patchwork, trying to see which terms should be dropped or added to make them into consistent equations. We find that every time they become consistent, they also become derivable from a Lagrangian.

We can start by simply dropping all terms coupling B to A in (2.4). The field equations become

$$\partial_{\mu} F^{\alpha\beta\mu\nu} - g A^{\alpha}{}_{\gamma\mu} F^{\gamma\beta\mu\nu} + g F^{\alpha}{}_{\gamma}{}^{\mu\nu} A^{\gamma\beta}{}_{\mu} = 0, \tag{4.1}$$

$$\partial_{\mu} (\partial^{\mu} B^{\alpha\nu} - \partial^{\nu} B^{\alpha\mu}) = 0, \tag{4.2}$$

which are derivable from the Lagrangian $\mathscr{L} = -\frac{1}{4}(F^{\alpha\beta}{}_{\mu\nu})^2 - \frac{1}{4}(\partial^{\mu} B^{\alpha\nu} - \partial^{\nu} B^{\alpha\mu})^2$. They are the field equations of gauge models for the Lorentz group $\mathscr{L}$ and for the Abelian translation group $T_{3,1}$. Their set would describe a model for the direct product $\mathscr{L} \otimes T_{3,1}$. Notice, however, that, as the fields $B^{\alpha}{}_{\mu}$ are Lorentz vector fields, they should in reality couple to a Lorentz gauge potential. We take this into account by treating $B^{\alpha}{}_{\mu}$ as a source field: usual derivatives are replaced by covariant ones and a source current appears in (4.1). As $B^{\alpha}{}_{\mu}$ is a vector, it is its rotational that goes into the covariant derivative $\tau^{\alpha}{}_{\mu\nu}$ given in (2.2). Also the divergence in (4.2) becomes covariant. Vertex consistency then fixes the source current, and the new equations are

$$\partial_{\mu} F^{\alpha\beta\mu\nu} - g A^{\alpha}{}_{\gamma\mu} F^{\gamma\beta\mu\nu} + g F^{\alpha}{}_{\gamma}{}^{\mu\nu} A^{\gamma\beta}{}_{\mu} = g \tau^{\alpha\mu\nu} B^{\beta}{}_{\mu}, \tag{4.3}$$

$$\partial_{\mu} \tau^{\alpha\mu\nu} - g A^{\alpha}{}_{\gamma\mu} \tau^{\gamma\mu\nu} = 0. \tag{4.4}$$

These equations are derivable from $\mathscr{L} = -\frac{1}{4}F^2 - \frac{1}{4}\tau^2$, from which it can be checked that the source current in (4.3) is, as it should be, the spin density. We have been treating $B^{\alpha}{}_{\mu}$ as "normal" vector fields with the canonical dimension $(\text{mass})^1$. In reality, they have a defective dimension, as is clear from the redefinition $A^{\alpha 5}{}_{\mu} = L^{-1} B^{\alpha}{}_{\mu}$ used in the contraction procedure. In order to correct this in the above equations, it is enough to add a factor L^{-1} to each $B^{\alpha}{}_{\mu}$ field (and consequently to every $\tau^{\alpha}{}_{\mu\nu}$). The only novelty will be a factor L^{-2} in the spin density.

We can now compare the resulting equations with the sourceless cases of (2.3) and (2.4); the only difference is the term $g F^{\alpha}{}_{\gamma}{}^{\mu\nu} B^{\gamma}{}_{\mu}$ in (2.4). If we simply add this term to (4.4), vertex inconsistency comes out, but now we can relate it to a simple cause: such a term is obtained from a Lagrangian $\mathscr{L}' = -(g/2) F_{\alpha\beta}{}^{\mu\nu} B^{\alpha}{}_{\mu} B^{\beta}{}_{\nu}$ when variations are taken with respect to $B_{\alpha\nu}$; however, $\mathscr{L}'$ should also contribute to (2.3) or (4.3) through its variations with respect to $A_{\alpha\beta\nu}$. This contribution to (2.3) reestablishes vertex consistency. The new Lagrangian,

$$\mathscr{L} = -\tfrac{1}{4} F_{\alpha\beta}{}^{\mu\nu} (F^{\alpha\beta}{}_{\mu\nu} + 2gL^{-2} B^{\alpha}{}_{\mu} B^{\beta}{}_{\nu}) - 4L^{-2} \tau_{\alpha}{}^{\mu\nu} \tau^{\alpha}{}_{\mu\nu}, \tag{4.5}$$

leads to a rather complicated theory. Then comes a beautiful point: this theory, as it is, is nonrenormalizable because of the graphs with four external B legs and exchange of two or more A's. When we look for the necessary counterterms, we find that $[-(g^2/4) B^{\alpha}{}_{\mu} B^{\beta}{}_{\nu} B_{\alpha}{}^{\mu} B_{\beta}{}^{\nu}]$ must be added to (4.5). This is quite natural for the four-legged graphs because they have a zero divergence degree. This situation is analogous to the case of scalar electrodynamics, where the renormalization of the higher order graphs with four external scalar legs, also of vanishing divergence degree, enforces the presence of a $\lambda\varphi^4$ term in the Lagrangian.[12] The addition of the B^4 term puts (4.5) into the form

$$\mathscr{L} = -\tfrac{1}{4}(F_{\alpha\beta}{}^{\mu\nu} + gL^{-2} B_{\alpha}{}^{\mu} B_{\beta}{}^{\nu})^2 - (L^{-2}/4)(\tau_{\alpha}{}^{\mu\nu})^2. \tag{4.6}$$

This is the same Lagrangian as (3.5). The added B^4 term leads to a cubic term in (2.4), just that one we have seen suppressed by contraction in Sec. III. Therefore, summing up, by adding to (2.3) and (2.4) the terms necessary to wash out the vertex inconsistency, and then adding a last term to make the model renormalizable, we arrive at a de Sitter theory.

V. FINAL COMMENTS

The absence of a Lagrangian is a most grievous flaw in a field theory. In the case considered here, the group contraction procedure can be used to show that the conjugate momenta of the translation gauge potentials are vanishing, so precluding a coherent canonical quantization. The existence of a Lagrangian for the Yang–Mills equation is closely related to the structure constants cyclic symmetry,[2] which fails for nonsemisimple groups. Such a symmetry is used to obtain the Feynman rules for gauge models,[13] which have consequently to be reexamined. We have seen that, for the Poincaré group, the very definition of a vertex becomes impossible and quantization, at least in a perturbative approach, unfeasible. The addition of counterterms required by consistency leads to an intricate theory. Interestingly enough, the addition of a B^4 term required by renormalizability turns the model into a gauge theory for the de Sitter group, which appears as the nearest coherently quantizable theory.

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