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Various Green functions of the Dirac equation with a magnetic-solenoid field (the superposition of the Aharonov–Bohm field and a collinear uniform magnetic field) are constructed and studied. The problem is considered in $2+1$ and $3+1$ dimensions for the natural extension of the Dirac operator (the extension obtained from the solenoid regularization). Representations of the Green functions as proper time integrals are derived. The nonrelativistic limit is considered. For the sake of completeness the Green functions of the Klein–Gordon particles are constructed as well. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1699483]

I. INTRODUCTION

In the present paper we continue our previous study[1–3] of the Dirac equation with a magnetic-solenoid field, constructing and studying various Green functions of this equation. We recall that the magnetic-solenoid field is the collinear superposition of the constant uniform magnetic field and the Aharonov–Bohm (AB) field. The AB field is a field of an infinitely long and infinitesimally thin solenoid. Recently the interest in such a field configuration has been renewed in connection with planar physics problems, quantum Hall effect, and the Aharonov–Bohm effect in cyclotron and synchrotron radiations.[4–9]

In principle, the Green functions can be constructed whenever complete sets of solutions of the Dirac equations are available. In this connection, one should recall that solutions of the Dirac equation with the magnetic-solenoid field in $2+1$ and $3+1$ dimensions were obtained in Ref. 1. The singularity of the AB field demands a special attention to the correct definition of the Dirac operator. The need for self-adjoint extensions in the case of the Dirac Hamiltonian with the pure AB field in $2+1$ dimensions was recognized in Refs. 10 and 11 where certain boundary conditions at the origin were established. The regularized case and peculiarities of the behavior of a spinning particle in the presence of the magnetic string were considered in Refs. 12 and 13. The problem of the self-adjoint extension of the Dirac operator with the magnetic-solenoid field was studied in Refs. 2, 3, and 14. In $2+1$ dimensions, a one-parametric family of self-adjoint Dirac Hamiltonians specified by the corresponding boundary conditions at the AB solenoid was constructed, and the spectrum and eigenfunctions for each value of the extension parameter were found. In $3+1$ dimensions, a two-parametric family of the self-adjoint Dirac Hamiltonians was constructed on the condition that the spin polarization is conserved. The corresponding spectrum and eigenfunctions for each value of the extension parameters were found as well. In Refs. 2 and 3 the procedure of solenoid regularization was also considered. The procedure implies considering the finite solenoid and then making its radius go to zero. This procedure specifies some particular boundary conditions. The values of the extension parameters corresponding to the solenoid regularization case were determined in $2+1$ and $3+1$ dimensions. Further, we call the corresponding extension the natural extension. Nonrelativistic propagators for the spinless and spin-1/2 particle

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moving in the pure AB field were considered mainly in the relation to the AB effect. The propagator of the spinless particle was found in Refs. 15, 16, and 17 as a sum of partial propagators corresponding to homotopically different paths in the covering space of the physical background. The nonrelativistic propagator of the spin-1/2 particle in the AB field for a particular value of the self-adjoint extension parameter was discussed in Ref. 18. The relativistic scalar case for the AB field was studied in Ref. 19. The propagators and the AB effect in general gauge theories were considered in Refs. 20 and 21. Recently, vacuum polarization effects in the AB field have aroused great interest, see, for example, Refs. 22 and 23 and references therein.

In the present paper, we construct and study the Green functions of the Dirac particle in the magnetic-solenoid field in $2+1$ and $3+1$ dimensions. The physical importance of the problem is stressed by the fact that the knowledge of the Green functions in such a configuration allows one to study quantum (and quantum field) effects in the magnetic-solenoid field on a regular base. A technical specificity of the problem is related to the necessity to take into account all the peculiarities related to the self-adjoint extension problem of the Dirac operator in the background under consideration. In Sec. II we consider the $(2+1)$-dimensional case in detail. Here, constructing the Green functions, we use the exact solutions of the Dirac equation that are related to the specific values of the extension parameter. These values correspond to the natural extension, see above. The representations of the Green functions as proper time integrals are derived. In addition, we calculate the nonrelativistic Green functions as well. In Sec. III we extend the results to the $(3+1)$-dimensional case. In the Appendix, for the sake of completeness, we present the Green functions of the relativistic scalar particle.

We note that the magnetic-solenoid field belongs to such type of fields that do not violate the vacuum stability. For such fields a unique stable vacuum exists, and quantum field definitions of the Green functions below hold true.[24] In particular, the causal propagator $S^c(x,x')$ and the anticausal propagator $S^{\bar{c}}(x,x')$ are defined by the expressions

$$S^c(x,x') = i\langle 0|T\hat{\psi}(x)\hat{\bar{\psi}}(x')|0\rangle, \tag{1}$$

$$S^{\bar{c}}(x,x') = i\langle 0|\hat{\psi}(x)\hat{\bar{\psi}}(x')T|0\rangle, \tag{2}$$

where $\hat{\psi}(x)$ is the quantum spinor field in the Furry representation, satisfying the Dirac equation with the magnetic-solenoid field, $|0\rangle$ is the vacuum in this representation. The symbol of the T-product acts on both sides: it orders the field operators to its right-hand side and antiorders them to its left-hand side. The functions $S^c(x,x')$, $S^{\bar{c}}(x,x')$ can be expressed via the functions $S^{\mp}(x,x')$,

$$S^c(x,x') = \theta(\Delta x^0)S^-(x,x') - \theta(-\Delta x^0)S^+(x,x'), \quad \Delta x^0 = x^0 - x^{0\prime}, \tag{3}$$

$$S^{\bar{c}}(x,x') = \theta(-\Delta x^0)S^-(x,x') - \theta(\Delta x^0)S^+(x,x'), \tag{4}$$

and the latter can be calculated via a complete set ${}_{\pm}\psi_a(x)$ of solutions of the Dirac equation with the magnetic-solenoid field as

$$S^{\mp}(x,x') = i\sum_a {}_{\pm}\psi_a(x)\,{}_{\pm}\bar{\psi}_a(x'). \tag{5}$$

The solutions with the subscript $(+)$ belong to the positive energy spectrum, whereas the solutions with the subscript $(-)$ belong to the negative energy spectrum. Via a all possible quantum numbers are denoted.

The Dirac equation with the magnetic-solenoid field has the form

$$(\gamma^\nu P_\nu - M)\psi(x) = 0. \tag{6}$$

Here $P_\nu = i\partial_\nu - qA_\nu(x)$, $x=(x^\nu)$, q is an algebraic charge, for electrons $q=-e<0$, M is the electron mass, and $A_\nu(x)$ are potentials of the magnetic-solenoid field. In the $(3+1)$-dimensional case $\nu=0,1,2,3$ and γ^ν are the corresponding gamma-matrices. In the $(2+1)$-dimensional case $\nu=0,1,2$ and in what follows, we employ the letter Γ to denote the gamma-matrices. We use for these matrices the following representation:

$$\Gamma^0=\sigma^3, \quad \Gamma^1=i\sigma^2, \quad \Gamma^2=-i\sigma^1,$$

where σ^i are the Pauli matrices. In cylindric coordinates (φ,r), $x^1=r\cos\varphi$, $x^2=r\sin\varphi$, the potentials of the magnetic-solenoid field have the form

$$A_0=0, \quad eA_1=[l_0+\mu+A(r)]\frac{\sin\varphi}{r}, \quad eA_2=-[l_0+\mu+A(r)]\frac{\cos\varphi}{r},$$

$$(A_3=0 \text{ in } 3+1), \quad A(r)=eBr^2/2. \tag{7}$$

Here B is the magnitude of the uniform magnetic field, and the magnitude B^{AB} of the AB field is given by the expression $B^{\mathrm{AB}}=\Phi\,\delta(x^1)\,\delta(x^2)$, where Φ is the AB-solenoid flux, $(l_0+\mu)=\Phi/\Phi_0$, $\Phi_0=2\pi/e$. It is supposed that l_0 is integer and $0\leqslant\mu<1$.

The functions $S^{\mp}(x,x')$ obey the Dirac equation (6), whereas the causal and anticausal propagators obey the nonhomogeneous Dirac equations:

$$(\gamma^\nu P_\nu - M)S^c(x,x') = -\delta(x-x'), \quad (\gamma^\nu P_\nu - M)S^{\bar{c}}(x,x') = \delta(x-x').$$

We note that the commutation function $S(x,x')$, the advanced $S^{\mathrm{adv}}(x,x')$ and the retarded $S^{\mathrm{ret}}(x,x')$ Green functions can be expressed in terms of $S^c(x,x')$, $S^{\bar{c}}(x,x')$ as follows:

$$S(x,x')=S^-(x,x')+S^+(x,x')=\mathrm{sgn}(\Delta x^0)[S^c(x,x')-S^{\bar{c}}(x,x')], \tag{8}$$

$$S^{\mathrm{adv}}(x,x')=-\theta(-\Delta x^0)S(x,x'), \quad S^{\mathrm{ret}}(x,x')=\theta(\Delta x^0)S(x,x'). \tag{9}$$

II. 2+1 DIMENSIONAL CASE

A. Sets of exact solutions

First we study the $(2+1)$-dimensional case, for which, as known,[2,3] the Dirac operator with the magnetic-solenoid field in $2+1$ dimensions possesses a one-parameter family of self-adjoint extensions. That provides a one-parameter family of boundary conditions at the origin. Following Refs. 2 and 3, we denote the extension parameter as Θ. Generally speaking, the AB symmetry is violated for the spinning particle, which is therefore sensible to the solenoid flux sign. As was demonstrated in Refs. 2 and 3, the values $\Theta=\pm\pi/2$ correspond to the natural extension, $\Theta=-\pi/2$ if the flux is positive and $\Theta=\pi/2$ if the flux is negative. Below we present a set of solutions ${}_\pm\psi_a(x)$ of (6) which we will use for Green function construction according to the formulas (5). We consider the problem separately for two values of the extension parameter.

We start with the case $\Theta=-\pi/2$. The positive energy spectrum is given by ${}_+\varepsilon$ and the negative energy spectrum is given by ${}_-\varepsilon$,

$${}_+\varepsilon=-{}_-\varepsilon=\sqrt{M^2+\omega}. \tag{10}$$

Both branches are determined by the spectrum of the quantity ω which is defined below. The solutions ${}_\pm\psi_a(x)$ can be expressed via the solutions $u(x)$ of the squared Dirac equation. The latter solutions have the form

$$\begin{gathered} {}_\pm u_{m,l,\sigma}(x)=e^{-i\,{}_\pm\varepsilon x^0}u_{m,l,\sigma}(x_\perp), \\ x_\perp=(x^1,x^2), \quad m=0,1,\ldots, \quad l=0,\pm1,\ldots, \quad \sigma=\pm1, \end{gathered} \tag{11}$$

where

$$u_{m,l,\sigma}(x_\perp)=\sqrt{\gamma}g_l(\varphi)\phi_{m,l,\sigma}(r)\upsilon_\sigma, \quad l\neq 0,$$

$$u_{m,0,+1}(x_\perp)=\sqrt{\gamma}g_0(\varphi)\phi_{m,0,+1}(r)\upsilon_{+1},$$

$$u_{m,0,-1}(x_\perp)=\sqrt{\gamma}g_0(\varphi)\phi^{ir}_{m,-1}(r)\upsilon_{-1}, \quad \gamma=e|B|,$$

and

$$g_l(\varphi)=\frac{1}{\sqrt{2\pi}}\exp\left\{i\varphi\left[l-l_0-\frac{1}{2}(1+\sigma^3)\right]\right\},$$

$$\upsilon_{+1}=\begin{pmatrix}1\\0\end{pmatrix}, \quad \upsilon_{-1}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The functions $\phi_{m,l,\sigma}(r)$, $\phi^{ir}_{m,-1}(r)$ are expressed via the Laguerre functions $I_{m+\alpha,m}(\rho)$ as

$$\begin{gathered}\phi_{m,l,\sigma}(r)=I_{m+|\nu|,m}(\rho), \quad \phi^{ir}_{m,-1}(r)=I_{m-\mu,m}(\rho),\\ \rho=\gamma r^2/2, \quad \nu=\mu+l-(1+\sigma)/2.\end{gathered} \tag{12}$$

We recall that the Laguerre functions $I_{m+\alpha,m}(\rho)$ are related to the Laguerre polynomials $L^\alpha_m(x)$ [8.970, 8.972.1 (Ref. 25)] as

$$I_{m+\alpha,m}(x)=\sqrt{\frac{m!}{\Gamma(m+\alpha+1)}}e^{-x/2}x^{\alpha/2}L^\alpha_m(x).$$

For the magnetic field $B>0$, the spectrum of ω corresponding to the functions $u_{m,l,\sigma}(x_\perp)$ is

$$\omega=\begin{cases}2\gamma(m+l+\mu), & l-(1+\sigma)/2\geqslant 0,\\ 2\gamma(m+(1+\sigma)/2), & l-(1+\sigma)/2<0,\end{cases} \tag{13}$$

except the functions $u_{m,0,-1}(x_\perp)$ for which the spectrum of ω is

$$\omega=2\gamma m. \tag{14}$$

Then the complete set ${}_\pm\psi_a$ with $a=(m,l)$ has the form

$${}_\pm\psi_{m,l}(x)=N(\Gamma P+M)\,{}_\pm u_{m,l,-1}(x). \tag{15}$$

The latter form provides correct expressions both for $\omega\neq 0$ and $\omega=0$, since the states with $\omega=0$ can only be expressed in terms of the spinors with $\sigma=-1$ [we note that ${}_+\psi\equiv 0$ for $\omega=0$, nevertheless it is convenient to remain in (11) u_+ with ${}_+\varepsilon=M$]. The normalization factor with respect to the usual inner product $(\psi,\psi')=\int\psi^\dagger(x)\psi'(x)\mathrm{d}\mathbf{x}$ reads

$$N=\begin{cases}[2|{}_\pm\varepsilon|(|{}_\pm\varepsilon|-M)]^{-1/2}, & \omega\neq 0,\\ [2M]^{-1}, & \omega=0.\end{cases}$$

The quantum number l characterizes the angular momentum of the particle, m is the radial quantum number, see Ref. 1.

For $B<0$ the spectrum of states differs nontrivially from the expressions given by Eqs. (13) and (14). Here ω corresponding to $u_{m,l,\sigma}(r)$ is

$$\omega=\begin{cases} 2\gamma(m-l+1-\mu), & l-(1+\sigma)/2<0, \\ 2\gamma(m+(1-\sigma)/2), & l-(1+\sigma)/2\geqslant 0, \end{cases} \tag{16}$$

except the functions $u_{m,0,-1}(x_\perp)$ for which the spectrum of ω is

$$\omega=2\gamma(m+1-\mu). \tag{17}$$

Now we go to the case with the extension parameter $\Theta=\pi/2$. We recall that one needs for self-adjoint extensions of the radial Dirac Hamiltonian only in the subspace $l=0$ to which we refer to as the critical subspace. Thus, the only solutions in the $l=0$ subspace must be subjected to the one of asymptotic condition from a one-parametric family of boundary conditions as $r\to 0$. By this reason for $\Theta=\pi/2$, the solutions only differ from (11) in the subspace $l=0$,

$$\begin{gathered} u_{m,0,+1}(x_\perp)=\sqrt{\gamma}g_0(\varphi)\phi^{ir}_{m,+1}(r)\upsilon_{+1}, \quad \phi^{ir}_{m,+1}(r)=I_{m+\mu-1,m}(\rho), \\ u_{m,0,-1}(x_\perp)=\sqrt{\gamma}g_0(\varphi)\phi_{m,0,-1}(r)\upsilon_{-1}, \end{gathered} \tag{18}$$

where the spectrum for $u_{m,0,+1}(x_\perp)$ is given as

$$\omega=2\gamma(m+\mu), \quad B>0, \tag{19}$$

$$\omega=2\gamma m, \quad B<0. \tag{20}$$

B. Construction of Green functions

The main point in constructing the Green functions is the summations in the representation (5). In the case under consideration, this summation can be done with the help of special relations which can be established for the solutions of the Dirac equation.

Let us start with the calculation of the Green functions for the extension parameter $\Theta=-\pi/2$ and $B>0$. In this case, taking into account that the eigenfunctions u of the equation $[(\Gamma P_\perp)^2+\omega]u=0$ corresponding to any $\omega\neq 0$ obey the equations

$$\begin{gathered} \Gamma P_\perp\, {}_\pm u_{m_+,l,-\sigma}(x)=-i\sqrt{\omega}\,{}_\pm u_{m_-,l,\sigma}(x), \quad l\leqslant 0, \quad P_\perp=(0,P_1,P_2), \\ \Gamma P_\perp\, {}_\pm u_{m,l,-\sigma}(x)=i\sqrt{\omega}\,{}_\pm u_{m,l,\sigma}(x), \quad l\geqslant 1, \quad m_\pm=m+(1\pm\sigma)/2, \end{gathered} \tag{21}$$

and the explicit form of the solutions ${}_\pm\psi_{m,l}$, one can verify that for $|\varepsilon|\neq M$ the following relations hold true:

$$\begin{gathered} {}_\pm\psi_{m,l}(x)\,{}_\pm\overline{\psi}_{m,l}(x')=(\Gamma P+M)\frac{1}{2\,{}_\pm\varepsilon}e^{-i\,{}_\pm\varepsilon\Delta x^0}\sum_{\sigma=\pm 1}\phi_{m_-,l,\sigma}(x_\perp,x'_\perp)\Xi_\sigma, \quad l\leqslant 0, \\ {}_\pm\psi_{m,l}(x)\,{}_\pm\overline{\psi}_{m,l}(x')=(\Gamma P+M)\frac{1}{2\,{}_\pm\varepsilon}e^{-i\,{}_\pm\varepsilon\Delta x^0}\sum_{\sigma=\pm 1}\phi_{m,l,\sigma}(x_\perp,x'_\perp)\Xi_\sigma, \quad l\geqslant 1, \end{gathered} \tag{22}$$

where

$$\begin{gathered} \phi_{m,l,\sigma}(x_\perp,x'_\perp)=\frac{\gamma}{2\pi}e^{i[l-l_0-(1+\sigma)/2]\Delta\varphi}I_{m+\alpha,m}(\rho)I_{m+\alpha,m}(\rho'), \\ \Delta\varphi=\varphi-\varphi', \quad \alpha=\begin{cases} \mu+l-(1+\sigma)/2, & l\geqslant 1, \\ -[\mu+l-(1+\sigma)/2], & l\leqslant 0, \end{cases} \quad \Xi_{\pm 1}=(1\pm\sigma^3)/2. \end{gathered} \tag{23}$$

The above relations and Eqs. (3) and (5) allow us to represent the causal Green function in the following form:

$$S^c(x,x')=(\Gamma P+M)\Delta^c(x,x'),$$
$$\Delta^c(x,x')=i\sum_{m,l,\sigma}\left[\theta(\Delta x^0)\frac{e^{-i_+\varepsilon\Delta x^0}}{2_+\varepsilon}-\theta(-\Delta x^0)\frac{e^{-i_-\varepsilon\Delta x^0}}{2_-\varepsilon}\right]\phi_{m,l,\sigma}(x_\perp,x'_\perp)\Xi_\sigma. \tag{24}$$

Then we can use the representations

$$\theta(\Delta x^0)\frac{e^{-i_+\varepsilon\Delta x^0}}{2_+\varepsilon}-\theta(-\Delta x^0)\frac{e^{-i_-\varepsilon\Delta x^0}}{2_-\varepsilon}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-ip_0\Delta x^0}}{\varepsilon^2-p_0^2-i\epsilon}\mathrm{d}p_0, \tag{25}$$

$$\frac{1}{\varepsilon^2-p_0^2-i\epsilon}=i\int_0^\infty e^{-i(\varepsilon^2-p_0^2)s}\,\mathrm{d}s, \tag{26}$$

in Eq. (24). Integrating over p_0, we obtain finally

$$\Delta^c(x,x')=\int_0^\infty f(x,x',s)\mathrm{d}s,$$
$$f(x,x',s)=\frac{1}{2(\pi s)^{1/2}}e^{(-i\Delta x_0^2/4s)}e^{i\pi/4}e^{-iM^2s}i\sum_{m,l,\sigma}e^{-i\omega s}\phi_{m,l,\sigma}(x_\perp,x'_\perp)\Xi_\sigma. \tag{27}$$

The path of the integration over s is deformed so that it goes slightly below the singular points $s_k=k\pi/\gamma$, $k=1,2,\ldots$.

Using (5), (22), and the representation

$$-\theta(-\Delta x^0)\frac{e^{-i_+\varepsilon\Delta x^0}}{2_+\varepsilon}+\theta(\Delta x^0)\frac{e^{-i_-\varepsilon\Delta x^0}}{2_-\varepsilon}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{-ip_0\Delta x^0}}{\varepsilon^2-p_0^2+i\epsilon}\mathrm{d}p_0,$$
$$\frac{1}{\varepsilon^2-p_0^2+i\epsilon}=i\int_{-0}^{-\infty}e^{-i(\varepsilon^2-p_0^2)s}\,\mathrm{d}s, \tag{28}$$

instead of (25) and (26) we obtain from (4),

$$S^{\bar c}(x,x')=(\Gamma P+M)\Delta^{\bar c}(x,x'),\quad \Delta^{\bar c}(x,x')=\int_{-0}^{-\infty}f(x,x',s)\mathrm{d}s, \tag{29}$$

where $f(x,x',s)$ is given by Eq. (27). The negative values for s are defined as $s=|s|e^{-i\pi}$, and the path of integration over s is deformed so that it goes slightly below the singular points $-s_k$.

We now consider the summations in (27). Applying the formula [8.976(1) (Ref. 25)] we can sum over m to get

$$\sum_{m=0}^{\infty}e^{-i2m\gamma s}I_{m+\alpha,m}(\rho)I_{m+\alpha,m}(\rho')=\exp\left\{\frac{i}{2}(\rho+\rho')\cot(\gamma s)\right\}\frac{e^{i\alpha\gamma s}e^{i\gamma s}}{2i\sin(\gamma s)}e^{-i\pi\alpha/2}J_\alpha(z),$$
$$z=\sqrt{\rho\rho'}/\sin(\gamma s), \tag{30}$$

where $J_\alpha(z)$ are the Bessel functions [8.402 (Ref. 25)], and for negative s we take $\arg s=-\pi+0$.

Similar results can be obtained for the case $B<0$. Here one should use the solutions corresponding to the spectrum of ω (16) and (17). Then these results can be united to obtain expressions which hold true for any sign of B,

$$f(x,x',s)=\sum_{l=-\infty}^{\infty} f_l(x,x',s), \quad f_l(x,x',s)=A(s)\sum_{\sigma=\pm 1}\Phi_{l,\sigma}(s)e^{-i\sigma eBs}\Xi_\sigma,$$
$$A(s)=\frac{eB}{8\pi^{3/2}s^{1/2}\sin(eBs)}\exp\left\{\frac{i\pi}{4}-iM^2s-il_0\Delta\varphi\right\}$$
$$\times\exp\left\{-\frac{i(\Delta x_0)^2}{4s}+\frac{ieB}{4}(r^2+r'^2)\cot(eBs)\right\}, \tag{31}$$

$$\Phi_{l,\sigma}(s)=e^{il_\sigma\Delta\varphi}e^{-i(l_\sigma+\mu)eBs}e^{-i\pi|l_\sigma+\mu|/2}J_{|l_\sigma+\mu|}(z), \quad l_\sigma=l-(1+\sigma)/2, \quad l\neq 0,$$
$$\Phi_{0,+1}(s)=e^{-i\Delta\varphi}e^{i(1-\mu)eBs}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z), \quad \Phi_{0,-1}(s)=e^{-i\mu eBs}e^{i\pi\mu/2}J_{-\mu}(z). \tag{32}$$

Now we consider the summation over l. One can see that the following relations hold true:

$$\sum_{l=1}^{\infty}\Phi_{l,-1}(s)=\sum_{l=1}^{\infty}\Phi_{l+1,+1}(s)=e^{-i\mu eBs}Y(z,\Delta\varphi-eBs,\mu),$$

$$\sum_{l=-1}^{-\infty}\Phi_{l,-1}(s)=\sum_{l=-1}^{-\infty}\Phi_{l+1,+1}(s)=e^{-i\mu eBs}Y(z,-\Delta\varphi+eBs,-\mu),$$

where

$$Y(z,\eta,\mu)=a_1(z)+\tilde{Y}(z,\eta,\mu), \quad \tilde{Y}(z,\eta,\mu)=\sum_{l=2}^{\infty}a_l(z), \quad a_l(z)=e^{i\eta l}(-i)^{l+\mu}J_{l+\mu}(z). \tag{33}$$

The evaluation of the sum in (33) can be done in a similar way to what was done in Ref. 26. There exist all $\partial_z a_l(z)$ on the half-line, $0<z<\infty$, and the relation [8.471 (2) (Ref. 25)], $\partial_z J_\nu(z)=[J_{\nu-1}(z)-J_{\nu+1}(z)]/2$, can be used. The series $\tilde{Y}(z,\eta,\mu)$ converges and the series of derivatives $\Sigma_{l=2}^{\infty}\partial_z a_l(z)$ converges uniformly in $(0,\infty)$. It is a sufficient condition to write down $\partial_z\tilde{Y}(z,\eta,\mu)=\Sigma_{l=2}^{\infty}\partial_z a_l(z)$. Thus, one arrives to a differential equation with respect to $Y(z,\eta,\mu)$,

$$\frac{d}{dz}Y(z,\eta,\mu)=-Y(z,\eta,\mu)i\cos\eta+\frac{1}{2}(-i)^{\mu}[-ie^{i\eta}J_\mu(z)+J_{1+\mu}(z)]. \tag{34}$$

that is true on the half-line, $0<z<\infty$. The solution of (34) reads

$$Y(z,\eta,\mu)=\frac{1}{2}(-i)^{\mu}\int_0^z e^{i(y-z)\cos\eta}[-ie^{i\eta}J_\mu(y)+J_{1+\mu}(y)]dy. \tag{35}$$

This is also valid for $Y(z,-\eta,-\mu)$.

It is useful to introduce the following function:

$$f_{nc}(x,x',s)=\sum_{l\neq 0}f_l(x,x',s).$$

It defines the part of the Green functions that is the same for all extensions. With the help of the function $Y(z,\eta,\mu)$ (33), (35) one can write

$$f_{nc}(x,x',s)=A(s)e^{-i\mu eBs}e^{-ieBs\sigma^3}\{Y(z,\Delta\varphi-eBs,\mu)+Y(z,-\Delta\varphi+eBs,-\mu)$$
$$+[e^{-i\pi\mu/2}J_{\mu}(z)-e^{-i(\Delta\varphi-eBs)}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z)]\Xi_{+1}\}. \tag{36}$$

The function $f_0(x,x',s)$ is specific for each extension. It is reasonable to mark it with a superscript that assumes the values of the extension parameter. Thus, for $\Theta=-\pi/2$,

$$f_0^{(-\pi/2)}(x,x',s)=A(s)e^{-i\mu eBs}[e^{-i\Delta\varphi}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z)\Xi_{+1}+e^{-ieBs\sigma^3}e^{i\pi\mu/2}J_{-\mu}(z)\Xi_{-1}]. \tag{37}$$

Accordingly, the function $f(x,x',s)$ acquires the same superscript,

$$f^{(-\pi/2)}(x,x',s)=f_{nc}(x,x',s)+f_0^{(-\pi/2)}(x,x',s). \tag{38}$$

For the extension parameter $\Theta=\pi/2$, one obtains

$$f_0^{(\pi/2)}(x,x',s)=A(s)e^{-i\mu eBs}[e^{-i\Delta\varphi}e^{-i\pi(\mu-1)/2}J_{\mu-1}(z)\Xi_{+1}+e^{-ieBs\sigma^3}e^{-i\pi\mu/2}J_{\mu}(z)\Xi_{-1}],$$
$$f^{(\pi/2)}(x,x',s)=f_{nc}(x,x',s)+f_0^{(\pi/2)}(x,x',s). \tag{39}$$

Besides, one can consider particles with "spin-down" polarization in $2+1$ dimensions. The corresponding wave functions $\psi^{(-1)}(x)$ can be presented as

$$\psi^{(-1)}(x)=\sigma^1(\Gamma P-M)u(x)\,,$$

where $u(x)$ are solutions (11) of the squared Dirac equation. The propagator related to such particles can be expressed in terms of the function $\Delta^c(x,x')$ (24),

$$S^c_{(-1)}(x,x')=-\sigma^1(\Gamma P-M)\Delta^c(x,x')\sigma^1\,.$$

At this point we should make some remarks.

One can see that there exists a simple relation between scalar Green functions and Green functions of the squared Dirac equation (for the above considered extensions). Consider this relation in the example of causal Green functions. First of all, we note that the Klein–Gordon equation differs from the squared Dirac equation by the Zeeman interaction term. Then we can see (remembering the origin of the quantum number l for both spinning and spinless particles) that the scalar propagator can be derived from $\Delta^c(x,x')$ by only retaining the terms with $\sigma=-1$ only. The term $eB\sigma^3$, which is responsible for the Zeeman interaction with the uniform magnetic field, has to be removed. The Zeeman interaction with the solenoid flux, influencing the terms with $l=0$, depends on the flux sign and can be repulsive or attractive. The repulsive contact interaction case is physically equivalent to the spinless case, since in both cases the corresponding wave functions vanish at the origin. The necessary boundary condition is realized for the extension parameter $\Theta=\pi/2$. Thus, one can obtain the scalar Green functions using the coefficients of Ξ_{-1} in $f_l(x,x',s)$ (31), (32) and $f_0^{(\pi/2)}(x,x',s)$ (39). By following such prescriptions, one arrives at the expression (A1) obtained by direct calculation.

In the spinless case there is no physically preferred orientation of the plane x^1x^2. Therefore, the solenoid flux direction does not matter, i.e., the AB symmetry, $l_0\to l_0+1$, is conserved. The direction of the uniform magnetic field does not matter as well. This can be observed from the explicit form of the Green functions (A1) where the change $B\to -B$ is equivalent to the choice of the opposite orientation of the plane, $l\to -l$, $\Delta\varphi\to-\Delta\varphi$, $\Phi\to-\Phi$. In the spinning case the given spin direction breaks the symmetry related to the plane orientation. The Zeeman interaction of the spin with the background violates the AB symmetry as well as the symmetry with respect to the change $B\to -B$.

As is known, influence of the solenoid flux on the particle is observed only when the flux is not equal to an integral number of quanta ($\mu\neq 0$). In this connection it is instructive to consider

the Green functions for the particular case $\mu=0$. We note that the part $f_{nc}(x,x',s)$ (36) of the function $f(x,x',s)$ is regular everywhere, while the part $f_0(x,x',s)$ is singular at the origin. Thus, taking the limit $\mu\to 0$ in (36) and using the relation $J_1(y)=-J_0'(y)$ we get

$$f_{nc}(x,x',s)=A(s)e^{-ieBs\sigma^3}\{e^{-iz\cos(\Delta\varphi-eBs)}-J_0(z)+[J_0(z)+ie^{-i(\Delta\varphi-eBs)}J_1(z)]\Xi_{+1}\}.$$

The corresponding expression for $f_0(x,x',s)$ can be obtained in the following way. We restrict the range of z to $0<\delta<z<\infty$, where $\delta\ll 1$. Then we take the limit $\mu\to 0$ and use the continuity of the Bessel functions with respect to its index. At the end we construct the analytic continuation of the obtained expressions over the interval $(0,\delta)$. Thus, starting from either (37) or (39) we get

$$f_0(x,x',s)=A(s)[-ie^{-i\Delta\varphi}J_1(z)\Xi_{+1}+e^{ieBs}J_0(z)\Xi_{-1}],$$

where the superscript is no longer necessary. Thus, the explicit form of $f(x,x',s)$ is

$$f(x,x',s)=\frac{eB}{8\pi^{3/2}s^{1/2}\sin(eBs)}\exp\left\{\frac{i\pi}{4}-\frac{i(\Delta x_0)^2}{4s}-iM^2s-ieBs\sigma^3\right\}$$
$$\times\exp\left\{-il_0\Delta\varphi+\frac{ieB}{4}(r^2+r'^2)\cot(eBs)-\frac{ieBrr'\cos(\Delta\varphi-eBs)}{2\sin(eBs)}\right\}. \quad (40)$$

Making a transformation to Cartesian coordinates in (40) and setting $l_0=0$, one can obtain the known result of the uniform magnetic field, see for example, Ref. 27.

C. Nonrelativistic case

Consideration of the Green functions in the background under question in the nonrelativistic case is important for various physical applications. Below we study this case in detail. The solutions of the Schrödinger equation for "spin-up" particles $(+)$ and antiparticles $(-)$ in the case $\Theta=-\pi/2$ read

$$ {}_+\phi_{m,l}(x)=e^{-iEx^0}\sqrt{\frac{\gamma}{2\pi}}e^{i(l-l_0-1)\varphi}\phi_{m,l,+1}(r),\quad E=\frac{\omega_{m,l,\sigma}}{2M},$$
$$ {}_-\phi_{m,l}(x)=e^{-iEx^0}\sqrt{\frac{\gamma}{2\pi}}e^{-i(l-l_0)\varphi}\phi_{m,l,-1}(r),\quad l\neq 0, \quad (41)$$

$$ {}_-\phi_{m,0}(x)=e^{-iEx^0}\sqrt{\frac{\gamma}{2\pi}}e^{il_0\varphi}\phi^{ir}_{m,-1}(r), \quad (42)$$

where the values $\omega_{m,l,\sigma}$ are defined by m,l,σ with the help of formulas (13), (14) for $B>0$, and (16), (17) for $B<0$. The solutions ${}_+\phi_{m,l}(x)$ $({}_-\phi_{m,l}(x))$ for the "spin-down" case can be obtained from the solutions ${}_-\phi_{m,l}(x)$ $({}_+\phi_{m,l}(x))$ for the "spin-up" case with the change $\varphi\to-\varphi$ in (41), (42).

The retarded Green functions for particles and antiparticles are defined as

$$S^{\mathrm{ret},(\pm)}(x,x')=\theta(\Delta x^0)\sum_l S_l^{(\pm)}(x,x'),\quad S_l^{(\pm)}(x,x')=i\sum_m {}_\pm\phi_{m,l}(x)\,{}_\pm\phi^*_{m,l}(x'),$$
$$S_{nc}^{(\pm)}(x,x')=\sum_{l\neq 0} S_l^{(\pm)}(x,x'), \quad (43)$$

where the part $S_{nc}^{(\pm)}(x,x')$ is the same for all extensions, whereas $S_0^{(\pm)}(x,x')$ is specific for each extension. Carrying out the summations in (43) one obtains

$$S_l^{(\pm)}(x,x')=A_{nr}(x,x')e^{\mp i\gamma\tau}e^{\pm i(l_\pm-l_0)\Delta\varphi}e^{-i|l_\pm+\mu|\gamma\tau}e^{-i\pi|l_\pm+\mu|/2}J_{|l_\pm+\mu|}(z_{nr}),$$

$$A_{nr}(x,x')=\frac{\gamma}{4\pi\sin(\gamma\tau)}\exp\left[\frac{i}{2}(\rho+\rho')\cot(\gamma\tau)\right], \tag{44}$$

$$S_{nc}^{(+)}(x,x')=A_{nr}(x,x')e^{-il_0\Delta\varphi}e^{-i(1+\mu)eB\tau}\{e^{-i\pi\mu/2}J_\mu(z_{nr})-e^{-i\Delta\varphi}e^{ieB\tau}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z_{nr})$$
$$+Y(z_{nr},\Delta\varphi-eB\tau,\mu)+Y(z_{nr},-\Delta\varphi+eB\tau,-\mu)\}, \tag{45}$$

$$S_{nc}^{(-)}(x,x')=A_{nr}(x,x')e^{il_0\Delta\varphi}e^{i(1-\mu)eB\tau}\{Y(z_{nr},-\Delta\varphi-eB\tau,\mu)+Y(z_{nr},\Delta\varphi+eB\tau,-\mu)\}, \tag{46}$$

$$z_{nr}=\sqrt{\rho\rho'}/\sin(\gamma\tau),\quad \tau=\Delta x^0/2M,\quad l_\pm=l-(1\pm1)/2,\quad l\neq0,$$

whereas for $l=0$,

$$S_0^{(+)(\mp\pi/2)}(x,x')=A_{nr}(x,x')e^{-i(l_0+1)\Delta\varphi}e^{-i\mu eB\tau}e^{\mp i\pi(1-\mu)/2}J_{\pm(1-\mu)}(z_{nr}), \tag{47}$$

$$S_0^{(-)(\mp\pi/2)}(x,x')=A_{nr}(x,x')e^{il_0\Delta\varphi}e^{i(1-\mu)eB\tau}e^{\pm i\pi\mu/2}J_{\mp\mu}(z_{nr}). \tag{48}$$

The Green function in the ''spin-down'' case can be obtained with the change $\Delta\varphi\to-\Delta\varphi$ in (44)–(48) and with the change $S^{(\pm)}$ by $S^{(\mp)}$ in all the functions $S(x,x')$ in (44)–(48). Thus, one can see that the Green functions for the nonrelativistic particle is irregular at $r=0$ when the contact interaction is attractive.

We note that for the limiting case $B=0$ (the uniform magnetic field is absent), $S_l^{(+)(-\pi/2)}(x,x')$ coincide with the known expression for the spinless particle,[15–17] which is natural in the case of a repulsive contact interaction. $S_l^{(+)(\pi/2)}(x,x')$ for $B=0$ coincide with the corresponding expressions obtained in Ref. 18.

III. 3+1 DIMENSIONAL CASE

To obtain the Green functions in $3+1$ dimensions we use the orthonormalized solutions ${}_\pm\Psi_{p_3,m,l,\sigma}(x)$ of the Dirac equation found in Refs. 2 and 3. The quantum numbers m, l have the same meaning as in the $(2+1)$-dimensional case, p_3 is the x^3 component of the momentum, and σ is the spin quantum number. The positive energy spectrum is given by ${}_+\varepsilon$ and the negative energy spectrum is given by ${}_-\varepsilon$. They both are expressed via the quantity ω as

$${}_+\varepsilon=-{}_-\varepsilon=\sqrt{M^2+p_3^2+\omega}. \tag{49}$$

The spectra of ω are given in (13), (14) for $B>0$, and in (16), (17) for $B<0$. For $\omega\neq0$, one can present the solutions ${}_\pm\Psi_{p_3,m,l,\sigma}$ in the following form:

$${}_\pm\Psi_{p_3,m,l,\sigma}(x)=N(\gamma^\nu P_\nu+M)\,{}_\pm U_{p_3,m,l,\sigma}(x),$$

$${}_\pm U_{p_3,m,l,\sigma}(x)=\frac{1}{\sqrt{2\pi}}\,e^{-i\,{}_\pm\varepsilon x^0-ip_3x^3}U_{m,l,\sigma}(x_\perp),$$

$$U_{m,l,\sigma}(x_\perp)=\begin{pmatrix}u_{m,l,\sigma}(x_\perp)\\ \sigma^3u_{m,l,\sigma}(x_\perp)\end{pmatrix},\quad N=[2|{}_\pm\varepsilon|(|{}_\pm\varepsilon|+p_3)]^{-1/2}, \tag{50}$$

whereas for $\omega=0$,

$${}_\pm\Psi_{p_3,0,l,-\xi}(x)=N(\gamma^\nu P_\nu+M)\,{}_\pm U_{p_3,0,l,-\xi}(x),\quad \xi=\mathrm{sgn}(B),$$

where $u_{m,l,\sigma}(x_\perp)$ are the two-spinors defined in (11).

We are going to construct the Green functions using the solutions that correspond to the natural extensions of the Dirac operator, i.e., for the extension parameters chosen as $\Theta_{+1}=\Theta_{-1}=\Theta$, and $\Theta=\pm\pi/2$. First we consider the case $\Theta=-\pi/2$, and $B>0$. We note that for $\omega\neq 0$,

$$\gamma^\nu P_{\perp\nu} U_{m,l,-\sigma} = i\sqrt{\omega}\, U_{m,l,\sigma}, \quad l\geqslant 1,$$
$$\gamma^\nu P_{\perp\nu} U_{m_+,l,-\sigma} = -i\sqrt{\omega}\, U_{m_-,l,\sigma}, \quad l\leqslant 0, \tag{51}$$

where $P_\perp=(0,P_1,P_2,0)$. The summations in (5) can be done similarly to the $(2+1)$-dimensional case by the help of some important relations derived by us for the solutions (50). Namely, for the states with a given $\omega\neq 0$, the following relations hold true:

$$\sum_{\sigma=\pm 1} {}_\pm\Psi_{p_3,m,l,\sigma}(x)\, {}_\pm\overline{\Psi}_{p_3,m,l,\sigma}(x') = \sum_{\sigma=\pm 1} \frac{1}{2\,{}_\pm\varepsilon}(\gamma^\nu P_\nu + M)\frac{1}{2}(1+\sigma\Sigma^3)\, {}_\pm\phi_{p_3,m,l,\sigma}(x,x'), \quad l\geqslant 1,$$
$$\sum_{\sigma=\pm 1} {}_\pm\Psi_{p_3,m_+,l,-\sigma}(x)\, {}_\pm\overline{\Psi}_{p_3,m_+,l,-\sigma}(x') = \sum_{\sigma=\pm 1} \frac{1}{2\,{}_\pm\varepsilon}(\gamma^\nu P_\nu + M)\frac{1}{2}(1+\sigma\Sigma^3)\, {}_\pm\phi_{p_3,m_+,l,-\sigma}(x,x'), \quad l\leqslant 0, \tag{52}$$

and for $\omega=0$, we have

$${}_\pm\Psi_{p_3,0,l,-1}(x)\, {}_\pm\overline{\Psi}_{p_3,0,l,-1}(x') = \frac{1}{2\,{}_\pm\varepsilon}(\gamma^\nu P_\nu + M)\frac{1}{2}(1-\Sigma^3)\, {}_\pm\phi_{p_3,0,l,-1}(x,x'),$$

where

$${}_\pm\phi_{p_3,m,l,\sigma}(x,x') = \frac{1}{2\pi} e^{-i\,{}_\pm\varepsilon\Delta x^0 - ip_3\Delta x^3}\phi_{m,l,\sigma}(x_\perp,x'_\perp), \quad \Delta x^3 = x^3 - x'^3. \tag{53}$$

The functions $\phi_{m,l,\sigma}(x_\perp,x'_\perp)$ are defined in (23). Therefore,

$$S^c(x,x') = (\gamma^\nu P_\nu + M)\Delta^c(x,x'),$$
$$\Delta^c(x,x') = i\sum_{m,l,\sigma}\int_{-\infty}^{\infty} dp_3 \frac{1}{2}(1+\sigma\Sigma^3)\Big[\theta(\Delta x^0)\frac{1}{2\,{}_+\varepsilon}\,{}_+\phi_{p_3,m,l,\sigma}(x,x') - \theta(-\Delta x^0)\frac{1}{2\,{}_-\varepsilon}\,{}_-\phi_{p_3,m,l,\sigma}(x,x')\Big]. \tag{54}$$

Applying the relations (25),(26), one obtains the proper time integral representation for Δ^c,

$$\Delta^c(x,x') = \int_0^\infty f(x,x',s)ds, \quad f(x,x',s) = \sum_{l=-\infty}^{\infty} f_l(x,x',s),$$
$$f_l(x,x',s) = D(s)\sum_{\sigma=\pm 1}\Phi_{l,\sigma}(s)e^{-i\sigma eBs}\frac{1}{2}(1+\sigma\Sigma^3),$$

$$D(s)=\frac{eB}{16\pi^2 s\sin(eBs)}\exp\left\{\frac{i}{4s}[(\Delta x_3)^2-(\Delta x_0)^2]-iM^2 s\right\}$$
$$\times\exp\left\{-il_0\Delta\varphi+\frac{ieB}{4}(r^2+r'^2)\cot(eBs)\right\}, \tag{55}$$

where $\Phi_{l,\sigma}(s)$ are defined in (32).

Carrying out similar calculations for $B<0$ one can verify that (55) is valid for both signs of B. Therefore, for any sign of B, we get

$$f_{nc}(x,x',s)=\sum_{l\neq 0} f_l(x,x',s)=D(s)e^{-ieBs(\mu+\Sigma^3)}\Big\{Y(z,\Delta\varphi-eBs,\mu)+Y(z,-\Delta\varphi+eBs,-\mu)$$
$$+[e^{-i\pi\mu/2}J_\mu(z)-e^{-i(\Delta\varphi-eBs)}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z)]\frac{1}{2}(1+\Sigma^3)\Big\},$$

$$f_0^{(-\pi/2)}(x,x',s)=\tfrac{1}{2}D(s)e^{-i\mu eBs}[e^{-i\Delta\varphi}e^{-i\pi(1-\mu)/2}J_{1-\mu}(z)(1+\Sigma^3)$$
$$+e^{ieBs}e^{i\pi\mu/2}J_{-\mu}(z)(1-\Sigma^3)],$$

$$f^{(-\pi/2)}(x,x',s)=f_{nc}(x,x',s)+f_0^{(-\pi/2)}(x,x',s), \tag{56}$$

Using the corresponding solutions for the case $\Theta=\pi/2$, we obtain

$$f_0^{(\pi/2)}(x,x',s)=\tfrac{1}{2}D(s)e^{-i\mu eBs}[e^{-i\Delta\varphi}e^{-i\pi(\mu-1)/2}J_{\mu-1}(z)(1+\Sigma^3)+e^{ieBs}e^{-i\pi\mu/2}J_\mu(z)(1-\Sigma^3)],$$

$$f^{(\pi/2)}(x,x',s)=f_{nc}(x,x',s)+f_0^{(\pi/2)}(x,x',s). \tag{57}$$

IV. SUMMARY

Various Green functions of the Dirac equation with the magnetic-solenoid field are constructed as sums over exact solutions of this equation. We stress that doing that we had to take into account all the peculiarities related to the self-adjoint extension problem of the Dirac operator in the background under consideration. Both $2+1$ and $3+1$ dimensional cases are considered. Compact form for the Green functions was obtained thanks to the important relations (22) and (52) derived by us for the exact solutions under consideration. The representations of the Green functions as proper time integrals are constructed. The kernels of the proper time integrals are represented both as infinite sums over the orbital quantum number l and as simple integrals. The Green functions are obtained for two natural self-adjoint extensions, one for the positive solenoid flux and the other one for the negative solenoid flux. The physical motivation for the choice of these extensions is their correspondence to the presence of the point-like magnetic field at the origin and their close relation to the MIT boundary conditions.[23,28,29] Thus, the considered cases are of most interest for applications. Other values of the extension parameter correspond to additional contact interactions,[30] and some of the values are of physical interest as well. To find a closed form of Green functions for the arbitrary value of the extension parameter is a more complicated task. The spectra of the corresponding extensions in the critical subspace are no longer periodic for such a situation that requires to apply more exquisite calculation methods. We suppose to consider this issue in the future.

In addition, the nonrelativistic Green functions are constructed. The latter Green functions are represented for all possible types of $2+1$ dimensional nonrelativistic particles.

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APPENDIX

For the sake of completeness we consider here the Green functions for the scalar particle. They are defined by Eqs. (3), (4), (8), and (9), where $S^{\mp}(x,x')$ read

$$S^{\mp}(x,x') = \pm i \sum {}_{\pm}\phi_n(x)\, {}_{\pm}\phi_n^*(x'),$$

and ${}_{\pm}\phi_n(x)$ form a complete set of orthonormalized solutions of the Klein–Gordon equation. Here we consider the natural extension of the Klein–Gordon operator for which solutions in $2+1$ dimensions and the related spectrum read[3]

$${}_{\pm}\phi_{m,l}(x) = \frac{1}{\sqrt{2\varepsilon}}\, e^{-i\,{}_{\pm}\varepsilon x^0} \sqrt{\frac{\gamma}{2\pi}}\, e^{i(l-l_0)\varphi} I_{m+|l+\mu|,m}(\rho),$$

$${}_{\pm}\varepsilon = \pm\sqrt{M^2+\omega}, \quad \omega = \gamma[1+2m+|l+\mu|+\xi(l+\mu)],$$

$$l = 0,\pm 1,\pm 2,\ldots, \quad m = 0,1,2,\ldots .$$

Using Eqs. (25), (26), (28), and (30), we calculate the causal and anticausal propagators. They have the form

$$S^c(x,x') = \int_0^{\infty} f^{sc}(x,x',s)\mathrm{d}s, \quad S^{\bar{c}}(x,x') = \int_{-0}^{-\infty} f^{sc}(x,x',s)\mathrm{d}s,$$

$$f^{sc}(x,x',s) = \sum_l f_l^{sc}(x,x',s)\,,$$

$$f_l^{sc}(x,x',s) = A(s) e^{il\Delta\varphi} e^{-i(l+\mu)eBs} e^{-i\pi|l+\mu|/2} J_{|l+\mu|}\,, \tag{A1}$$

$$f^{sc}(x,x',s) = A(s) e^{-i\mu eBs}[e^{-i\pi\mu/2} J_{\mu}(z) + Y(z,\Delta\varphi - eBs,\mu) + Y(z,-\Delta\varphi + eBs,-\mu)],$$

where $A(s)$ is given in (31), and $Y(z,\eta,\mu)$ in (33), (35). The expression (A1) can be generalized for the $(D+1)$-dimensional case, where D is the number of spacial dimensions, with the substitution $A(s)$ in (A1) by $A^{(D)}(s)$,

$$A^{(D)}(s) = A(s)\exp\left\{\frac{i}{4s}\sum_{k=3}^{D}(\Delta x_k)^2\right\}\left(\frac{e^{-i\pi/2}}{4\pi s}\right)^{(D-2)/2}, \quad D \geqslant 3.$$

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