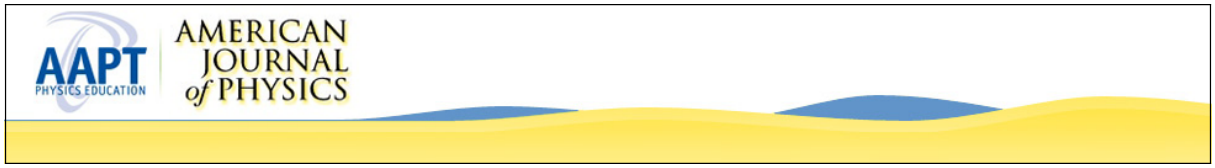

n+1 dimensional Dirac equation and the Klein paradox

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$n+1$ dimensional Dirac equation and the Klein paradox

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In a recent article in this journal, Nitta *et al.*[1] have presented both a derivation of the Dirac equation in $1+1$ dimensions and its solution for the step potential. Furthermore, numerical simulations for the scattering of a wave packet under the conditions of the Klein paradox were presented. The purpose of this comment is to clarify two points. First, the Lorentz structure of the potential and its connection with the Klein paradox. Second, the connection between the number of space dimensions and the number of spinor components.

In Appendix A of Ref. 1, the Dirac equation for a free particle in $1+1$ dimensions is derived as it is usually done in the literature for $3+1$ dimensions.[2] The authors found

$$\alpha^2=\beta^2=1, \quad \alpha\beta+\beta\alpha=0, \tag{1}$$

concluding that α and β are reduced to 2×2 matrices and that any two of the three Pauli matrices can satisfy these relations. They chose $\alpha=\sigma_x$ and $\beta=\sigma_z$ and declared "*In the presence of the scalar potential* $V(x)$, *the* $1+1$ *dimensional Dirac equation is extended to the form*

$$\left[i\hbar\frac{\partial}{\partial t}-V(x)\right]\Psi(x,t) = \left[c\sigma_x\left(-i\hbar\frac{\partial}{\partial x}\right)+\sigma_z m_0c^2\right]\Psi(x,t) \quad (27).'' \tag{2}$$

In addition, it is argued that "*For the case of 2 or 3 dimensions, we have to use the Dirac equation with ordinary* 4×4 *Dirac matrices and 4-component spinors because there appears the spin degree of freedom.*" In the main body of the paper the authors presented their calculations for the reflection and transmission amplitudes. (It should be noted in passing that these quantities are indeed amplitudes but not coefficients as the authors mistakenly stated. Needless to say, the sum of the coefficients should be equal to one when there exists a transmitted wave):

$$R=\frac{a-b}{a+b}, \quad T=\frac{2a}{a+b}, \tag{3}$$

where

$$a=\frac{\sqrt{E^2-(m_0c^2)^2}}{E+m_0c^2}, \tag{4}$$

$$b=\frac{\sqrt{(E-V_0)^2-(m_0c^2)^2}}{E-V_0+m_0c^2}. \tag{5}$$

Thus, they concluded that for $V_0>E+m_0c^2$ there is the Klein paradox.

The first point to be elucidated is that the potential in the extended form of the Dirac equation is not a scalar potential as stated by Nitta *et al*. Under a Lorentz transformation, the potential in Eq. (2) transforms like the energy, i.e., the time component of a Lorentz vector. On the other hand, a scalar potential should appear in the Dirac equation multiplied by σ_z in order to transform itself under a Lorentz transformation in the same way as the mass of the particle, i.e., a Lorentz scalar. This would affect Eq. (5) modifying Eq. (4) by the substitution $m_0\to m_0+V_0/c^2$ instead of $E\to E-V_0$, leading to no Klein paradox in the presence of a pure scalar potential. I think it is important to mention that in $1+1$ dimensions there are only three linearly independent Lorentz structures for the potential: scalar, vector, and pseudoscalar. This happens because there are only four linearly independent 2×2 matrices. This *quid pro quo* between scalar and vector potentials has also appeared recently in this journal in a paper by Holstein,[3] where the Klein paradox for the Klein–Gordon and the Dirac equations was analyzed. In discussing subbarrier relativistic effects in $3+1$ dimensions,[4] Anchishkin also unnecessarily regarded the time component of a four-vector potential as a scalar potential. It is obvious from the above discussion that erroneous terms for potentials in relativistic equations may cause confusion to the unwary.

The second point regards the dimensionality of space. For the generic $n+1$ dimensions, it can be derived that the Hermitian square matrices α_i and β satisfy the relations $\alpha_i^2=\beta^2=1$, $\{\alpha_i,\beta\}=0$, and $\{\alpha_i,\alpha_j\}=2\delta_{ij}$, where $i=1,2,...,n$. It can also be derived that $\mathrm{Tr}(\alpha_i)=\mathrm{Tr}(\beta)=0$ and that their eigenvalues are ±1, so one can conclude that α_i and β are even-dimensional matrices. For $n=1$ and $n=2$ one can choose the 2×2 Pauli matrices satisfying the same algebra as α_i and β, resulting in two-component spinors in both cases. For $n=3$ and higher dimensions, though, that is not possible any more because there are more matrices required by the algebra than Pauli matrices at one's disposal. This is the reason why one has to appeal to 4×4 matrices and four-component spinors in $3+1$ dimensions. It is true that there is no spin in the $1+1$ dimensional case because there is no angular momentum in one spatial dimension. Otherwise, in $2+1$ dimensions there are only perpendicular projections of the angular momentum.

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Erratum: "Field pattern of a magnetic dipole" [Am. J. Phys. 68, 577–578 (2000)]

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Mc Tavish[1] derived the result

$$\left(\frac{x}{D}\right)^2+\left(\frac{y}{D}\right)^2=\left(\frac{x}{D}\right)^{4/3} \tag{1}$$

for the field lines of a magnetic dipole. Unfortunately, in the paper the function $(x/D)^2+(y/D)^2=(x/D)^{3/4}$ was plotted for several values of the parameter D, instead of Eq. (1). A corrected version of that figure is shown in Fig. 1.

In Fig. 1, Eq. (1) is plotted for the values of $D=0.25$, 0.5, and 1.0. It can be seen that the correct curves (solid line with full circles) are quite different from those originally plotted (solid line) for these values of D, showing a much less circular form.

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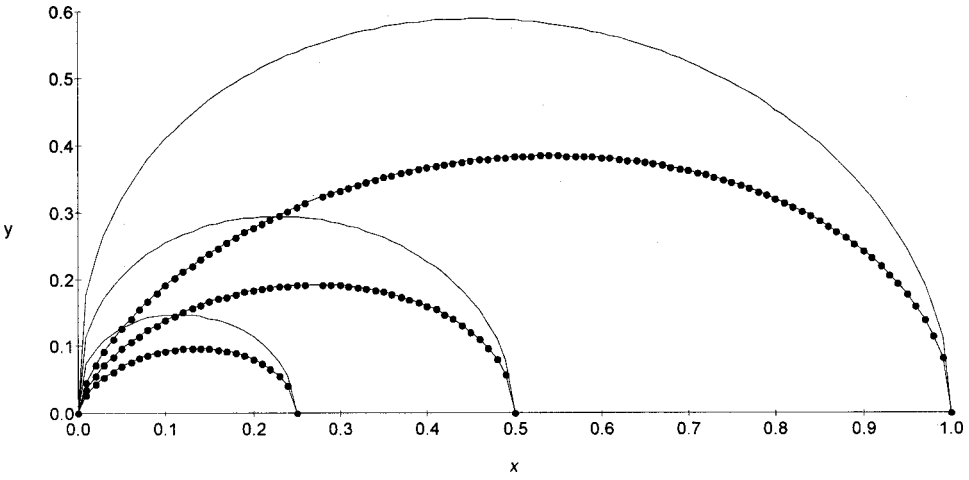

Fig. 1. Corrected figure: The field lines for the magnetic dipole.

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