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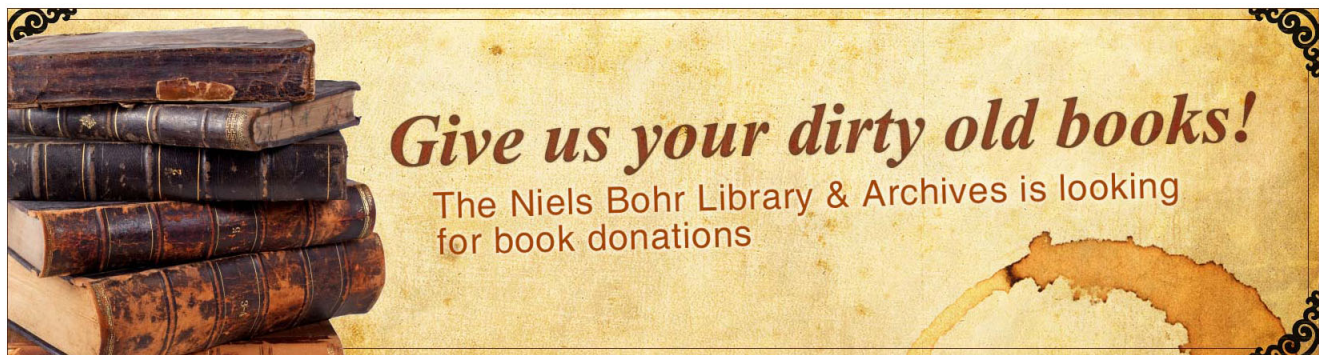
On Duffin–Kemmer–Petiau particles with a mixed minimal-nonminimal vector coupling and the nondegenerate bound-states for the one-dimensional inversely linear background

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The problem of spin-0 and spin-1 bosons in the background of a general mixing of minimal and nonminimal vector inversely linear potentials is explored in a unified way in the context of the Duffin–Kemmer–Petiau theory. It is shown that spin-0 and spin-1 bosons behave effectively in the same way. An orthogonality criterion is set up and it is used to determine uniquely the set of solutions as well as to show that even-parity solutions do not exist. © *2010 American Institute of Physics.*
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I. INTRODUCTION

The first-order Duffin–Kemmer–Petiau (DKP) formalism[1–4] describes spin-0 and spin-1 particles and has been used to analyze relativistic interactions of spin-0 and spin-1 hadrons with nuclei as an alternative to their conventional second-order Klein–Gordon and Proca counterparts. The DKP formalism enjoys a richness of couplings not capable of being expressed in the Klein–Gordon and Proca theories.[5,6] Although the formalisms are equivalent in the case of minimally coupled vector interactions,[7–9] the DKP formalism opens news horizons as far as it allows other kinds of couplings which are not possible in the Klein–Gordon and Proca theories. Nonminimal vector potentials, added by other kinds of Lorentz structures, have already been used successfully in a phenomenological context for describing the scattering of mesons by nuclei.[10–17] Nonminimal vector coupling with a quadratic potential,[18] with a linear potential,[19] and mixed space and time components with a step potential[20,21] and a linear potential[22] have been explored in the literature. See also Ref. 22 for a comprehensive list of references on other sorts of couplings and functional forms for the potential functions. In Ref. 22 it was shown that when the space component of the coupling is stronger than its time component, the linear potential, a sort of vector DKP oscillator, can be used as a model for confining bosons.

The problem of a particle subject to an inversely linear potential in one spatial dimension ($\sim|x|^{-1}$), known as the one-dimensional hydrogen atom, has received considerable attention in the literature (for a rather comprehensive list of references, see Ref. 23). This problem presents some conundrums regarding the parities of the bound-state solutions. This problem was also analyzed with the Klein–Gordon equation for a Lorentz vector coupling.[24,25] By using the technique of continuous dimensionality, the problem was approached with the Schrödinger, Klein–Gordon, and Dirac equations.[26] In this last work, it was concluded that the Klein–Gordon equation, with the interacting potential considered as a time component of a vector, provides unacceptable solutions while the Dirac equation has no bounded solutions at all. On the other hand, in a more recent work,[23] the authors use connection conditions for the eigenfunctions and their first derivatives across the singularity of the potential and conclude that only the odd-parity solutions of the

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Schrödinger equation survive. The problem was also sketched for a Lorentz scalar potential in the Dirac equation in Refs. 27 and 28, for a general mixing of vector and scalar couplings in the Dirac equation[29] and in the Klein–Gordon equation,[30] and for a pseudoscalar coupling in the Dirac equation.[31]

   The main purpose of the present article is to report on the properties of the DKP theory with time components of minimal and nonminimal vector inversely linear potentials for spin-0 and spin-1 bosons in a unified way. This sort of mixing, beyond its potential physical applications, shows to be a powerful tool to obtain a deeper insight about the nature of the DKP equation and its solutions as far as it explores the differences between minimal and nonminimal couplings. The problem is mapped into an exactly solvable Sturm–Liouville problem of a Schrödinger-like equation. The effective potential resulting from the mapping has the form of the Kratzer potential[32] and the closed form solution for the bound-states is uniquely determined by requiring orthonormalizability. The results imply that even-parity solutions to the one-dimensional DKP hydrogen atom do not exist.

II. VECTOR COUPLINGS IN THE DKP EQUATION

   The DKP equation for a free boson is given by[4] (with units in which $\hbar = c = 1$)

$$(i\beta^\mu \partial_\mu - m)\psi = 0, \tag{1}$$

where the matrices β^μ satisfy the algebra $\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\lambda\nu}\beta^\mu$ and the metric tensor is $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. That algebra generates a set of 126 independent matrices whose irreducible representations are a trivial representation, a five-dimensional representation describing the spin-0 particles and a ten-dimensional representation associated to spin-1 particles. The second-order Klein–Gordon and Proca equations are obtained when one selects the spin-0 and spin-1 sectors of the DKP theory. A well-known conserved four-current is given by $J^\mu = \bar{\psi}\beta^\mu \psi / 2$, where the adjoint spinor $\bar{\psi}$ is given by $\bar{\psi} = \psi^\dagger \eta^0$ with $\eta^0 = 2\beta^0 \beta^0 - 1$. The time component of this current is not positive definite but it may be interpreted as a charge density. Then the normalization condition $\int d\tau J^0 = \pm 1$ can be expressed as

$$\int d\tau \bar{\psi}\beta^0 \psi = \pm 2, \tag{2}$$

where the plus (minus) sign must be used for a positive (negative) charge.

   With the introduction of vector interactions, the DKP equation can be written as

$$(i\beta^\mu \partial_\mu - m - \beta^\mu A_\mu^{(1)} - i[P,\beta^\mu]A_\mu^{(2)})\psi = 0, \tag{3}$$

where P is a projection operator ($P^2 = P$ and $P^\dagger = P$) in such a way that $\bar{\psi}[P,\beta^\mu]\psi$ behaves like a vector under a Lorentz transformation as does $\bar{\psi}\beta^\mu \psi$. Once again $\partial_\mu J^\mu = 0$.[22] Notice that the vector potential $A_\mu^{(1)}$ is minimally coupled but not $A_\mu^{(2)}$. If the terms in the potentials $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are time-independent, one can write $\psi(\vec{r},t) = \phi(\vec{r})\exp(-iEt)$, where E is the energy of the boson, in such a way that the time-independent DKP equation becomes

$$[\beta^0(E - A_0^{(1)}) + i\beta^i(\partial_i + iA_i^{(1)}) - (m + i[P,\beta^\mu]A_\mu^{(2)})]\phi = 0. \tag{4}$$

In this case, $J^\mu = \bar{\phi}\beta^\mu \phi / 2$ does not depend on time, so that the spinor ϕ describes a stationary state. Note that the time-independent DKP equation is invariant under a simultaneous shift of E and $A_0^{(1)}$, such as in the Schrödinger equation, but the invariance does not maintain regarding E and $A_0^{(2)}$. It can be shown (see Ref. 22) that any two stationary states with distinct energies and subject to the boundary conditions

$$\int d\tau \partial_i (\bar{\phi}_\kappa \beta^i \phi_{\kappa'}) = 0 \tag{5}$$

are orthogonal in the sense that $\int d\tau \bar{\phi}_\kappa \beta^0 \phi_{\kappa'} = 0$ for $E_\kappa \neq E_{\kappa'}$. In addition, in view of (2) the spinors ϕ_κ and $\phi_{\kappa'}$ are said to be orthonormal if

$$\int d\tau \bar{\phi}_\kappa \beta^0 \phi_{\kappa'} = \pm 2\delta_{E_\kappa E_{\kappa'}}. \tag{6}$$

The charge-conjugation operation changes the sign of the minimal interaction potential, i.e., changes the sign of $A_\mu^{(1)}$. This can be accomplished by the transformation $\psi \to \psi_c = \mathcal{C}\psi = CK\psi$, where K denotes the complex conjugation and C is a unitary matrix, such that $C\beta^\mu = -\beta^\mu C$. The matrix that satisfies this relation is $C = \exp(i\delta_C)\eta^0\eta^1$. The phase factor $\exp(i\delta_C)$ is equal to ± 1, thus $E \to -E$. Note also that $J^\mu \to -J^\mu$, as should be expected for a charge current. Meanwhile, C anticommutes with $[P, \beta^\mu]$ and the charge-conjugation operation entails no change on $A_\mu^{(2)}$. The invariance of the nonminimal vector potential under charge-conjugation means that it does not couple to the charge of the boson. In other words, $A_\mu^{(2)}$ does not distinguish particles from anti-particles. Hence, whether one considers spin-0 or spin-1 bosons, this sort of interaction cannot exhibit Klein's paradox.

For the case of spin 0, we use the representation for the β^μ matrices given by[33]

$$\beta^0 = \begin{pmatrix} \theta & \bar{0} \\ \bar{0}^T & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} \tilde{0} & \rho_i \\ -\rho_i^T & \mathbf{0} \end{pmatrix}, \quad i = 1,2,3, \tag{7}$$

where

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8}$$

$\bar{0}$, $\tilde{0}$, and $\mathbf{0}$ are 2×3, 2×2, and 3×3 zero matrices, respectively, while the superscript T designates matrix transposition. Here the projection operator can be written as[5] $P = (\beta^\mu \beta_\mu - 1)/3 = \mathrm{diag}(1,0,0,0,0)$. In this case P picks out the first component of the DKP spinor. The five-component spinor can be written as $\psi^T = (\psi_1, \ldots, \psi_5)$ in such a way that the time-independent DKP equation for a boson constrained to move along the x-axis decomposes into

$$\left(\frac{d^2}{dx^2} + k^2 \right)\phi_1 = 0,$$

$$\phi_2 = \frac{1}{m}(E - A_0^{(1)} + iA_0^{(2)})\phi_1,$$

$$\phi_3 = \frac{i}{m}\frac{d\phi_1}{dx}, \quad \phi_4 = \phi_5 = 0, \tag{9}$$

where

$$k^2 = (E - A_0^{(1)})^2 - m^2 + (A_0^{(2)})^2. \tag{10}$$

Meanwhile,

$$J^0 = \frac{E - A_0^{(1)}}{m} |\phi_1|^2, \quad J^1 = \frac{1}{m} \mathrm{Im}\left(\phi_1^* \frac{d\phi_1}{dx} \right). \tag{11}$$

It is worthwhile to note that J^0 becomes negative in regions of space where $E < A_0^{(1)}$ (a circumstance associated to Klein's paradox) and that $A_\mu^{(2)}$ does not intervene explicitly in the current. With spinors satisfying (5), i.e.,

$$\left(\frac{d\phi_{1\kappa}^*}{dx} \phi_{1\kappa'} - \phi_{1\kappa}^* \frac{d\phi_{1\kappa'}^*}{dx} \right) \Bigg|_{x=x_{\inf}}^{x=x_{\sup}} = 0, \tag{12}$$

where $[x_{\inf}, x_{\sup}]$ is the range of x, orthonormalization formula (6) becomes

$$\int_{-\infty}^{+\infty} dx \frac{\frac{E_\kappa + E_{\kappa'}}{2} - A_0^{(1)}}{m} \phi_{1\kappa}^* \phi_{1\kappa'} = \pm \, \delta_{E_\kappa E_{\kappa'}} \tag{13}$$

regardless $A_\mu^{(2)}$. Equation (13) is in agreement with the orthonormalization formula for the Klein-Gordon theory in the presence of a minimally coupled potential.[34] This is not surprising because, after all, both DKP equation and Klein–Gordon equation are equivalent under minimal coupling.

For the case of spin 1, the β^μ matrices are[35]

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \bar{0}^T & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{14}$$

where s_i are the 3×3 spin-1 matrices $(s_i)_{jk} = -i\varepsilon_{ijk}$, e_i are the 1×3 matrices $(e_i)_{1j} = \delta_{ij}$, and $\bar{0} = (0 \ 0 \ 0)$, while $\mathbf{I}$ and $\mathbf{0}$ designate the 3×3 unit and zero matrices, respectively. In this representation $P = \beta^\mu \beta_\mu - 2 = \mathrm{diag}(1,1,1,1,0,0,0,0,0,0)$, i.e., P projects out the four upper components of the DKP spinor. With the spinor written as $\psi^T = (\psi_1, \ldots, \psi_{10})$, and partitioned as

$$\psi_I^{(+)} = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi_I^{(-)} = \psi_5,$$

$$\psi_{II}^{(+)} = \begin{pmatrix} \psi_6 \\ \psi_7 \end{pmatrix}, \quad \psi_{II}^{(-)} = \psi_2,$$

$$\psi_{III}^{(+)} = \begin{pmatrix} \psi_{10} \\ -\psi_9 \end{pmatrix}, \quad \psi_{III}^{(-)} = \psi_1, \tag{15}$$

the one-dimensional time-independent DKP equation can be expressed as

$$\left(\frac{d^2}{dx^2} + k^2 \right) \phi_I^{(\sigma)} = 0,$$

$$\phi_{II}^{(\sigma)} = \frac{1}{m} (E - A_0^{(1)} + i\sigma A_0^{(2)}) \phi_I^{(\sigma)},$$

$$\phi_{III}^{(\sigma)} = \frac{i}{m}\frac{d\phi_I^{(\sigma)}}{dx}, \quad \phi_8 = 0, \tag{16}$$

where k is again given by (10), and σ is equal to $+$ or $-$. Now the components of the four-current are

$$J^0 = \frac{E - A_0^{(1)}}{m}\sum_\sigma |\phi_I^{(\sigma)}|^2, \quad J^1 = \frac{1}{m}\mathrm{Im}\sum_\sigma \phi_I^{(\sigma)\dagger}\frac{d\phi_I^{(\sigma)}}{dx}, \tag{17}$$

and orthonormalization formula (6) takes the form

$$\int_{-\infty}^{+\infty} dx \frac{\frac{E_\kappa + E_{\kappa'}}{2} - A_0^{(1)}}{m}\sum_\sigma \phi_{I\kappa}^{(\sigma)\dagger}\phi_{I\kappa'}^{(\sigma)} = \pm\, \delta_{E_\kappa E_{\kappa'}}. \tag{18}$$

Just as for scalar bosons, $J^0 < 0$ for $E < A_0^{(1)}$ and $A_\mu^{(2)}$ does not appear in the current. Similarly, $A_\mu^{(2)}$ do not manifest explicitly in the orthonormalization formula. The prescribed orthonormalization expression is well-founded provided

$$\sum_\sigma \left(\frac{d\phi_{I\kappa}^{(\sigma)\dagger}}{dx}\phi_{I\kappa'}^{(\sigma)} - \phi_{I\kappa}^{(\sigma)\dagger}\frac{d\phi_{I\kappa'}^{(\sigma)}}{dx}\right)\Bigg|_{x=x_{\inf}}^{x=x_{\sup}} = 0. \tag{19}$$

Comparison between the two sets of formulas for the spin-0 and spin-1 sectors of the DKP theory evidences that vector bosons and scalar bosons behave in a similar way.

III. THE INVERSELY LINEAR POTENTIAL

Now we are in a position to use the DKP equation with specific forms for vector interactions. Let us focus our attention on time components of minimal and nonminimal vector potentials in the inversely linear form, viz.,

$$A_0^{(1)} = -\frac{g_1}{|x|}, \quad A_0^{(2)} = -\frac{g_2}{|x|}, \tag{20}$$

where the coupling constants and g_1 and g_2 are real parameters. In this case the first equations of (9) and (16) transmute into

$$-\frac{1}{2m}\frac{d^2\Phi}{dx^2} + V_{\mathrm{eff}}\Phi = E_{\mathrm{eff}}\Phi, \tag{21}$$

where Φ is equal to ϕ_1 for the scalar sector, and to $\phi_I^{(\sigma)}$ for the vector sector, with

$$V_{\mathrm{eff}} = -\frac{q}{|x|} + \frac{\alpha}{x^2}, \quad E_{\mathrm{eff}} = \frac{E^2 - m^2}{2m} \tag{22}$$

and

$$q = \frac{E}{m}g_1, \quad \alpha = -\frac{g_1^2 + g_2^2}{2m}. \tag{23}$$

Therefore, one has to search for bounded solutions in an effective Kratzer-like potential for $g_1 \neq 0$ or in an inversely quadratic potential for the case of a pure nonminimal vector potential ($g_1 = 0$). Inasmuch as the origin is a singular point of (21), one could expect the existence of singular solutions for Φ. In all the circumstances the effective potential presents a singularity at the origin given by $-1/x^2$. It is worthwhile to note at this point that the singularity $-1/x^2$ will never expose

the particle to collapse to the center[36] on the condition that α is greater than the critical value,

$$\alpha_c = -\frac{1}{8m}. \tag{24}$$

In the following, this necessary condition for the existence of bound-state solutions will be obtained in an alternative way. Note that the effective potential could bind the particle only if E_{eff} <0, corresponding to energies in the range $|E|<m$.

The Schrödinger equation with the Kratzer-like potential is an exactly solvable problem and its solution, for an attractive inversely linear term plus a repulsive inverse-square term in the potential, can be found on textbooks.[36–38] Since we need solutions involving either a repulsive or an attractive term in the inversely linear potential plus an attractive inversely quadratic potential, the calculation including this generalization is presented. Since V_{eff} is invariant under reflection through the origin $(x \rightarrow -x)$, eigenfunctions with well-defined parities can be built. Thus, one can concentrate attention on the positive half-line and impose boundary conditions on Φ at $x=0$ and $x=\infty$. Normalizability requires $\Phi(\infty)=0$ and the boundary condition at the origin will come into the existence by demanding orthogonality. As $x \rightarrow 0$, when the term $1/x^2$ dominates, the solution behaves as x^s, where s is a solution of the algebraic equation,

$$s(s - 1) - 2m\alpha = 0, \tag{25}$$

viz.,

$$s = \frac{1}{2}(1 \pm \sqrt{1 + 8m\alpha}). \tag{26}$$

Due to the twofold possibility of signs for s, it seems the solution of our problem cannot be uniquely determined. However, the sine qua non condition for orthogonality as dictated by (12) and (19) can be recast into a form similar to that one of the nonrelativistic case,[23,39]

$$\lim_{x \rightarrow 0}\left(\Phi_\kappa^* \frac{d\Phi_{\kappa'}}{dx} - \frac{d\Phi_\kappa^*}{dx}\Phi_{\kappa'} \right) = 0, \tag{27}$$

and there results that the allowed values for s are restricted to $\text{Re}(s) > 1/2$. Therefore, $\alpha > \alpha_c$ and the minus sign in (26) must be ruled out. That is to say that s is a real quantity in the open interval with $1/2 < s < 1$, or equivalently $0 < g_1^2 + g_2^2 < 1/4$. Under those conditions, the singular possibility for Φ is kept away and $|\Phi|^2/|x|$ behaves better than x^{-1} at the origin so that the square-integrability of Φ, even if $g_1 = 0$, is ensured. This tells us that the behavior of Φ at very small x implies into the homogeneous Dirichlet condition $\Phi(0) = 0$. We shall now distinguish the cases $g_1 = 0$ and $g_1 \neq 0$.

A. $g_1 = 0$

Defining

$$z = 2\sqrt{-2mE_{\text{eff}}}\,x, \tag{28}$$

where the quantity under the radical sign is either positive or negative, one obtains a special case of Whittakers differential equation,[40]

$$\Phi'' + \left(-\frac{1}{4} - \frac{2m\alpha}{z^2} \right)\Phi = 0. \tag{29}$$

The prime denotes differentiation with respect to z. The normalizable asymptotic form of the solution as $z \rightarrow \infty$ is $e^{-z/2}$ with $z > 0$. Notice that this asymptotic behavior rules out the possibility $E_{\text{eff}} > 0$, as has been pointed out already based on qualitative arguments. The exact solution can now be written as

$$\Phi = z^s w(z) e^{-z/2}, \tag{30}$$

where w is a regular solution of the confluent hypergeometric equation (Kummer's equation),[40]

$$zw'' + (b - z)w' - aw = 0, \tag{31}$$

with the definitions

$$a = s, \quad b = 2s. \tag{32}$$

The general solution of (31) is expressed in terms of the confluent hypergeometric functions (Kummer's functions) $_1F_1(a,b,z)$ [or $M(a,b,z)$] and $_2F_0(a,1+a-b,-1/z)$ [or $U(a,b,z)$],

$$w = A \, _1F_1(a,b,z) + B z^{-a} \, _2F_0\left(a, 1+a-b, -\frac{1}{z}\right), \quad b \neq -\tilde{n}, \tag{33}$$

where $\tilde{n}$ is a non-negative integer. Due to the singularity of the second term at $z=0$, only choosing $B=0$ gives a behavior at the origin which can lead to square-integrable solutions. Furthermore, the requirement of finiteness for Φ at $z=\infty$ implies that the remaining confluent hypergeometric function ($_1F_1(a,b,z)$) should be a polynomial. This is because $_1F_1(a,b,z)$ goes as e^z as z goes to infinity unless the series breaks off. This demands that $a=-n$, where n is also a non-negative integer. This requirement combined with (32) implies that the existence of bound-state solutions for pure inversely quadratic potentials is out of question.

B. $g_1 \neq 0$

As for $g_1 \neq 0$, it is convenient to define the dimensionless quantity γ,

$$\gamma = q \sqrt{-\frac{m}{2E_{\text{eff}}}}, \tag{34}$$

and using (21)–(23), with z defined in (28), one obtains the complete form for Whittaker's equation,[40]

$$\Phi'' + \left(-\frac{1}{4} + \frac{\gamma}{z} - \frac{2m\alpha}{z^2}\right)\Phi = 0. \tag{35}$$

Because $\alpha \neq 0$, the normalizable asymptotic form of the solution as $z \to \infty$ is again given by $e^{-z/2}$ and $E_{\text{eff}} < 0$, i.e., $|E| < m$. The solution for all z can be again expressed as in (30), but now w is the regular solution of Kummer's equation with

$$a = s - \gamma, \quad b = 2s. \tag{36}$$

Then w is expressed as $_1F_1(a,b,z)$ and in order to furnish normalizable Φ, the confluent hypergeometric function must be a polynomial. This demands that $a=-n$, where n is a non-negative integer in such a way that $_1F_1(a,b,z)$ is proportional to the associated Laguerre polynomial $L_n^{(b-1)}(z)$, a polynomial of degree n. This requirement, combined with (36), also implies into quantized energies,

$$E = \varepsilon(g_1)m\left\{1 + \left[\frac{g_1}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - (g_1^2 + g_2^2)}}\right]^2\right\}^{-1/2}, \quad n = 0, 1, 2, \ldots, \tag{37}$$

where ε, the sign function, is there because $Eg_1 > 0$ due to the fact that $\gamma = n + s > 0$ ($q > 0$).

On the half-line, Φ is given by

$$\Phi(z) = N z^s e^{-z/2} L_n^{(2s-1)}(z), \quad n = 0, 1, 2, \ldots, \tag{38}$$

where N is a constant related to the normalization. Eigenfunctions on the whole line with well-defined parities can be built. Those eigenfunctions can be constructed by taking symmetric and antisymmetric linear combinations of Φ. These new eigenfunctions possess the same eigenenergy, then, in principle, there is a twofold degeneracy. Nevertheless, the matter is a little more complicated because the effective potential presents a singularity. Since Φ vanishes at the origin, the symmetric combination of Φ presents a discontinuous first derivative at the origin. In fact, Φ is not differentiable at the origin (recall that near the origin, Φ behaves like x^s with $1/2 < s < 1$). In the specific case under consideration, the effect of the singularity of the potential on $\Phi' = d\Phi/dx$ can be evaluated by integrating (21) from $-\delta$ to $+\delta$ and taking the limit $\delta \to 0$. The connection condition among $\Phi'(+\delta)$ and $\Phi'(-\delta)$ can be summarized as

$$\Phi'(+\delta) - \Phi'(-\delta) = 2m \int_{-\delta}^{+\delta} dx V_{\text{eff}} \Phi. \tag{39}$$

Substitution of (38) into (39) allows us to conclude that only the odd-parity combination furnishes a consistent result. This happens because the right-hand side of (39) vanishes for an odd eigenfunction, as it should do. For an even eigenfunction, though, the right-hand side of (39) should equal $-2\Phi'(-\delta)$ for arbitrary g_1 and g_2, but it does not. Therefore, we are forced to conclude that the Φ must be an odd-parity function. As an unavoidable conclusion, the bound-state solutions are nondegenerate.

IV. CONCLUSIONS

We succeed in searching for exact DKP bounded solutions for massive particles by considering a mixing of minimal and nonminimal vector inversely linear potentials for spin-0 and spin-1 bosons in a unified way. The solution of the DKP-Coulomb problem was uniquely determined by requiring orthonormalizability. As a bonus, the appropriate boundary conditions on Φ were proclaimed. A pure nonminimal coupling does not hold bound-states. For $g_1 \neq 0$, there is an infinite set of bound-state solutions either for particles (in the range $0 < E < m$) or for antiparticles (in the range $-m < E < 0$). The spectrum does not distinguish the sign of g_2, but E goes to $-E$ as $g_1 \to -g_1$ as it has already been anticipated by the charge-conjugate transformation arguments in Sec. II. No matter the signs of the potentials or how strong they are, the particle and antiparticle levels neither meet nor dive into the continuum. Thus, there is no room for the production of particle-antiparticle pairs. This all means that Klein's paradox never manifests. The regime of weak coupling ($0 < g_1 \ll 1/2$ and $|g_2| \ll 1/2$) runs in the nonrelativistic limit, viz., $E - m \simeq -m g_1^2 / [2(n+1)^2]$. This nonrelativistic limit, where only the g_1-dependence survives, corresponds to the energy levels for particles in a nonrelativistic one-dimensional Coulomb potential.[23] Invariably, the spectrum is nondegenerate and the eigenfunction behaves as an odd-parity function.

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