

Suppressing Fermi Acceleration in a Driven Elliptical Billiard

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(Received 16 December 2009; published 1 June 2010)

We study dynamical properties of an ensemble of noninteracting particles in a time-dependent elliptical-like billiard. It was recently shown [Phys. Rev. Lett. **100**, 014103 (2008)] that for the non-dissipative dynamics, the particle experiences unlimited energy growth. Here we show that inelastic collisions suppress Fermi acceleration in a driven elliptical-like billiard. This suppression is yet another indication that Fermi acceleration is not a structurally stable phenomenon.

DOI: [10.1103/PhysRevLett.104.224101](https://doi.org/10.1103/PhysRevLett.104.224101)

PACS numbers: 05.45.Pq, 05.45.Tp

Fermi acceleration (FA) is a phenomenon that occurs when a classical particle acquires unlimited energy upon collisions with a heavy and moving wall. The original idea is due to Fermi [1] who assumed that the enormous energy of the cosmic particles comes from interactions with moving magnetic clouds. After that many different 1D Fermi accelerator models were studied [2–5]. Basically they are composed of a classical particle which experiences collisions with a moving wall. A source of returning for a next collision can be a fixed wall [3,4], a gravitational field [5], or both [6]. A simple generalization to 2D is to consider the dynamics of the particle inside a billiard domain that, depending on the shape of the boundary, demonstrates regular [7], mixed [8], or fully chaotic dynamics [9]. Applications of billiards to physical problems include superconducting [10] and confinement of electrons in semiconductors by electric potentials [11,12], ultracold atoms trapped in a laser potential [13–16], mesoscopic quantum dots [17], reflection of light from mirrors [18], waveguides [19,20], and microwave billiards [21,22].

If the boundary is time dependent, the Loskutov-Ryabov-Akinshin (LRA) conjecture [23] claims that chaotic dynamics for a billiard with the static boundary is a sufficient condition to produce FA if a time perturbation of the boundary is introduced. This conjecture was confirmed in many models [24–26]. Recently, however, [27] a specific perturbation in the boundary of an integrable elliptical billiard led to the observation of a tunable FA. The result discussed in [27] was a break of two paradigms: (i) it was expected [28] that the elliptical billiard, which is integrable for static boundary and therefore demonstrates the most regular dynamics, does not exhibit FA; and (ii) since the static version of the elliptical billiard does not have chaotic dynamics, then the LRA conjecture [23] should be extended.

In this Letter we show that the mechanism which produces FA in the time-dependent elliptical-like billiard can be broken by nonelastic collisions. Since the destruction is observed for very small dissipation, one can conjecture that

FA is not a structurally stable phenomenon. We consider the dynamics of an ensemble of noninteracting particles in a time-dependent elliptical-like billiard. Our results show that initial conditions chosen along the separatrix curve of the billiard with a static boundary lead the particle to exhibit FA. Thus the LRA conjecture can be extended to the existence of a heteroclinic orbit in the phase space instead of the existence of a set with chaotic dynamics. The mechanism which produces FA, as discussed in [27], is the successive crossings by the particle of a neighborhood of a separatrix curve in the static case, which under time perturbation to the boundary turns into a stochastic layer. Such crossings change the dynamics of the particle from rotation to libration (or vice versa) and causes the kinetic energy of the particle to fluctuate. These fluctuations increase with time and lead to an anomalous diffusion and consequently to FA. Here we demonstrate that inelastic collisions of the particle with the boundary break down the mechanism of FA and therefore suppress the unlimited energy gain of the bouncing particle. The dissipation stops the successive crossings of the particle of the stochastic layer. This suppression confirms a conjecture [29] for suppression of FA in 2D billiards under inelastic collisions.

The model under study consists of a classical particle confined to a closed domain whose boundary changes in time according to the following equation in polar coordinates

$$R(\theta, e, a, t) = \frac{1 - e^2[1 + a \cos(t)]^2}{1 + e[1 + a \cos(t)] \cos(q\theta)}, \quad (1)$$

where e is the eccentricity of the ellipse, $q \geq 1$ is an integer, a is the amplitude of the time perturbations, θ is the angular coordinate, and t is the time. If $a = 0$ and $q = 1$, the results for the static case are recovered where there are two conserved quantities: (1) the kinetic energy of the particle [7] and (2) the angular momentum about the two foci [8], which is equal to

$$F(\alpha, \theta) = \frac{\cos^2(\alpha) - e^2 \cos^2(\phi)}{1 - e^2 \cos^2(\phi)}. \quad (2)$$

The dynamics of the model is described by the following implicit 4D mapping. We start with the initial condition $(\theta_n, \alpha_n, V_n, t_n)$, where α_n is the angle between the trajectory of the particle and the tangent line to the boundary of the billiard table with the angular coordinate θ_n , $V_n > 0$ is the velocity of the particle and t_n is the moment of the n th collision of the particle with the boundary. Figure 1 illustrates the corresponding coordinate angles for a typical orbit in the elliptical billiard. Given the initial condition, the dynamics of the particle is given via the relations $X(t) = X(\theta_n, t_n) + |\vec{V}_n| \cos(\alpha_n + \phi_n)(t - t_n)$ and $Y(t) = Y(\theta_n, t_n) + |\vec{V}_n| \sin(\alpha_n + \phi_n)(t - t_n)$ where $\phi_n = \arctan(Y'(\theta_n, t_n)/X'(\theta_n, t_n))$, $X' = dX/d\theta$, and $Y' = dY/d\theta$. The new angular coordinate θ_{n+1} is obtained by following the trajectory of the particle via molecular dynamics until the moment $t = t_n + \Delta t$ where Δt satisfies the equation

$$\sqrt{X^2(\Delta t) + Y^2(\Delta t)} = \frac{1 - e^2[1 + a \cos(\Delta t)]^2}{1 + e[1 + a \cos(\Delta t)] \cos(q\theta)}. \quad (3)$$

The moment of the next collision is obtained as

$$t_{n+1} = t_n + \sqrt{\frac{[X(\theta_{n+1}) - X(\theta_n)]^2 + [Y(\theta_{n+1}) - Y(\theta_n)]^2}{|\vec{V}_n|}}. \quad (4)$$

The reflection rule for the collision of the particle with the boundary is given via the relations

$$\vec{V}'_{n+1} \cdot \vec{T}_{n+1} = \vec{V}'_n \cdot \vec{T}_{n+1}, \quad (5)$$

$$\vec{V}'_{n+1} \cdot \vec{N}_{n+1} = -\gamma \vec{V}'_n \cdot \vec{N}_{n+1}, \quad (6)$$

where \vec{T} and \vec{N} are the unit tangent and normal vectors, respectively, the prime indicates that the velocity of the particle is measured in the reference frame of the moving wall, and $\gamma \in [0, 1]$ is the restitution coefficient. For $\gamma =$

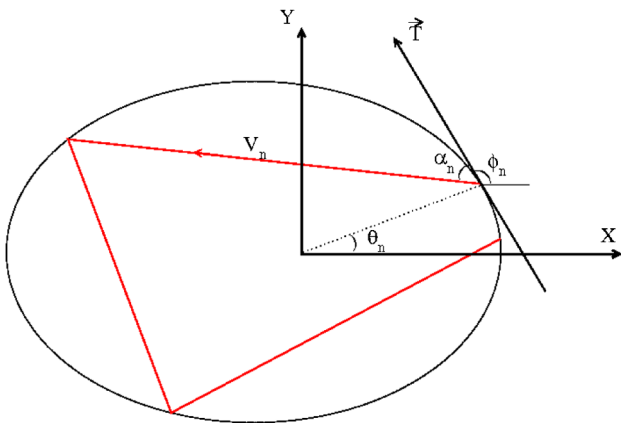


FIG. 1 (color online). Sketch of the elliptical billiard.

1 one gets the case of the elastic collisions. Based on Eqs. (5) and (6), the components of the particle's velocity after collision are given by

$$\vec{V}_{n+1} \cdot \vec{T}_{n+1} = |\vec{V}_n| [\cos(\alpha_n + \phi_n) \cos(\phi_{n+1})] + |\vec{V}_n| [\sin(\alpha_n + \phi_n) \sin(\phi_{n+1})], \quad (7)$$

$$\begin{aligned} \vec{V}_{n+1} \cdot \vec{N}_{n+1} = & -\gamma |\vec{V}_n| [\sin(\alpha_n + \phi_n) \cos(\phi_{n+1})] \\ & - \gamma |\vec{V}_n| [-\cos(\alpha_n + \phi_n) \sin(\phi_{n+1})] \\ & + (1 + \gamma) \frac{dR(t)}{dt} [\sin(\theta_{n+1}) \cos(\phi_{n+1})] \\ & - (1 + \gamma) \frac{dR(t)}{dt} [\cos(\theta_{n+1}) \sin(\phi_{n+1})], \end{aligned} \quad (8)$$

where dR/dt is the velocity of the moving boundary at the $(n + 1)$ th collision. The velocity of the particle after this collision is

$$V_{n+1} = \sqrt{(\vec{V}_{n+1} \cdot \vec{T}_{n+1})^2 + (\vec{V}_{n+1} \cdot \vec{N}_{n+1})^2}, \quad (9)$$

and the coordinate angle α_{n+1} equals

$$\alpha_{n+1} = \arctan([\vec{V}_{n+1} \cdot \vec{N}_{n+1}] / [\vec{V}_{n+1} \cdot \vec{T}_{n+1}]). \quad (10)$$

Figure 2 shows the phase space for the static boundary for $q = 1$ overlapped with a stochastic layer around the separatrix which emerges when time perturbations to the boundary are introduced. $F > 0$ correspond to the rotator orbits and $F < 0$ to the librator ones. The average velocity of the particle as a function of n is shown in Fig. 3 for different control parameters. The initial conditions used were $\alpha_0 = \pi/2$, $\theta_0 = \pi$, which correspond to the location of the heteroclinic point along the separatrix for the static

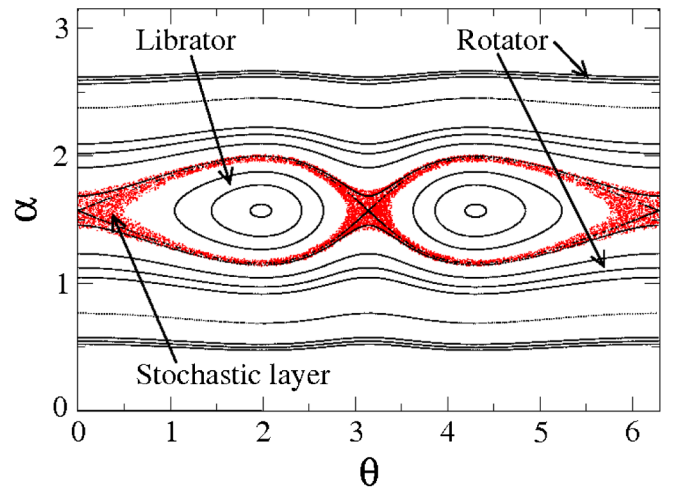


FIG. 2 (color online). Phase space of the static billiard overlapped with a stochastic layer created by the time perturbations. The control parameters used were static case $e = 0.4$, $q = 1$ and time-dependent perturbation $e = 0.4$, $a = 0.01$, and $\gamma = 1$ with $V_0 = 1$ and 10^4 collisions with the boundary.

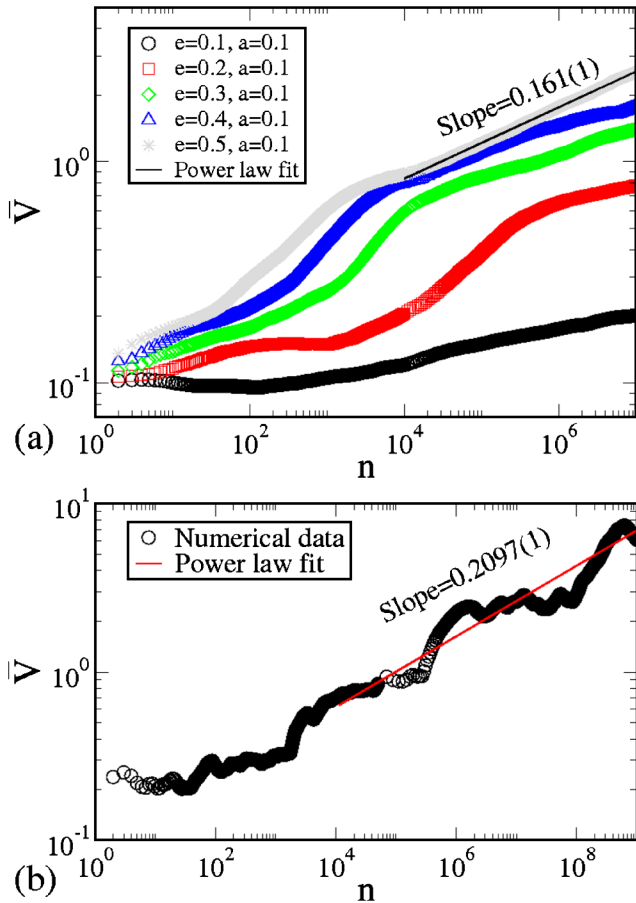


FIG. 3 (color online). (a) $\bar{V} \times n$ for an ensemble of 100 particles for $q = 1$. The parameters are labeled in the figure. (b) $\bar{V} \times n$ for a single particle. The control parameters were $e = 0.5$, $a = 0.1$, and $\gamma = 1$. The slope obtained is 0.2097(1).

boundary with $q = 1$, $V_0 = 10^{-1}$ and 100 uniformly distributed $t_0 \in [0, 2\pi]$. The control parameters are labeled in the figure. One can see that after an initial transient and the regime of a fast growth marked by a strongly chaotic behavior of F [see Fig. 5(b)], the curves stabilize in a regime of a constant growth. A power law fitting for the control parameter $e = 0.5$ and $a = 0.1$ gives the slope 0.161(1). Note, however, that the time perturbations used in this paper are rather different from the ones used in [27], which lead to different growth exponents. The simulations were run up to 10^7 collisions with the boundary. A longer simulation for a single initial condition up to 10^9 collisions with the boundary is shown in Fig. 3(b). The slope of the growth for this curve is slightly larger and a power law fitting gives 0.2097(1), which clearly confirms the presence of FA. These results allow us to extend the LRA conjecture [23] and replace chaotic dynamics for the static boundary to the existence of a heteroclinic orbit in the phase space. When the time perturbations are introduced, the separatrix turns into a stochastic layer [27] therefore leading the particle to experience FA.

The behavior observed for the elliptical domain, i.e., $q = 1$ can be extended to other shapes of the boundary.

Moreover, if very thin stochastic layers exist, the introduction of time dependence of the boundary enlarges them, therefore leading the particle to exhibit FA. Consider, for example, $q = 3$. For the static case, i.e., $a = 0$, the control parameter $e_c = 1/(q^2 - 1)$ marks a change when the boundary exhibits nonconcave pieces. The invariant spanning curves in the phase space are therefore destroyed for $e \geq e_c$. Figure 4(a) shows the phase space for $q = 3$. One can clearly see two symmetric chains of period-three orbits separated from a very thin stochastic layer that, for the definition of the pixels of the figure, rather look like separatrix curves. One of them is around $\alpha = \pi/3$ and the other is around $\alpha = 2\pi/3$. There are also three chains of period-six orbits where one of them is near $\alpha = \pi/2$ and the other two are near the top and near bottom, respectively. An overlap with the stochastic layers for the time-dependent boundary is shown in Fig. 4(b). Therefore, one is lead to believe that FA will take place, and it indeed does.

We now discuss the effect of inelastic collisions when $\gamma < 1$. As the particle hits the boundary, there is a loss of energy upon collision which changes drastically the dynamics of the particle. Namely, the particle wanders in the

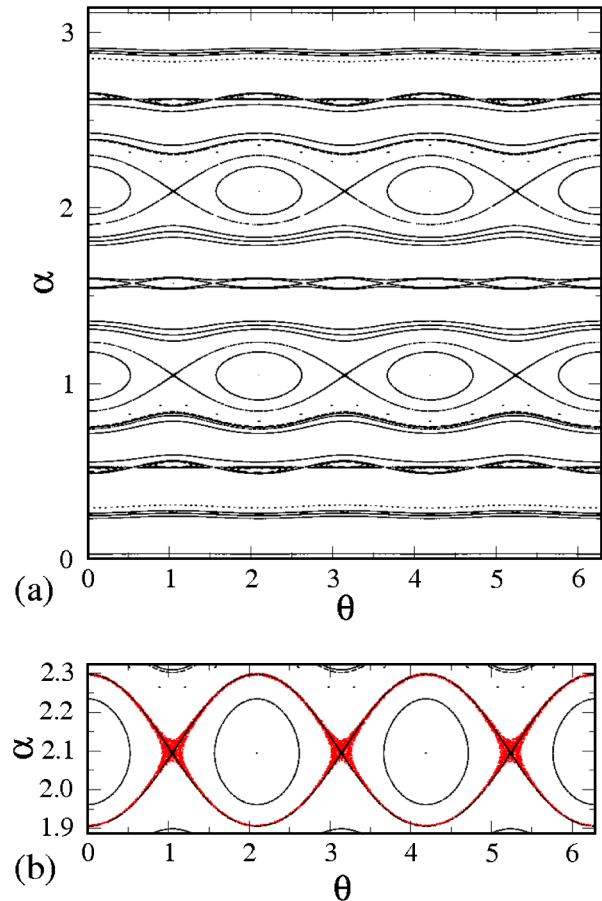


FIG. 4 (color online). (a) Phase space for $q = 3$ and $e = 0.01$. (b) Enlargement of the upper period-three chain (black) overlapped by the large stochastic layer [gray (red)] created by the time dependence. We used $a = 0.01$.

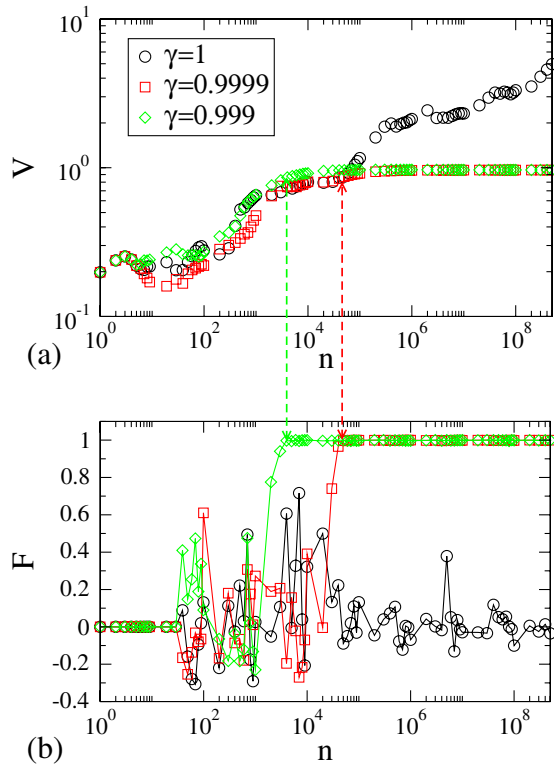


FIG. 5 (color online). (a) Plot of $V_n \times n$ for different damping coefficients, as labeled in the figure. (b) Plot of $F_n \times n$ for the same control parameters as in (a).

stochastic layer for a while. After that, it escapes and stays trapped either in librator or rotator orbits. Figure 5(a) shows the velocity of the particle as a function of n , for $q = 1$ and for three different damping coefficients: $\gamma = 1$ (non-dissipative case), $\gamma = 0.9999$, and $\gamma = 0.999$. One can see that even for a small dissipation, the regime of the energy growth is interrupted, therefore leading the particle's energy to a constant plateau. Figure 5(b) shows, in a log \times linear plot, the time evolution of the observable F [see Eq. (2)]. One can see that for $\gamma = 1$, the value of F fluctuates around 0 up to 5×10^8 collisions. The dynamics of the particle for two considered values of $\gamma < 1$, demonstrates a trapping in the rotator orbits after some hundreds of collisions. However the particle can also evolve towards librator orbits. These results confirm that FA is suppressed because of a breakdown of the mechanism producing FA. Since the mapping changes very little with the inelastic collisions, one can claim that FA is not a structurally stable phenomenon.

To conclude, we have shown that suppression of FA by the introduction of even very small dissipation is possible for dynamical systems in which the static (unperturbed) situation demonstrate completely regular (integrable) be-

havior. Therefore FA seems to be not a structurally stable phenomenon.

The work of E. D. L. was partially supported by CNPq, FAPESP, and FUNDUNESP. He also acknowledges support for a visit to Georgia Tech from CAPES/FULBRIGHT. L. B. was partially supported by NSF Grant No. DMS-0900945.

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