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R. Aldrovandi

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Extended gauge theories

R. Aldrovandi[a)]
Laboratoire de Physique Théorique (Unité Associée au CNRS n° 769), Institut Henri Poincaré, Université Pierre et Marie Curie, 11, rue Pierre et Marie Curie, 75231 Paris Cedex 05, France

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A scheme inspired in Lie algebra extensions is introduced that enlarges gauge models to allow some coupling between space-time and gauge space. Everything may be written in terms of a generalized covariant derivative including usual differential plus purely algebraic terms. A noncovariant vacuum appears, introducing a natural symmetry breaking, but currents satisfy conservation laws alike those found in gauge theories.

I. INTRODUCTION

The phenomenological successes of the standard model encompassing the Weinberg–Salam model and QCD brought about an optimistic hope that gauge theories would provide the clue to the whole question of fundamental processes. It was anticipated that gravitation would also submit to a joint picture and a more general, unified theory would be found including and surpassing both the standard model and General Relativity. However, as the years went by, it was realized that such high expectations were not being fulfilled. The large number of free parameters in electroweak theory, the difficulties with grand unification, the lasting frustation concerning the quark confinement problem and the dogged resistence of gravitation to submit to the standard gauge scheme have progressively changed those feelings. Experimental results are more then enough a guarantee that gauge theories are fundamental indeed, but theory would say that they must be somehow enlarged, the present dominating trends being those involving strings and supersymmetry. We wish here to examine the first steps into another, quite different kind of generalization of gauge theories, inspired in the theory of Lie algebra extensions. Experience in gravitational gauge modeling strongly suggests a coupling between space-time and gauge space, while gauge models presuppose a strict local separation between them. The aim will be to find an acceptable compromise between these two conflicting positions.

The general ideas are exposed in Sec. II, where attention is called to the fact that independence between space-time and gauge space is equivalent to the adjoint character of the gauge potential. It follows in Sec. III a purely descriptive resumé on the subject of Lie algebra extensions,[1,2] in reality an adaptation of material on group extensions[3,4] which is more abundant in the physical literature. In the next section some homological language[5] is introduced and the previous results translated into it. The tone is rather pedagogical, introducing algebraic terminology via analogy with the supposedly known language of differential forms. Although modern treatments of the subject make use of general modules,[6] we prefer to follow here the physically more intuitive aproach using representations (or better, actions). This also allows us the consideration of the case in which neither algebra reduces to a module of the other and leads to a difference with respect to usual treatments, with the use of algebra-valued cochains instead of the module-valued ones. "Noncommutative" modules are extensively used in noncommutative geometry,[7] so that our approach is more akin in spirit to cyclic cohomology. Progressive introduction of bundle language hopefully paves the way to a later comparison with the geometrical approach to gauge theories. In order to keep notation and language at a reasonable level of simplicity, we adopt a rather free way of speaking, forgetting about sections, pull-backs, etc. whenever they are not essential. An algebraic derivative appears, in terms of which cohomology of representations has a treatment formally similar to the cohomology of differential forms. Group extensions have been used[3] in the sixties to provide the formal proof of the so-called no-go theorems, or the theorems of McGlinn type.[8] Such theorems forbade coupling between internal and space-time symmetries at the algebraic level, an interdiction circumvented by gauge theories through the introduction of local vector fields. A brief outline on vector fields on manifolds and of the bundle structure of gauge theories is given in Sec. V. In this case, of course, usual differentials are also at work, and it turns out that everything can be written in terms of generalized derivatives, sums of usual and algebraic derivatives. Finally, the general field equations leading to what we claim to be the natural generalizations of Yang–Mills equations are given in Sec. VI. Generalized source currents satisfy, rather surprisingly, a conservation law analogous to the covariant divergenceless of currents in gauge theories.

II. GAUGE SPACE AND SPACE-TIME

Separation between space-time and internal space is inherent to the subjacent geometrical structure in the gauge scheme, a differentiable fiber bundle which is a smooth manifold combining "internal" gauge space and space-time in such a way that the total space is locally a direct product of both. In very simple words, disregarding sections and pull-backs, around any point of the bundle there exists a neighborhood on which a "separated" basis $\{X_\mu\} = \{X_a, X_i\}$ is defined for the local vector fields, the first m (= space-time dimension) fields X_a representing a basis for space-time

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fields and the remaining X_i representing a basis for the gauge group algebra. In such "direct product" basis[9] the members will have commutation relations of the form

$$[X_a,X_b]=f^c{}_{ab}X_c,\quad [X_a,X_i]=0,\quad [X_k,X_i]=C^j{}_{ki}X_j. \tag{2.1}$$

The right-hand term in the first relation represents merely the possible anholonomicity of the space-time basis, which we shall keep for a while because it gives a similar aspect to the two algebras involved. The vanishing of the second commutator signals the separation between space-time and the gauge space, the direct-product character of the association of the algebras generated by the X_a's and the X_j's: In pedant language, the fields X_j's respond to the action of the X_a's according to the null representation. Equations (2.1) summarize the basic, underlying background into which the gauge potentials are to be inserted.

In the presence of a connection (or gauge potential) $\alpha = X_j\alpha^j$ the X_a's are replaced by the covariant derivatives $X'_a = X_a - \alpha^j{}_a X_j$, in terms of which the commutation relations become

$$[X'_a,X'_b]=f^c{}_{ab}X'_c - F^j{}_{ab}X_j,\quad [X'_a,X_i]=0,\quad [X_k,X_i]=C^j{}_{ki}X_j. \tag{2.2}$$

This new basis is sometimes called the "horizontal lift" basis. The fields $\{X_i\}$ keep forming an ideal of the total field algebra and a representation of the gauge group algebra. The coefficients

$$F^k{}_{ab}=X_a(\alpha^k{}_b)-X_b(\alpha^k{}_a)-f^c{}_{ab}\alpha^k{}_c - C^k{}_{ij}\alpha^i{}_a\alpha^j{}_b \tag{2.3}$$

represent the connection curvature (gauge field strength). The vanishing of the second commutator requires now that

$$X_i(\alpha^j{}_a)=C^j{}_{ki}\alpha^k{}_a, \tag{2.4}$$

a condition also used to obtain (2.3). Of course, this only says that α belongs to the adjoint representation and is the expected behavior of a connection under infinitesimal gauge transformations, but it is important to notice this relation between α's adjointness and the independence between "internal" gauge space and space-time.

Fields on manifolds generate one-dimensional local transformation groups. A field Y responds to the action generated by a field X according to the Lie derivative $L_X Y=[X,Y]$ and the commutation relations describe the actions of transformations generated by the fields on each other. The second relation says that space-time fields do not "feel" gauge transformations and gauge fields do not "feel" space-time transformations. Anholonomicity being unessential, the X_a in (2.1) may be seen as representing the generators of the group of translations on space-time. The presence of a connection modifies these translation generators to X'_a, thereby altering space-time homogeneity. Through the Lie derivative, the presence of the fields X_i will affect every tensor on the bundle. Echos of such effect will arrive at the associated bundles, formed by space-time combined with group multiplets to which particles are attributed. Each multiplet will carry a representation of the gauge group. The covariant derivative on such associated bundles will have, instead of the fields X_i, their representative operators acting on the given representation. The gauge generators X_i, kept unmodified, do not respond to space-time transformations at all. We would of course expect something different were we to build a gauge model for the space-time symmetries themselves.

In reality, the local underlying geometrical structure of gauge theories is completely fixed by the above commutation relations, because of the Jacobi identities. The Jacobi identity for three space-time fields X'_a, X'_b, X'_c gives the Bianchi identity

$$\begin{aligned}[][X'_a,[X'_b,X'_c]]+[X'_c,[X'_a,X'_b]]+[X'_b,[X'_c,X'_a]]\\ = -\{X'_a(F^j{}_{bc})+X'_c(F^j{}_{ab})+X'_b(F^j{}_{ca})\\ +f^d{}_{ab}F^j{}_{cd}+f^d{}_{ca}F^j{}_{bd}+f^d{}_{bc}F^j{}_{ad}\}X=0,\end{aligned} \tag{2.5}$$

or, in invariant language,

$$dF+[\alpha,F]=0. \tag{2.6}$$

The Jacobi identity for three fields X'_a, X'_b, and X_i gives

$$\begin{aligned}[][X'_a,[X'_b,X_i]]+[X_i,[X'_a,X'_b]]+[X'_b,[X_i,X'_a]]\\ = -\{X_i(F^j{}_{ab})+C^j{}_{ik}F^k{}_{ab}\}X_j=0,\end{aligned} \tag{2.7}$$

which simply states the covariance of the curvature under gauge transformations: also F belongs to the adjoint representation. Consequently, the commutation rules (2.2) do contain the basic geometrical background, to which of course dynamics is to be added. We can introduce dynamics through a Lagrangian or by the "duality rule" which states[10] that the field equations are, in the sourceless case, just (2.5), (2.6) written for the dual $\tilde{F}$ of F,

$$\begin{aligned}X_aF^{jab}+f^c{}_a{}^bF^{ja}{}_c - C^j{}_{ki}\alpha^k{}_aF^{iab}\\ = X'_aF^{jab}+f^c{}_a{}^bF^{ja}{}_c=0,\end{aligned}$$

which is the same as

$$d\tilde{F}+[\alpha,\tilde{F}]=0. \tag{2.8}$$

Noether source currents are then added to the right-hand side. This rule has the advantage of giving the correct field equations even when no Lagrangian is present,[11] as is the case when the gauge group is non-semisimple.[12] From (2.8) follows a severe constraint on the source current J,

$$d\tilde{J}-[\alpha,\tilde{J}]=0. \tag{2.9}$$

This property ensures the gauge invariance of the total (gauge field plus sources) system.

The commutation relations above suppose a very special association, on the bundle, of the two algebras $\{X_a\}$ and $\{X_i\}$ of the representatives of space-time fields and gauge group left-invariant fields. Algebra associations are the object of the theory of Lie algebra extensions. We shall see below how this theory, besides providing some new insight on gauge theories in general, suggests a modification of the above relations to

$$[X'_a,X'_b]=f^c{}_{ab}X'_c - F^j{}_{ab}X_j - \beta^j{}_{ab}X_j,\quad [X'_a,X_i]=C^j{}_{ai}X_j, \tag{2.10}$$

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$$[X_k,X_i] = C^j{}_{ki}X_j.$$

The second relation admits now a coupling between gauge and space-time transformations, of which the additional "curvature" β is a measure. The (modified) translation generators act on the gauge generators no more through the null representation, but according to a representation determined by the coefficients $C^j{}_{ai}$. Some coupling of the sort must exist if gravitation is to be given by a gauge theory related to space-time symmetries, obviously not indifferent to translations.

III. LIE ALGEBRA EXTENSIONS

Let L be a Lie algebra with generators J_a and commutation relations

$$[J_a,J_b] = f^c{}_{ab}J_c. \tag{3.1}$$

Consider a vector space V on which a representation ρ of L is defined. ρ is a mapping of L into the set of automorphisms of V,

$$\rho: L \to \mathrm{Aut}(V),$$

$$\rho: J_a \to \rho(J_a).$$

It is helpful to take a basis $\{X_j\}$ on V and see each $\rho(J_a)$ as a matrix with elements

$$[\rho(J_a)]^j{}_i = C^j{}_{ai}, \tag{3.2}$$

with the representation acting according to

$$\rho(J_a)(X_i) = C^j{}_{ai}X_j. \tag{3.3}$$

More precisely, ρ is an *action* of L on V, that is, a mapping

$$\rho: L \otimes V \to V,$$

leading each pair (J_a, X_i) into $C^j{}_{ai}X_j \in V$. It is preferable to use the word *action* instead of *representation* (which we shall nevertheless be using rather freely for its intuitive value), since $J_a \to \rho(J_a)$ is not *a priori* a homomorphism. When ρ is a homomorphism, V is the *carrier space* of the representation. As long as V is simply a vector space, V is an *L module*, but we shall below drop this condition.

Let us recall that a Lie algebra L consists of an underlying vector space on which an antisymmetric internal operation $[,]_L$ is defined which satisfies the Jacobi identity. In particular, any vector space like V above may be considered as a commutative Lie algebra, generated by matrices X_i such that

$$[X_i,X_j]_V = 0. \tag{3.4}$$

It is more economical to consider once and for all the general case coming out when V is not simply a vector space, or a commutative algebra, but a nontrivial algebra by itself. This means that we have instead

$$[X_i,X_j]_V = C^k{}_{ij}X_k, \tag{3.5}$$

with $C^k{}_{ij}$ some structure constants. Of course, V is then no more a simple L module and we actually have the action of an algebra on the other (which modern authors prefer to call an *operation*[13]). A particular example occurs when $L \equiv V$ and the matrices (3.2) with elements $[\rho(J_a)]^c{}_b = f^c{}_{ab}$ generate the adjoint representation of L.

Consider now the direct sum $L \oplus V$ of the underlying vector space of L and the underlying vector space of V. How can we combine the algebras L and V to get a larger Lie algebra E with underlying vector space $L \oplus V$? The direct product case is well known, which appears when, once absorbed into E, both L and V constitute subalgebras ignoring each other: every element of L commutes with every element of V. The general answer is, however, that L and V may be combined in various ways to give different algebras E (the *extended algebras*), depending mainly on the action of L on E.

We want to obtain E as an "*extension of L by V*" and now proceed to discuss under which conditions E deserves such a name. To begin with, once L and V are "immersed" in the larger vector space $L \oplus V$, the operation $[,]_E$ in E does not necessarily coincide with those of L and V. Furthermore, once $[,]_E$ is defined, there will be conditions on the structure coefficients, coming from the Jacobi identities. And then, when such conditions are satisfied, the particular algebra E obtained depends on the action of L on V, on how the original representation behaves when considered in the enlarged space.

A first condition, justifying the expression "extension *by V*," is that V be simply *included* in $L \oplus V$. More formally, the mapping i: $V \to E$ is an inclusion and preserves the algebra V. This means that i is an algebra isomorphism and the E bracket $[,]_E$ will, when restricted to the $\{X_j\}$, coincide with that of (3.5):

$$[X_i,X_j]_E = [X_i,X_j]_V = C^k{}_{ij}X_k \in i(V). \tag{3.6}$$

This is the action of $i(V)$ on $i(V)$ itself. We shall identify $i(V) \equiv V$ as long as it does not lead to confusion. Concerning the insertion of L into E, things are more interesting when the mapping

$$\sigma: L \to E,$$

$$\sigma: J_a \to X_a,$$

taking L into E is not necessarily an algebra homomorphism. Because we wish to consider the extensions coming through the given representation ρ, the automorphisms representing the action of L on $i(V)$ are written

$$[X_a,X_i]_E = \rho(J_a)(X_i) = C^j{}_{ai}X_j \in i(V). \tag{3.7}$$

This expression and (3.6) say that $i(V)$ is a normal subalgebra (an ideal) of the extended algebra. The commutation rules of E will be (3.6), (3.7) and the expression for $[X_a,X_b]_E$. The X_a's would provide a *linear* (or *vector*) representation of L if they just mimic the behavior of the J_a's,

$$[X_a,X_b]_E = f^c{}_{ab}X_c, \tag{3.8}$$

but in fact this is not necessarily the case for a general action of L on $\sigma(L)$. The general relation emerging is of the form

$$[X_a,X_b]_E = f^c{}_{ab}X_c - \beta^j{}_{ab}X_j, \tag{3.9}$$

the last term measuring the homomorphism breaking. This expression may be viewed as depicting automorphisms of E acting on the X_a's. In this case, when acting on its own representatives X_a, the automorphisms induced by the action of L do not necessarily yield results restricted to $\sigma(L)$. This subspace is not necessarily a closed subalgebra: the $\beta^j{}_{ab}$'s are

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precisely some constant (in the present purely algebraic context) components measuring the departure of the action from $\sigma(L)$. For linear representations, $\beta^j{}_{ab}=0$ and σ will be a Lie algebra homomorphism. When V is commutative, representations satisfying (3.9) are usually called *projective* representations in Physics literature.[14,15] We shall eventually use the same terminology in the present more general context.

Despite the appearence of (3.9), L is not entirely "lost into E." Some of its identity remains because a mapping may always be defined:

$$\pi_*:E\to L,$$

which is such that

$$\pi_*[i(V)]=0. \qquad (3.10)$$

Borrowing from fiber bundle language, we say that π_* is the *projection* of the *complete space* E into the *base space* L, characterizing $i(V)$ as the *vertical space*. σ plays a role similar to a bundle *section*. Extra structures, analogous to connections and able to characterize horizontal spaces, will be seen to exist in the next chapter. It should be noticed, however, that we shall presently apply all this to the tangent field algebras of differential bundles and we should be careful with this language. Differential bundles are defined in such a way that only the null representation (with all $C^j{}_{ai}=0$) of L on V is concerned.

The enlarged algebra E will then be given by (3.6), (3.7), and (3.9). We shall from now on drop the subindex in $[,]_E$. In order to characterize E as a Lie algebra, the operation $[,]$ must satisfy the Jacobi identity. In a way analogous to (3.2) the V generators X_i may be taken as matrices of elements $(X_i)^k{}_j=C^k{}_{ij}$ automatically satisfying (3.6), which expresses the inner automorphisms of the subalgebra V. This corresponds of course to the adjoint representation of V. As already said, it is helpful to think also of the X_a's as matrices with elements $(X_a)^i{}_j=C^i{}_{aj}$, although this presupposes the representaton to be linear. Applied to two members of the set $\{X_i\}$ and one of $\{X_a\}$, the Jacobi identity gives (3.7) in terms of such matrices. For one X_i and two X_a's, the identity is

$$\begin{aligned}0&=[X_a,[X_b,X_i]]+[X_i,[X_a,X_b]]+[X_b,[X_i,X_a]]\\&=\{C^k{}_{aj}C^j{}_{bi}-C^k{}_{bj}C^j{}_{ai}-f^c{}_{ab}C^k{}_{ci}+\beta^j{}_{ab}C^k{}_{ji}\}X_k.\end{aligned} \qquad (3.11)$$

The right-hand side would seem to give (3.9) in matrix version, but we should be attentive: when X_i belongs to the center of the algebra V, $C^k{}_{ji}=0$, and the matrix representation gives (3.8) instead, due to its essentially linear character. When $\beta_{ab}=\beta^j{}_{ab}X_j$ has values only in the center of V, σ is a homomorphism and the extension is said to be *central*. This is in particular the case when V is commutative. Notice also that a direct product (all $C^j{}_{ai}=0$, the null representation in reality), by (3.11), can only happen when β_{ab} belongs to the center. When applied to three members of $\{X_a\}$ the Jacobi identity gives finally a new condition: the $\beta^j{}_{ab}$'s are no more independent but must submit to

$$\begin{aligned}&C^j{}_{ai}\beta^i{}_{bc}+C^j{}_{ci}\beta^i{}_{ab}+C^j{}_{bi}\beta^i{}_{ca}\\&\quad+f^d{}_{ab}\beta^j{}_{cd}+f^d{}_{ca}\beta^j{}_{bd}+f^d{}_{bc}\beta^j{}_{ad}=0.\end{aligned} \qquad (3.12)$$

Recalling that, in the present algebraic context, multiplying by the matrix elements $C^j{}_{ai}$ represents the action of the generator X_a, comparison with (2.5) makes of this condition a kind of "algebraic Bianchi identity."

We have freedom to choose other basis for the algebra E in the vector space $L\oplus V$, say,

$$X_a\to X'_a=h^b{}_aX_b-\alpha^i_aX_i. \qquad (3.13)$$

The "fourlegs" $h^b{}_a$ will only change the anholonomicity. The change leading to interesting information is simply

$$X_a\to X'_a=X_a-\alpha^i_aX_i, \qquad (3.14)$$

the $\alpha^i{}_a$'s being some constants. The commutation relations become

$$[X'_a,X'_b]=f^c{}_{ab}X'_c-\beta'^j{}_{ab}X_j, \qquad (3.15)$$

$$[X'_a,X_i]=C'^j{}_{ai}X_j, \qquad (3.16)$$

$$[X_i,X_j]=C^k{}_{ij}X_k, \qquad (3.17)$$

where

$$\beta'^j{}_{ab}=\beta^j{}_{ab}+K^j{}_{ab}, \qquad (3.18)$$

with

$$K^j{}_{ab}=C^j{}_{ai}\alpha^i{}_b-C^j{}_{bi}\alpha^i{}_a-\alpha^j{}_cf^c{}_{ab}-C^j{}_{ki}\alpha^k{}_a\alpha^i{}_b \qquad (3.19)$$

and

$$C'^j{}_{ai}=C^j{}_{ai}-\alpha^k{}_aC^j{}_{ki}, \qquad (3.20)$$

which is simply the matrix version of (3.14).

Given the $C^j{}_{ai}$'s, any set of $\beta^j{}_{ab}$'s satisfying condition (3.12) will make of E a Lie algebra. Consequently, there is in principle a fair choice of possible extended algebras E, each one corresponding to a different action of L on $i(V)$ [choice of the $C^j{}_{ai}$'s in (3.7)] and on $\sigma(L)$ [choice of the $\beta^j{}_{ab}$'s in (3.9)]. In reality, many of these choices are equivalent between each other. A set $(C^j{}_{ai},\beta^j{}_{ab})$ will be equivalent to a direct product if an α exists which, once used in (3.14), causes the vanishing of the $C^{j\,\prime}{}_{ai}$'s and of the $\beta^{j\,\prime}{}_{ab}$'s, leading back to (2.1). If an α exists such that $K^j{}_{ab}=-\beta^j{}_{ab}$, a semidirect product results. The extension is said to be *trivial*.

The attentive reader will have noticed the close formal analogy of (3.12) and the first three terms in (3.19) with the exterior derivations of a two-form β and a one-form α in an anholonomic basis $\{X_a\}$. It is as if (3.12) said $d\beta=0$ and (3.19) that $K=d\alpha-[\alpha,\alpha]$. This analogy is sound indeed. To show its full meaning, as well as to examine equivalences between extensions, it is advisable to resort to a homological language, which we shall now briefly introduce. This language, akin to that of differential forms, will have the further advantage of being quite appropriate to the later discussion of differentiable vector field algebras.

IV. INTO COHOMOLOGICAL LANGUAGE

We take on the space $(L\oplus V)^*$ dual to $L\oplus V$ the basis $\{\omega^a,\ \omega^i\}$, formed by the covectors ω^a, ω^i such that $\omega^a(X_b)=\delta^a{}_b$, $\omega^a(X_j)=0$ and $\omega^i(X_j)=\delta^i{}_j$, $\omega^i(X_a)=0$. These covectors constitute, by exterior product, a basis for all the antisymmetric covariant tensors on $L\oplus V$. They are exterior forms on the vector space $L\oplus V$, but will here be

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considered as acting on the algebra. When applied to algebras and taken in an algebraic context, exterior forms are *cochains*. The first-order covectors like ω^a and ω^i above will be one-cochains, second-order tensors like $\omega^a \wedge \omega^j$ are two-cochains, etc. Cochains may also be defined on groups, group representations, and other non-necessarily linear objects but Lie algebras are linear spaces and their cochains relatively simpler. Differential forms are particular linear cochains and many of their well-known properties come in reality from that. A derivative operation $\mathbf{d}$ is defined which takes a p-cochain into a $(p+1)$-cochain,

$$\mathbf{d}:\Omega_p \to (\mathbf{d}\Omega)_{p+1}, \tag{4.1}$$

by

$$\begin{aligned}(p+1)(\mathbf{d}\Omega)(X_1,X_2,...,X_{p+1}) = {} & X_1\Omega(X_2,X_3,...,X_{p+1}) - X_2\Omega(X_1,X_3,...,X_{p+1}) \\ & + X_3\Omega(X_1,X_2,X_4,...,X_{p+1}) - \cdots + (-)^p X_{p+1}\Omega(X_1,X_2,X_4,...,X_p) \\ & - \Omega([X_1,X_2],X_3,...,X_{p+1}) + \Omega([X_1,X_3],X_2,...,X_{p+1}) + \\ & - \Omega([X_1,X_4],X_2,...,X_{p+1}) + \cdots + (-)^p\Omega([X_1,X_{p+1}],X_2,...,X_p) \\ & + \Omega(X_1,[X_2,X_3],...,X_{p+1}) + \cdots + (-)^p\Omega(X_1,X_2,X_3,...,[X_p,X_{p+1}]).\end{aligned} \tag{4.2}$$

An expression like $X_1\Omega(X_2,X_3,...,X_{p+1})$ means that X_1 is to act on the result of $\Omega(X_2,X_3,...,X_{p+1})$. We shall be interested in algebra-valued cochains, following a line akin to cyclic cohomology. On algebra-valued cochains, the actions of generators appearing in (4.2) are the algebra operations, that is, the brackets: for instance, $X_1\Omega(X_2,X_3,...,X_{p+1}) = [X_1,\Omega(X_2,X_3,...,X_{p+1})]$. The same must be kept in mind when using the exterior product, although in that case also their complete antisymmetry must be accounted for. As examples,

$$\omega\wedge\omega(X_1,X_2) = [\omega(X_1),\omega(X_2)], \tag{4.3}$$

$$\begin{aligned}\omega\wedge\beta(X_1,X_2,X_3) = {} & [\omega(X_1),\beta(X_2,X_3)] \\ & + [\omega(X_3),\beta(X_1,X_2)] \\ & + [\omega(X_2),\beta(X_3,X_1)].\end{aligned} \tag{4.4}$$

A direct calculation shows the validity of the Poincaré lemma $\mathbf{d}^2 = \mathbf{d}\,\mathbf{d} \equiv 0$. We shall say that a cochain γ_p satisfying $\mathbf{d}\gamma_p = 0$ is *closed*, or is a *p-cocycle*. If another cochain σ_{p-1} exists such that $\gamma_p = \mathbf{d}\sigma_{p-1}$, γ_p is *exact*, or is a *p-coboundary*. Every coboundary is a cocycle, but not vice versa.

Let us come back to the cochains ω^a and ω^i. Using (4.2) for $\mathbf{d}\omega^c(X_a,X_b)$, $\mathbf{d}\omega^c(X_i,X_j)$ and $\mathbf{d}\omega^c(X_a,X_j)$ we find that the $\{\omega^a\}$ obey the Maurer–Cartan equations

$$\mathbf{d}\omega^c = -\tfrac{1}{2}f^c{}_{ab}\omega^a\wedge\omega^b. \tag{4.5}$$

The ω^a's constitute a basis for the cochains on the X_a's. Let us define the V-valued two-cochain

$$\beta = X_j\beta^j = \tfrac{1}{2}X_j\beta^j{}_{ab}\omega^a\wedge\omega^b \tag{4.6}$$

and the one-cochain

$$\omega = X_i\omega^i, \tag{4.7}$$

which will be important in the following. If we calculate $\mathbf{d}\beta$ (X_a, X_b, X_c), it is immediate to recognize the expression of the Jacobi identity (3.12), with the X_a's acting on the X_j's according to (3.7). Consequently, that identity simply states the closure of β in the L subspace, $\mathbf{d}\beta$ $(X_a,X_b,X_c) = 0$. This is a first case showing the meaning of the "analogy" noticed at the end of the previous chapter.

Treatments of the subject stop usually at this point, with the closure of β in the L sector, and go directly to the discussion of equivalence classes of representations. Because we shall be interested later in the relations with complete fiber bundles, we extend the result to the complete algebra E. It is trivial to do it. As

$$\omega\wedge\beta(X_a,X_b,X_c) = \beta\wedge\omega(X_a,X_b,X_c) = 0,$$

we may write

$$\begin{aligned}&\{\mathbf{d}\beta - \omega\wedge\beta + \beta\wedge\omega\}(X_a,X_b,X_c) \\ &\quad = \{\mathbf{d}\beta - [\omega,\beta]\}(X_a,X_b,X_c) = 0.\end{aligned}$$

As the expression $\{\mathbf{d}\beta - [\omega,\beta]\}$ vanishes also for any other combination of E basis elements (X_1, X_2, X_3) as arguments,

$$D_\omega\beta := \{\mathbf{d}\beta - [\omega,\beta]\} = 0. \tag{4.8}$$

This is a dual version of (3.12) with the advantage of being invariant, holding in any basis on E. In the same token, by comparing the results for all possible kinds of arguments (X_1, X_2), we arrive at

$$\beta = D_\omega\omega := \mathbf{d}\omega - \omega\wedge\omega. \tag{4.9}$$

β measures a breaking of the Maurer–Cartan equations for the $\{\omega^j\}$. Actually, as $\beta(X_i,X_j) = 0$, they keep holding in the V subalgebra, but no more on the whole E. Once immersed in a larger algebra, the ω^i acquire a new role. In both expressions above D_ω is a "covariant derivative." Going back to the bundle language, the cochain ω has a role quite similar to that of a connection: it takes values on V, it is such that $\omega(X_i) = X_i$ and it vanishes when applied to the *horizontal* vectors X_a. Furthermore, (4.9) tells us that the two-cochain β plays the role of its "curvature," for which (4.8) expresses the "Bianchi identity." Recall that on fiber bundles the curvature measures exactly the departure of horizontal fields from constituting a closed subalgebra, just what β does here.

Actually, all the Jacobi identities above, (3.11), (3.12), and the unwritten one which simply gave (3.7) in matrix terms, can be summarized in a unique expression generaliz-

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ing (4.8). It is enough to introduce the canonical cochain for the total algebra,

$$W = X_a \omega^a + X_k \omega^k, \tag{4.10}$$

for which

$$\mathbf{d}W - W \wedge W = 0. \tag{4.11}$$

The identities already obtained come as parts of this expression. In this way all the identities of Sec. III are translated into the homological language.

Consider the V-valued one-cochain

$$\alpha = X_j \alpha^j = X_j \alpha^j{}_a \omega^a. \tag{4.12}$$

It takes values on the vertical V space. Defining also the two-cochain

$$K = X_j K^j = \tfrac{1}{2} X_j K^j{}_{ab} \omega^a \wedge \omega^b, \tag{4.13}$$

the K of expression (2.19) may be written as a "modified covariant derivative" of α,

$$K = \mathbf{D}_\omega \alpha := \mathbf{d}\alpha - \omega \wedge \alpha - \alpha \wedge \omega - \alpha \wedge \alpha. \tag{4.14}$$

But for the last squared term, this has the same aspect of the covariant derivative of a one-form α according to a connection ω. In (3.19), the term $\alpha^j{}_c f^c{}_{ab}$ is simply an effect of the anholonomity of the basis $\{X_a\}$ and the first three terms there are simply $\mathbf{d}\alpha$. The covariant derivative $\mathbf{D}_\omega$ is just the mapping $\mathbf{d}$ of the algebra E along the horizontal space. As to (3.18), it is now

$$\beta' = \beta + K. \tag{4.15}$$

In reality, $\omega' = \omega + \alpha$ defines a new "connection", with $\beta' = \mathbf{d}\omega' - \omega' \wedge \omega'$ and $d\beta' + [\omega', \beta'] = 0$. As $\omega'(X'_a) = 0$, ω' is just that "connection" which "declares" as horizontal the X'_a.

If some α exists for which $\beta = -\mathbf{D}_\omega \alpha$, the homomorphism-breaking term in (3.9) may be made to vanish by transforming to the new basis $\{X'_a\}$. The projective representation reduces to a linear representation. The algebra E is a *trivial extension* of L by V. So, the condition for a representation to reduce to a linear representation is that the two-cocycle β be covariantly exact (a covariant coboundary). It happens then that $\sigma(L)$ is a subalgebra, the composition $(\pi_* \circ \sigma)$ is the identical automorphism of L and there is no intersection between $i(V)$ and $\sigma(L)$, in which case the algebra E is a *semidirect product*, σ is a *splitting* and the extension is *decomposable*. A direct calculation shows that

$$\mathbf{D}_\omega K \equiv 0, \tag{4.16}$$

a second "Bianchi identity" for this case.

Actually, for algebra-valued cochains the differential operator may be written as $d = W^\mu \wedge X_\mu = \omega^a \wedge X_a + \omega^i \wedge X_i$, with X_μ acting on the right through the bracket operation. This is of course reminiscent of the usual rule $d = dx^i \wedge \partial / \partial x^i$ of differential calculus.

We have seen that extensions are characterized by the action of L on $i(V)$ and on $\sigma(L)$. The equivalences between choices of sets $(C^j{}_{ai}, \beta^j{}_{ab})$ referred to at the end of Sec. III can now be given a precise treatment. Cohomology groups can be introduced in a way analogous to those defined for differential forms. Closed V-valued p-cochains constitute a group $Z^p(E,V)$; V-valued p-coboundaries form another group, $B^p(E,V)$. The pth cohomology group is then the quotient $H^p(E,V) = Z^p(E,V)/B^p(E,V)$ and will contain equivalence classes of chains differing by covariant derivatives as in (4.15). The dimension of $H^p(E,V)$ is the pth Betti number $b^p(E,V)$. When the second Betti number $b^2(E,V)$ vanishes, every projective representation is equivalent to a linear representation, the algebra extensions are all trivial. In particular, by Whitehead theorems, this is always the case for semisimple Lie algebras when V is a module. In the general case, to β and β' related by (4.15) will correspond the same representation. Actions or representations will correspond to cohomology classes of the two-forms β, elements of the cohomology group $H^2(E,V)$. Each equivalence class will be labeled by a cocycle β. *Central* extensions, as said above, appear when β belongs to the center of V. All trivial extensions are central extensions.

The above purely algebraic case appears when the $\alpha^j{}_c$'s and $\beta^j{}_{ab}$'s are constants. Things are more complicated and interesting when Lie algebras of fields on differentiable manifolds are involved. Usual derivatives are then to be added to the above "algebraic" derivatives and a more complex pattern emmerges.

V. FIELD ALGEBRAS ON MANIFOLDS

We are slowly approaching the conditions of gauge theories, ready however to stop some steps before reaching them. In a gauge theory, the gauge group acts on the basic bundle through vector fields which are the representatives of its Lie algebra generators. But also the group of translations on space-time is there represented, although in a biased way: relations (2.1) and (2.2) tell us that translations only "act" on the gauge generators through the null representation. The action of groups on manifolds leads, in general, to highly nonlinear representations but we shall here only consider slight departures from the linear case, as suggested by Lie algebra extensions. Notice also that, although the cohomological approach is global on the algebras, it will here be applied to the local algebras of tangent fields and so be essentially local.

We have to take into account the action[16] of two groups on a manifold M. Consider the action of one group, a Lie group G which may be any of them. Suppose the Lie algebra G' of the group G to be given by (3.1) in the basis of generators $\{J_a\}$. The Lie algebra generators J_a are locally represented by vector fields X_a on M. The whole set of fields on M constitute an infinite Lie algebra $\Xi(M)$ and $X_a \in \Xi(M)$. A representation ρ in terms of fields will now be a mapping ρ: $G' \to \Xi(M)$, ρ: $J_a \to X_a = \rho(J_a)$. The carrier space will be $C^\infty(M)$, the space of infinitely differentiable functions on M (in reality, differentiability to a few orders is enough). The representation would be linear if (3.8) holds. In the present case, by Frobenius' theorem, this would mean that at each point x of M the X_a's span a subspace of the tangent space $T_x M$ locally tangent to a submanifold of M locally diffeomorphic to the group G. In the general case, although such a subspace of $T_x M$ always exists, there will be no submanifold to which it is tangent. Consider a local vector basis $\{X_\mu; \mu = 1,2,...,n = \dim M\}$ around some $x \in M$ and suppose

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it can be chosen so as to include the X_a's plus some other fields $\{X_i\}$: $\{X_\mu\} = \{X_a, X_i\}$. The X_a's do not constitute a subalgebra of the whole field Lie algebra around x and extra terms appear as in (3.9).

To get some preliminary insight, consider first the nearest departures from linear representations, the central projective representations for which the commutation relations are

$$[X_a, X_b] = f^c{}_{ab} X_c - \beta_{ab}, \tag{5.1}$$

with β_{ab} some functions on M. In this case, the action of the Lie algebra on functions of $C^\infty(M)$ will include, besides the action of fields, multiplications by functions. This case, which corresponds to extensions by $C^\infty(M)$ itself, is specially important when M is the phase space of some dynamical system[17] and in some approaches to quantization.[18] The Jacobi identity for the X_a's imposes

$$X_a(\beta_{bc}) + X_c(\beta_{ab}) + X_b(\beta_{ca}) + f^d{}_{ab}\beta_{cd} + f^d{}_{ca}\beta_{bd} + f^d{}_{bc}\beta_{ad} = 0. \tag{5.2}$$

Under the change

$$X_a \to X'_a = X_a - \alpha_a, \tag{5.3}$$

where now $\alpha_a \in C^\infty(M)$, the commutation relations become

$$[X'_a, X'_b] = f^c{}_{ab} X'_c - \beta'_{ab}, \tag{5.4}$$

where

$$\beta'_{ab} = \beta_{ab} + \{X_a(\alpha_b) - X_b(\alpha_a) - \alpha_c f^c{}_{ab}\}. \tag{5.5}$$

We may take on the space tangent to M around a point x the covector basis $\{\omega^\mu\} = \{\omega^a, \omega^i\}$ dual to $\{X_\mu\} = \{X_a, X_i\}$. The forms ω^a such that $\omega^a(X_b) = \delta^a{}_b$ will obey the Maurer–Cartan equations and constitute a basis for the dual space $\rho(G')^*$. Defining the two-form $\beta = (1/2)\,\beta_{ab}\omega^a \wedge \omega^b$ would put the Jacobi identity under the form $d\beta = 0$, that is, β is a cocycle. Defining also the one-form $\alpha = \alpha_a \omega^a$, which is such that $\alpha_a = \alpha(X_a)$, transformation (5.5) becomes $\beta' = \beta + d\alpha$. If some α exists for which $\beta = -d\alpha$, the extra term in (5.4) may be made to vanish by transforming according to (5.3) and the projective representation reduces to a linear representation. So, the condition for a projective representation to reduce to a linear representation is that the closed two-form β (a two-cocycle) be exact (a coboundary). The one-form α has the properties of an Abelian connection, of which $d\alpha$ is the curvature. When no α exists for which $\beta = -d\alpha$, β is a closed nonexact form. To β and β' related by $\beta' = \beta + d\eta$ will correspond the same projective representation. Consequently, projective representations will correspond to the (cohomology) classes of the two-forms β. They will correspond to extensions of the algebra $\{X_a\}$ by the algebra $C^\infty(M)$ of real functions on M with the usual product operation. Extensions are equivalent if related by $\beta' = \beta + d\alpha$ and so each equivalence class will be labeled by a cocycle β. A trivial extension will correspond to a linear representation. The exterior derivative "d" appearing above is restricted to the algebra $\{X_a\}$. We notice the emergence of the connectionlike form α, the linear case happening when β is an Abelian "curvature." The structure coefficients play the role of an anholonomicity of the basis $\{X_a\}$. The above case appears (with $C^j{}_{ai} = 0$) in the action of the gauge group on the functional space of gauge potentials.[19] The $\{X_a\}$ stand for the representatives of gauge transformation generators on that space, α and β being functionals with the connections as arguments: α is an anomaly and $d\alpha = 0$ is the Wess–Zumino condition.[15]

Let us now proceed to the more general case in which β is vector valued with values along the remaining fields $\{X_i\}$ of the basis $\{X_\mu\}$: $\beta_{ab} = X_i \beta^i{}_{ab}$. We think to have made the point we wished with the anholonomous coefficients, giving the two algebras a similar appearance for the discussion of extensions. We shall drop them from now on. The basis $\{X_\mu\}$ will have commutation relations of the form

$$[X_a, X_b] = -\beta^j{}_{ab} X_j, \tag{5.6}$$

$$[X_a, X_i] = C^j{}_{ai} X_j, \tag{5.7}$$

$$[X_k, X_i] = C^j{}_{ki} X_j. \tag{5.8}$$

As in (3.2), we may introduce matrices C_a with elements $(C_a)^j_i = C^j{}_{ai}$. The three commutation relations above describe the actions of local transformations generated by the set of fields $\{X_a, X_i\}$ on itself. Once more, we must take into account that such relations only define a Lie algebra if the Jacobi identities are satisfied. In the present case, in principle, things are more complicated, because the structure coefficients are no more constants. The matrix elements of C_a are point dependent and the fields act on them, leading to extra terms in the previous Jacobi identities. Some extra suppositions will be necessary to get a simple structure. As there are things to be learned from them, we shall write their full expressions:

$$[X_a,[X_b,X_c]] + [X_c,[X_a,X_b]] + [X_b,[X_c,X_a]] = -\{X_a(\beta^j{}_{bc}) + X_c(\beta^j{}_{ab}) + X_b(\beta^j{}_{ca}) + C^j{}_{bk}\beta^k{}_{ca} + C^j{}_{ck}\beta^k{}_{ab} + C^j{}_{ak}\beta^k{}_{bc}\}X_j = 0. \tag{5.9}$$

This generalizes (3.12) in the presence of fields. Recall that in expressions like (3.12) and (3.19) the factors of type $C^j{}_{ak}$ have the role of a "matrix derivative." There, the only representation at work was that given by the matrix action in (3.3). Here, we have two actions at work, the matricial and that of the fields X_a. We may use the notational device $X^*_a(Z^n) = X_a(Z^n) + C^n{}_{aj}Z^j$, for any Z with a higher index of the kind i, j, k, as such a combination will appear frequently. With this notation the condition becomes

$$X^*_a(\beta^j{}_{bc}) + X^*_c(\beta^j{}_{ab}) + X^*_b(\beta^j{}_{ca}) = 0. \tag{5.10}$$

It is also interesting to introduce the notation $X^*_i(Z^n) = X_i(Z^n) + c^n{}_{ij}Z^j$. These X^*_i measure departures from covariance, as in (2.4), $X^*_i(\alpha^n{}_a) = 0$, for a connection α and (2.7), $X^*_i(F^n{}_{ab}) = 0$, for its curvature. Furthermore, they constitute by themselves a representation, as

$$[X^*_k, X^*_i] = C^j{}_{ki} X^*_j. \tag{5.11}$$

We want an extension of gauge theories, preserving as far as possible its properties. The structure constants $c^n{}_{ij}$ are in reality only "constant" on the gauge group, but we shall take them constant also on space-time, $X_a(c^n{}_{ij}) = 0$. The resulting structure will have similarities with the gauge theories but the matrix representations will be at work besides the

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fields X_a and X_i. The next Jacobi identity is

$$[X_a,[X_b,X_i]] + [X_i,[X_a,X_b]] + [X_b,[X_i,X_a]] = \{X_a(C^k{}_{bi}) - X_b(C^k{}_{ai}) + C^k{}_{aj}C^j{}_{bi} - C^k{}_{bj}C^j{}_{ai} - X_i(\beta^k{}_{ab}) - \beta^j{}_{ab}C^k{}_{ji}\}X_k = 0, \quad (5.12)$$

so that the condition is

$$X_i^*(\beta^k{}_{ab}) = X_a^*(C^k{}_{bi}) - X_b^*(C^k{}_{ai}). \quad (5.13)$$

The covariance breaking of β is measured by a kind of "curl" of the C_a's. The last nontrivial identity is

$$[X_a,[X_i,X_j]] + [X_j,[X_a,X_i]] + [X_i,[X_j,X_a]] = \{X_a(C^k{}_{ij}) + X_j(C^k{}_{ai}) - X_i(C^k{}_{aj}) + C^k{}_{an}C^n{}_{ij} - C^k{}_{in}C^n{}_{aj} - C^n{}_{al}C^k{}_{nj}\}X_k = 0. \quad (5.14)$$

Collecting the terms conveniently, we get

$$X_j^*(C^k{}_{ai}) - X_i^*(C^k{}_{aj}) + C^n{}_{ij}C^k{}_{an} = 0. \quad (5.15)$$

The simplest solution for these identities comes out when we take all $C^k{}_{ai}$'s and $\beta^k{}_{ab}$'s constant. In this case, the matrices (C_a) and (C_i) give by themselves a representation of (5.6)–(5.8). There is however another interesting solution for these equations. Due to (5.11), Eq. (5.15) is automatically satisfied by

$$C^k{}_{aj} = X_j^*(\gamma^k{}_a), \quad (5.16)$$

γ being an object whose covariance breaking is just measured by the structure coefficients $C^k{}_{aj}$. Looking then for a solution for (5.10) and (5.13), we find that they are both satisfied by

$$\beta^k{}_{ab} = X_a\gamma^k{}_b - X_b\gamma^k{}_a - C^k{}_{ij}\gamma^i{}_a\gamma^j{}_b, \quad (5.17)$$

which has the appearance of a curvature but does not belong to the adjoint representation.

We have seen that, differently from the $\{\omega^a\}$, the $\{\omega^i\}$ did not satisfy the Maurer–Cartan equations. Life in $\{\omega^\mu\}$-space is really dual to that in $\{X_\mu\}$ space. Unlike their dual partners X_i, the ω^i's do not close an algebra under the exterior product. The $\gamma^i{}_a$'s may be seen as the components of the pull-back of the ω^i's to the dual space L^*. As the ω^a's may be identified with their own pullbacks,

$$\sigma^*(\omega^i) = \gamma^i{}_a\omega^a.$$

Equations (5.6)–(5.8), with their consequences just given, provide the new geometrical stage set, in replacement of (2.1). There is now a coupling between space-time and gauge transformations.

VI. EXTENDED GAUGE THEORIES

Let us see now how a connection fares in the new environment. Under the change

$$X_a \to X'_a = X_a - \alpha^j{}_a X_j, \quad (6.1)$$

if we suppose that α is indeed a connection, $X_i^*(\alpha^k{}_a) = 0$, the commutation relations become

$$[X'_a,X'_b] = -\mathscr{F}^j{}_{ab}X_j, \quad (6.2)$$

$$[X'_a,X_i] = C^k{}_{ai}X_k, \quad (6.3)$$

$$[X_k,X_i] = C^j{}_{ki}X_j. \quad (6.4)$$

Here,

$$\mathscr{F}^k{}_{ab} = \beta^j{}_{ab} + F^k{}_{ab} + C^k{}_{aj}\alpha^j{}_b - C^k{}_{bj}\alpha^j{}_a, \quad (6.5)$$

$$F^k{}_{ab} = X_a\alpha^k{}_b - X_b\alpha^k{}_a - C^k{}_{ij}\alpha^i{}_a\alpha^j{}_b. \quad (6.6)$$

We may write $\mathscr{F}^k{}_{ab}$ in many different though equivalent ways, corresponding to different rearrangements of the terms. For instance,

$$\mathscr{F}^k{}_{ab} = \beta^j{}_{ab} + F^{*k}{}_{ab}, \quad (6.7)$$

where

$$F^{*k}{}_{ab} = X_a^*\alpha^k{}_b - X^*{}_b\alpha^k{}_a - C^k{}_{ij}\alpha^i{}_a\alpha^j{}_b, \quad (6.8)$$

extends (3.18)–(3.19) to the case in the presence of fields. We have so the vaccum field plus the usual field strength written with modified derivatives. If solution (5.16) is still available, the new total field strength can be written

$$\mathscr{F}^k{}_{ab} = F^k{}_{ab} + X'_a\gamma^k{}_b - X'_b\gamma^k{}_a - C^k{}_{ij}\gamma^i{}_a\gamma^j{}_b. \quad (6.9)$$

To the usual gauge field F is added β with the normal derivatives of (5.17) changed to covariant derivatives according to the connection α. We should keep in mind that covariant derivatives have different expressions when acting on objects belonging to different representations. Because of their origin from (5.16), the γ's respond to the X^* representation of (5.11), so that in (6.9) $X'_a\gamma^k{}_b = X_a\gamma^k{}_b - \alpha^j{}_aX_j^*(\gamma^k{}_b)$. We have so an usual field strength plus the nontrivial vacuum with a peculiar minimal coupling of the vacuum potential with the gauge potential. Another point of view would define a new "gauge potential" as $\alpha' = \alpha + \gamma$ and

$$\mathscr{F}^k{}_{ab} = X_a\alpha'^k{}_b - X_b\alpha'^k{}_a - C^k{}_{ij}\alpha'^i{}_a\alpha'^j{}_b - \alpha^j{}_aX_j^*(\gamma^k{}_b) + \alpha^j{}_bX_j^*(\gamma^k{}_a).$$

Finally,

$$\mathscr{F}^k{}_{ab} = [X_a + \gamma^i{}_aX_i](\alpha^k{}_b) - [X_b + \gamma^i{}_bX_i](\alpha^k{}_a) - C^k{}_{ij}\alpha^i{}_a\alpha^j{}_b + [X_a - \alpha^i{}_aX_i](\gamma^k{}_b) - [X_b - \alpha^i{}_bX_i](\gamma^k{}_a) - C^k{}_{ij}\gamma^i{}_a\gamma^j{}_b.$$

Here, (minus) γ behaves as a connection, "derives" α and is derived by it.

Before going further, let us interprete some aspects of gauge theories in the light of the above picture. Not only $C^k{}_{ai}$ but also β vanish in gauge theories. We are now in condition to understand the meaning of that. In gauge theories, the forms ω^k appearing in (4.5) are the Maurer–Cartan or canonical forms of the gauge group. As β is related to them by (4.9), it is forcibly null by the Maurer–Cartan equations. As is well known,[20] the canonical forms, once pulled back to space-time, give the usual $g^{-1}dg$ contribution of the vacuum to the gauge potential. Consequently, β is the vacuum field. We have seen in Chap. IV that the presence of a connection was simply an addition to the initial cochain ω. It is in reality $\omega' = \omega + \alpha$ which is the usual physical connection, β' the field strength. They are "counted from" ω and β. Once β is zero, ω can be gauged off and α is the remaining connection. The covariant derivative becomes D_α and K in (4.14) acquires its usual expression $d\alpha - \alpha \wedge \alpha$. The direct product appears as a very particular case of trivial extension. In the present case, the coupling between space-time and

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gauge space modifies the derivatives $X_a \to X^*_a$ and creates a broken, nontrivial vacuum β, whose nonadjoint character is an effect of a possible "rotation" of the C_a's. Let us examine the Jacobi identities: we obtain

$$X^*_j(C^k{}_{ai}) - X^*_i(C^k{}_{aj}) + C^n{}_{ij}C^k{}_{an} = 0, \tag{6.10}$$

which is (5.15) again, and says that indeed a solution like (5.16) still works. The Bianchi identity is now

$$X^{*'}{}_a(\mathscr{F}^j{}_{bc}) + X^{*'}{}_c(\mathscr{F}^j{}_{ab}) + X^{*'}{}_b(\mathscr{F}^j{}_{ca}) = 0. \tag{6.11}$$

Finally, the adjointness breaking becomes

$$X^*_i(\mathscr{F}^k{}_{ab}) = X^{*'}{}_a(C^k{}_{bi}) - X^{*'}{}_b(C^k{}_{ai}). \tag{6.12}$$

So, nothing really quite unexpected comes out at first from the addition of a connection: the identities are the same as before the introduction of the connection, but corrected to the covariant derivatives. With respect to gauge theories, the novelty is the change of the derivatives X_a into the differential-algebraic X^*_a, rather reminiscent of Fock–Ivanenko derivative, and the presence of a noninvariant vacuum. If we use the duality rule to obtain the field equations, the new extended Yang–Mills equation generalizing (2.8) is

$$X^{*'}{}_a\mathscr{F}^{jab} = J^{jb}, \tag{6.13}$$

J^{jb} being a source current. In gauge models only those currents are acceptable which satisfy (2.9). They must be covariantly irrotational, $X'{}_bJ^{jb} = 0$, as it follows from (2.8) and the covariance of F^{jab} that $X'_bX'_aF^{jab} = 0$. We would not *a priori* expect nothing of the sort in the extended case, as $\mathscr{F}^{jab}$ is not covariant and $X^{*'}{}_a$ is a new, strange derivative. Nevertheless, a quite analogous relation turns out to be true:

$$\begin{aligned} X^{*'}{}_bX^{*'}{}_a\mathscr{F}^{jab} &= -\tfrac{1}{2}[X^{*'}{}_a, X^{*'}{}_b]\mathscr{F}^{jab} \\ &= \tfrac{1}{2}\mathscr{F}^k{}_{ab}X_k\mathscr{F}^{jab} \\ &\quad - \tfrac{1}{2}[X^{*'}{}_aC^j{}_{bk} - X^{*'}{}_bC^j{}_{ak}]\mathscr{F}^{kab}. \end{aligned}$$

Use of (6.12) reduces this to $\frac{1}{2}\mathscr{F}^k{}_{ab}\, C^j{}_{ki}\mathscr{F}^{iab} = 0$. The requirement that the extended algebra remain a Lie algebra is strong enough to cause the variance of $\mathscr{F}$ to be just compensated by the behavior of the $C^j{}_{ak}$'s. Currents should consequently satisfy a generalized irrotationality

$$X^{*'}{}_bJ^{jb} = 0, \tag{6.14}$$

and the theory retains more symmetry than would be expected at first sight. In gauge models, the constraints (2.9) on source fields lay behind the Ward identities instrumental in the procedure of order by order renormalization,[21] so that (6.14) would allow some hopes concerning renormalizability.

VII. FINAL COMMENTS AND SPECULATIONS

The general case of the extended scheme here presented seems rather involved. Simple special cases should be analyzed. An example of the type (6.9), corresponding to a gauge "potential" with a noncovariant part, has been examined elsewhere[22] from a more particular point of view: the space-time translation group itself is "gaugefied," and the whole theory becomes a model of the Einstein–Cartan type. The $C^k{}_{ai}$'s acquire a status of connection and the field equation (6.13) turns out to be Einstein's equation for the corresponding curvature.

Let us say a few words on the case in which the gauge potential is added to an initial background given by (3.7)–(3.9) with all $C^k{}_{ai}$'s and $\beta^k{}_{ab}$'s constant, the matrices C_a and C_i realizing a representation of (5.6)–(5.8). Everything gets greatly simplified and it is found that

$$X^*_i(F^{*k}{}_{ab}) = 0,$$

so that $F^{*k}{}_{ab}$ as given by (6.8) belongs to the adjoint representation and is the natural candidate to the new physical field strength. From (6.13), it comes out that it obeys the Yang–Mills equation

$$X^{*'}{}_a(F^{*kab}) = J^{kb}.$$

When the matrices C_a, $(C_a)^k{}_i = (C^k{}_{ai})$, are antisymmetric and the algebra of the X_i's is semisimple, such equations come from a Lagrangian of the usual type, $-\frac{1}{4}F^{*kab}F^*{}_{kab}$. We have a model similar indeed to a gauge model, the difference being in the additional algebraic parts added to the derivatives. An initial nontrivial background, present before a gauge potential is introduced, would at first sight violate the no-go theorems, but in fact semidirect products would be admitted.

Some coupling between space-time and gauge space must exist if gravitation is to be described by a gauge theory related to space-time symmetries. We have given the general lines of a scheme modifying gauge theories in that direction. Indulging ourselves in a rather wild speculation, we might say that there is another, still more pretentious, possibility. Quark confinement, an extreme case of translational invariance breaking, has up to now been looked for as a dynamical (or mixed dynamical topological) effect, but it does pressupose so strong an effect of gauge fields on space-time behavior that we might wonder whether its origin could not be of a more primitive character, involving some basic "misbehavior" of the gauge program like the one in the extended scheme above. The presence of algebra-valued cochains, natural generalizations of the differential forms of gauge models, leads to derivatives of mixed differential-algebraic character and points to noncommutative geometrical effects which could have some role to play in confinement and shielding. Of course, the main interest in a gaugelike theory for gravitation lies in the possibility of renormalizability. And despite the background vacuum noncovariance, currents in the above extended scheme submit to conditions analogous to those ensuring invariance and renormalizability to gauge theories.

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