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Constrained KP models as integrable matrix hierarchies

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We formulate the constrained KP hierarchy (denoted by $\text{cKP}_{K+1,M}$) as an affine $\widehat{sl}(M+K+1)$ matrix integrable hierarchy generalizing the Drinfeld–Sokolov hierarchy. Using an algebraic approach, including the graded structure of the generalized Drinfeld–Sokolov hierarchy, we are able to find several new universal results valid for the cKP hierarchy. In particular, our method yields a closed expression for the second bracket obtained through Dirac reduction of any untwisted affine Kac–Moody current algebra. An explicit example is given for the case $\widehat{sl}(M+K+1)$, for which a closed expression for the general recursion operator is also obtained. We show how isospectral flows are characterized and grouped according to the semisimple *non-regular* element E of $sl(M+K+1)$ and the content of the center of the kernel of E . © 1997 American Institute of Physics.

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I. INTRODUCTION

The constrained KP (cKP) hierarchy occupies one of the central positions in the current study of integrable hierarchies. This is mainly due to the fact that it represents a direct generalization of the KdV models and includes an impressive list of partial differential soliton equations^{1–7}. Also important are the relationships of the cKP hierarchy to several physically relevant models (like Toda models and discrete matrix models).

Let us recapitulate the most general form of the Lax operator belonging to the cKP hierarchy:

$$L = D^{K+1} + \sum_{l=0}^{K-1} u_l D^l + \sum_{i=1}^M \Phi_i D^{-1} \Psi_i, \quad (1.1)$$

and subjected to the following flow evolution equations:

$$\frac{\partial L}{\partial t_n} = [(L^{n/(K+1)})_+, L]. \quad (1.2)$$

We will denote the hierarchy defined by (1.1) and (1.2) as $\text{cKP}_{K+1,M}$. There are several different parametrizations (obtained by acting with various Miura maps) of the coefficients u_l, Φ_i, Ψ_i in (1.1), defining various reformulations of the $\text{cKP}_{K+1,M}$ hierarchy. It is, for instance, known that the Lax operator from (1.1) can be rewritten as a ratio $L = L_{M+K+1}/L_M$ of two purely differential operators L_{M+K+1} and L_M of orders $M+K+1$ and M , respectively.

Here we present a different parametrization governed by the Zakharov–Shabat equation associated with the $\widehat{sl}(M+K+1)$ algebra. So, instead of working with calculus of the pseudo-differential operators, we work here with the generalized Drinfeld–Sokolov matrix hierarchy^{8–13} associated in our case with the semisimple *nonregular* element E of $\widehat{sl}(M+K+1)$. The outcome of our construction is that to a given $sl(N+1)$ algebra one can associate various scalar Lax representations of the $\text{cKP}_{K+1,M}$ hierarchies with $M+K=N$ and $M, K \geq 1$. The special case of

$K=0, M=N$ has been treated in Ref. 14 and shown to correspond to the generalized NLS hierarchy,¹⁵ which in turn generalizes the AKNS hierarchy^{16–18} for which $K=0, M=1$.

The paper is organized as follows. In Section II the connection between the generic matrix eigenvalue problem we are interested in and the pseudo-differential Lax operator of the cKP type is established. Section III provides the algebraic foundation, within the generalized Drinfeld–Sokolov hierarchy for our model, with subsection III A dealing with the example of the $\widehat{sl}(M+K+1)$ algebra. Section IV examines the Zakharov–Shabat equation for the problem and provides the construction of the recurrence operator. In Section V the second bracket of the cKP _{$K+1, M$} hierarchy is obtained as a Dirac bracket, where the matrix hierarchy is considered as a constrained system. We conclude with Section VI, suggesting few possible applications and extensions of our results.

II. MATRIX EIGENVALUE PROBLEM AND cKP LAX OPERATORS

Consider the matrix eigenvalue problem

$$L\Psi = (D + A + \lambda E)\Psi = 0 \quad (2.1)$$

for the $(M+K+1) \times (M+K+1)$ Lax matrix operator $L = D + A + \lambda E$ given by

$$L = \begin{pmatrix} D & 0 & \cdots & 0 & q_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & D & 0 & \cdots & q_2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & & & D & q_M & 0 & \cdots & \cdots & \cdots & 0 \\ r_1 & r_2 & \cdots & r_M & D-v_1 & \lambda & 0 & \cdots & \cdots & 0 \\ 0 & & \cdots & 0 & 0 & D-v_2 & \lambda & 0 & \cdots & \vdots \\ 0 & & \cdots & 0 & 0 & 0 & D-v_3 & \lambda & \cdots & \vdots \\ \vdots & & \cdots & 0 & 0 & \cdots & 0 & \ddots & \ddots & 0 \\ \vdots & & \cdots & 0 & 0 & \cdots & 0 & \ddots & \ddots & \lambda \\ 0 & & \cdots & 0 & \lambda & 0 & \cdots & 0 & \cdots & D-v_{K+1} \end{pmatrix} \quad (2.2)$$

and acting in (2.1) on the $(M+K+1)$ column Ψ such that $\Psi^T = (\psi_1, \psi_2, \dots, \psi_{M+K+1})$. D is a differential operator which acts on the function f according to $[D, f] = f'$. We impose the condition $\sum_{i=1}^{K+1} v_i = 0$. Similar matrix operators have appeared in, e.g., Refs. 12 and 19.

We write explicitly the linear problem (2.2) as

$$\begin{aligned} \partial \psi_i + q_i \psi_{M+1} &= 0, \quad i = 1, \dots, M, \\ \sum_{i=1}^M r_i \psi_i + (\partial - v_1) \psi_{M+1} + \lambda \psi_{M+2} &= 0, \\ (\partial - v_r) \psi_{M+r} + \lambda \psi_{M+r+1} &= 0, \quad r = 2, \dots, K, \\ \lambda \psi_{M+1} + (\partial - v_{K+1}) \psi_{M+K+1} &= 0. \end{aligned} \quad (2.3)$$

Equation (2.3) gives rise to $K+1$ scalar Lax eigenvalue equations,

$$L_j \psi_{M+j} = (-\lambda)^{K+1} \psi_{M+j}; \quad j = 1, \dots, K+1, \quad (2.4)$$

where the scalar Lax operator is given by ($r = 2, \dots, K+1$)

$$\begin{aligned}
 L_r &= (D - v_{r-1})(D - v_{r-2}) \cdots (D - v_2) \left(D - v_1 - \sum_{i=1}^M r_i D^{-1} q_i \right) (D - v_{K+1}) \cdots (D - v_r), \\
 L_1 &= (D - v_{K+1})(D - v_K) \cdots (D - v_2) \left(D - v_1 - \sum_{i=1}^M r_i D^{-1} q_i \right).
 \end{aligned} \tag{2.5}$$

For all $K+1$ values of j the corresponding Lax operator L_j can be cast in the form of the Lax operator (1.1) in $\mathbf{ckP}_{K+1,M}$ hierarchy. All of the Lax operators (2.5) can be associated with the one-matrix eigenvalue problem (2.2). The question therefore arises whether the above reduction of the matrix eigenvalue problem determines uniquely the scalar Lax operator. We will now answer this problem of potential ambiguity by showing the equivalence between all the Lax operators from (2.5) in the sense of the Darboux–Bäcklund (DB) symmetry.

The following similarity transformations connect the neighboring Lax operators from (2.5):

$$(D - v_{r-1})^{-1} L_r (D - v_{r-1}) = T_{r-1}^{-1} L_r T_{r-1} = L_{r-1}, \quad r = 3, \dots, K+2, \tag{2.6}$$

with the “periodic condition” $L_{K+2} = L_1$. In Eq. (2.6) we have introduced the operator $T_j = \Phi_j D \Phi_j^{-1}$ with $\Phi_j \equiv \exp(\int v_j)$ to emphasize the Darboux–Bäcklund character of the above similarity transformation. In addition we have the following eigenvalue equation holding for each Lax operator L_j :

$$L_j \Phi_j = 0, \quad j = 2, \dots, K+1. \tag{2.7}$$

Assume now that L_j satisfies the Lax flow equation (1.2). Applying it to the equation (2.7) we find that $(\partial_{t_n} - (L_j^{n/(K+1)})_+) \Phi_j$ is annihilated by L_j and we therefore expect, without loosing generality, to have the following identity $(\partial_{t_n} - (L_j^{n/(K+1)})_+) \Phi_j = \alpha(t_2, t_3, \dots) \Phi_j$, where the proportionality coefficient α depends only on times t_i with $i \geq 2$. Comparing both sides of this identity we find that $\alpha = 0$ and therefore Φ_j is an eigenfunction of L_j , meaning that

$$\partial_{t_n} e^{(\int v_j)} = (L_j^{n/(K+1)})_+ e^{(\int v_j)}. \tag{2.8}$$

Recall now that the DB transformation $L \rightarrow TLT^{-1}$, where $T = \Phi D \Phi^{-1}$ with an eigenfunction Φ preserves the form of the Lax equation (1.2), i.e., the DB transformed Lax operator satisfies the same evolution equation as the original Lax operator (see, e.g., Refs. 20, 6, and 7). Since $L_j = T_{j-1} L_{j-1} T_{j-1}^{-1}$ and we have Eq. (2.8), we conclude that all the Lax operators from (2.5) are equivalent belonging to the same “multiplet” from the DB symmetry point of view.

III. CONSTRUCTION OF HIERARCHIES

In this section we provide the basic ingredients for the construction of the type of integrable hierarchies we are going to consider. The discussion is based on the method of references 8–12.

Let $\hat{\mathcal{S}}$ be an affine Kac–Moody algebra, and \mathcal{S} be the finite dimensional simple Lie algebra associated to it. The integral gradations of $\hat{\mathcal{S}}$ are defined by vectors $\mathbf{s} = (s_0, s_1, \dots, s_r)$,²¹ where s_i are non-negative relatively prime integers, and $r \equiv \text{rank } \mathcal{S}$. The corresponding grading operator is given by

$$Q_s \equiv \sum_{a=1}^r s_a \frac{2\lambda_a \cdot H^0}{\alpha_a^2} + N_s d, \tag{3.1}$$

where H_a^0 , $a = 1, 2, \dots, r$, are the Cartan subalgebra generators of \mathcal{S} , λ_a its fundamental weights satisfying $2\lambda_a \cdot \alpha_b / \alpha_b^2 = 2\delta_{ab}$, with α_a being the simple roots of \mathcal{S} . d is the usual derivation of

$\hat{\mathcal{G}}$, responsible for the homogeneous gradation of $\hat{\mathcal{G}}$, corresponding to $\mathbf{s}_{\text{hom}} = (1, 0, 0, \dots, 0)$. In addition, we have, $N_{\mathbf{s}} \equiv \sum_{i=0}^r s_i m_i^\psi$, $\psi = \sum_{a=1}^r m_a^\psi \alpha_a$, $m_0^\psi = 1$, where ψ is the highest positive root of \mathcal{G} . So, we have

$$\hat{\mathcal{G}} = \oplus_{n \in \mathbb{Z}} \hat{\mathcal{G}}_n(\mathbf{s}), \quad [Q_{\mathbf{s}}, \hat{\mathcal{G}}_n(\mathbf{s})] = n \hat{\mathcal{G}}_n(\mathbf{s}), \quad [\hat{\mathcal{G}}_m(\mathbf{s}), \hat{\mathcal{G}}_n(\mathbf{s})] \subset \hat{\mathcal{G}}_{m+n}(\mathbf{s}). \quad (3.2)$$

Introduce the Lax matrix operator

$$L \equiv \partial_x + E + A, \quad (3.3)$$

where E is a semisimple element of $\hat{\mathcal{G}}$, lying in $\hat{\mathcal{G}}_1$, and A is a potential belonging to the subalgebra $\hat{\mathcal{G}}_0$. The construction works equally well with E belonging to any subspace $\hat{\mathcal{G}}_n$, $n > 0$, and A having grade components ranging from 0 to $n - 1$. However, such general setting will not be needed in what follows.

The fact that E is semisimple means that $\hat{\mathcal{G}}$ can be decomposed into the kernel and image of the adjoint action of E

$$\hat{\mathcal{G}} = \text{Ker}(\text{ad } E) + \text{Im}(\text{ad } E). \quad (3.4)$$

As a consequence of Jacobi identity one has

$$[\text{Ker}(\text{ad } E), \text{Ker}(\text{ad } E)] \subset \text{Ker}(\text{ad } E), \quad [\text{Ker}(\text{ad } E), \text{Im}(\text{ad } E)] \subset \text{Im}(\text{ad } E). \quad (3.5)$$

Using the fact that E is semisimple, one can perform a gauge transformation to rotate the Lax operator into $\text{Ker}(\text{ad } E)$. Consider

$$L_0 \equiv U L U^{-1} \equiv \partial_x + E + \sum_{j=-\infty}^0 K^{(j)} \equiv \partial_x + E + K_0, \quad (3.6)$$

where U is an exponentiation of negative grade generators, $U = \exp \sum_{j=1}^{\infty} T^{(-j)}$, with $T^{(-j)} \in \hat{\mathcal{G}}_{-j}(\mathbf{s})$. Decomposing (3.6) into $Q_{\mathbf{s}}$ eigensubspaces, we get equations of the form $K^{(j)} = -[E, T^{(j-1)}] + X^{(j)}$, where $X^{(j)}$ depends on $T^{(m)}$ s for $m > j - 1$. Therefore, starting from the highest grade component, one can choose $T^{(j-1)}$ to exactly cancel the component of $X^{(j)}$ in $\text{Im}(\text{ad } E)$. Consequently, one gets $K^{(j)} \in \text{Ker}(\text{ad } E)$. Notice that the choice of $T^{(j-1)}$ is not unique, since its component in $\text{Ker}(\text{ad } E)$ is not relevant in the cancellation of the $\text{Im}(\text{ad } E)$ component of $X^{(j)}$. In addition, $T^{(j-1)}$ is determined as a local polynomial of the components of the potential A and its x -derivatives.

The flow equations for the hierarchies are constructed in a quite simple way.^{9,10} Consider a constant element $b^{(N)}$, with grade N ($N > 0$), belonging to the center of $\text{Ker}(\text{ad } E)$. Then, from the considerations above one gets that $b^{(N)}$ commutes with L_0 , and so, from (3.6) one has

$$[L, U^{-1} b^{(N)} U] = 0 \quad (3.7)$$

or

$$[L, (U^{-1} b^{(N)} U)_{\geq 0}] = -[L, (U^{-1} b^{(N)} U)_{< 0}], \quad (3.8)$$

where $(\cdot)_{\geq 0}$ and $(\cdot)_{< 0}$ mean non-negative and negative grade components, respectively. One observes that the l.h.s. of (3.8) has components with grades varying from 0 to $N + 1$, and the r.h.s. has grades varying from $-\infty$ to 0. Consequently, both sides of (3.8) have to lie on the subalgebra $\hat{\mathcal{G}}_0$. It is therefore consistent to introduce, for each element $b^{(N)}$ at the center of $\text{Ker}(\text{ad } E)$, with grade N ($N > 0$) a flow equation as

$$\frac{dL}{dt_{b^{(N)}}} = \frac{dA}{dt_{b^{(N)}}} \equiv [L, B_{b^{(N)}}], \quad (3.9)$$

where

$$B_{b^{(N)}} \equiv (U^{-1}b^{(N)}U)_{\geq 0} \equiv \sum_{j=0}^N B_{b^{(N)}}^{(j)}, \quad B_{b^{(N)}}^{(j)} \in \hat{\mathcal{S}}_j(\mathbf{s}). \quad (3.10)$$

Notice that $B_{b^{(N)}}$ is a polynomial of the components of A and its x -derivatives.

From (3.6) and (3.9) one gets

$$\frac{dL_0}{dt_{b^{(N)}}} = [L_0, \tilde{B}_{b^{(N)}}], \quad \text{with } \tilde{B}_{b^{(N)}} \equiv UB_{b^{(N)}}U^{-1} + \frac{dU}{dt_{b^{(N)}}}U^{-1}. \quad (3.11)$$

In fact, $\tilde{B}_{b^{(N)}}$ lies in $\text{Ker}(\text{ad } E)$. In order to see that, we denote $\tilde{B}_{b^{(N)}} = \tilde{B}_{b^{(N)}}^K + \tilde{B}_{b^{(N)}}^T$, with $\tilde{B}_{b^{(N)}}^T \in \text{Im}(\text{ad } E)$ and $\tilde{B}_{b^{(N)}}^K \in \text{Ker}(\text{ad } E)$. Then, splitting (3.11) in its $\text{Ker}(\text{ad } E)$ and $\text{Im}(\text{ad } E)$ components one gets

$$\frac{dK_0}{dt_{b^{(N)}}} - \partial_x \tilde{B}_{b^{(N)}}^K = [K_0, \tilde{B}_{b^{(N)}}^K] \quad (3.12)$$

and

$$\partial_x \tilde{B}_{b^{(N)}}^T + [E + K_0, \tilde{B}_{b^{(N)}}^T] = 0. \quad (3.13)$$

The highest grade component of (3.13) is $[E, (\tilde{B}_{b^{(N)}}^T)_N] = 0$, with $(\tilde{B}_{b^{(N)}}^T)_N \equiv \tilde{B}_{b^{(N)}}^T \cap \hat{\mathcal{S}}_N(\mathbf{s})$. Since there is no intersection between $\text{Ker}(\text{ad } E)$ and $\text{Im}(\text{ad } E)$, it follows that $(\tilde{B}_{b^{(N)}}^T)_N = 0$. Following this reasoning one concludes that $\tilde{B}_{b^{(N)}}^T = 0$, and so $\tilde{B}_{b^{(N)}}$ given in (3.11) lies in $\text{Ker}(\text{ad } E)$.

Notice that if $\text{Ker}(\text{ad } E)$ is Abelian (as is a case when E is regular), then (3.12) constitutes a local conservation law.

The flows defined in (3.9) commute, as a consequence of the fact that $\tilde{B}_{b^{(N)}} \in \text{Ker}(\text{ad } E)$, and that $b^{(N)}$ belongs to the center of $\text{Ker}(\text{ad } E)$. Indeed, those facts imply that $[d/dt_{b^{(N)}} - \tilde{B}_{b^{(N)}}]b^{(M)} = 0$. Conjugating with U , one gets $(d/dt_{b^{(N)}})(U^{-1}b^{(M)}U) = [B_{b^{(N)}}, U^{-1}b^{(M)}U]$. Taking the positive grade part and subtracting the same relation with $b^{(M)}$ and $b^{(N)}$ interchanged, one gets

$$\frac{dB_{b^{(M)}}}{dt_{b^{(N)}}} - \frac{dB_{b^{(N)}}}{dt_{b^{(M)}}} = [B_{b^{(N)}}, U^{-1}b^{(M)}U]_{\geq 0} - [B_{b^{(M)}}, U^{-1}b^{(N)}U]_{\geq 0}. \quad (3.14)$$

But $[B_{b^{(N)}}, U^{-1}b^{(M)}U]_{\geq 0} = [B_{b^{(N)}}, B_{b^{(M)}}] + [B_{b^{(N)}}, (U^{-1}b^{(M)}U)_{<0}]_{\geq 0}$. Since $b^{(N)}$ and $b^{(M)}$ commute, it follows that $[(U^{-1}b^{(M)}U)_{\geq 0}, U^{-1}b^{(N)}U] = -[(U^{-1}b^{(M)}U)_{<0}, U^{-1}b^{(N)}U]$. Taking the positive grade part of it one gets $[B_{b^{(M)}}, U^{-1}b^{(N)}U]_{\geq 0} = [B_{b^{(N)}}, (U^{-1}b^{(M)}U)_{<0}]_{\geq 0}$. Therefore one concludes that

$$\frac{dB_{b^{(M)}}}{dt_{b^{(N)}}} - \frac{dB_{b^{(N)}}}{dt_{b^{(M)}}} + [B_{b^{(M)}}, B_{b^{(N)}}] = 0 \quad (3.15)$$

and due to Eq. (3.9) that is a sufficient condition for the flows to commute:

$$\left[\frac{d}{dt_{b^{(M)}}}, \frac{d}{dt_{b^{(N)}}} \right] L = 0. \quad (3.16)$$

Notice that the gauge transformations

$$L \rightarrow e^S L e^{-S}, \quad B_{b^{(N)}} \rightarrow e^S B_{b^{(N)}} e^{-S}, \quad \frac{dS}{dt_{b^{(N)}}} = 0, \quad S \in \mathcal{K} \equiv \hat{\mathcal{G}}_0 \cap \text{Ker}(\text{ad } E) \quad (3.17)$$

are symmetries of the flows equations (3.9), in the sense that they preserve the form of the Lax operator L . Associated to such symmetries we have conserved quantities. Indeed, the component of zero grade on the r.h.s. of (3.8) is $[E, (U^{-1} b^{(N)} U)_{-1}]$. But that implies that the l.h.s. of (3.8), and consequently both sides of (3.9), have no components in $\text{Ker}(\text{ad } E)$. Then

$$\frac{dA^{\mathcal{K}}}{dt_{b^{(N)}}} = 0, \quad A^{\mathcal{K}} \equiv A \cap \text{Ker}(\text{ad } E). \quad (3.18)$$

Therefore, if we choose $A^{\mathcal{K}} = 0$ at $t_{b^{(N)}} = 0$, it will remain zero for all times. That is a reduction procedure which we will use below to obtain the constrained KP hierarchies from the above formalism. We shall decompose the potential $A \in \hat{\mathcal{G}}_0$ as

$$A \equiv A_0 + A^{\mathcal{K}} \quad (3.19)$$

with $A^{\mathcal{K}} \in \mathcal{K}$, and A_0 lying in the complement \mathcal{M} of \mathcal{K} in $\hat{\mathcal{G}}_0$. A_0 contains therefore the dynamical variables of the integrable hierarchy.

Since we are working with loop algebras (vanishing central term) it is useful to work with finite matrix representations. The commutation relations for $\hat{\mathcal{G}}$ can be written as

$$[T_a^m, T_b^n] = f_{ab}^c T_c^{m+n}, \quad [d, T_a^m] = m T_a^m, \quad (3.20)$$

where $T_a^0 \equiv T_a$, $a = 1, 2, \dots, \dim \mathcal{G}$, are the generators of the finite simple Lie algebra \mathcal{G} , and f_{ab}^c are its structure constants. If one has a (finite) matrix representation of \mathcal{G} , then one can construct a representation of $\hat{\mathcal{G}}$ by replacing

$$T_a^m \rightarrow z^m T_a, \quad (3.21)$$

where z is a complex parameter. However, in some calculations we will be interested in another representation of such type, where the powers of the complex parameter count the grade w.r.t. Q_s defined in (3.1). Accordingly we replace

$$T_a^m \rightarrow \lambda^l T_a, \quad l = g_a + m N_s, \quad (3.22)$$

where λ is a complex parameter, and

$$\left[\sum_{b=1}^r s_b \frac{2\lambda_b \cdot H^0}{\alpha_b^2}, T_a \right] = g_a T_a. \quad (3.23)$$

Notice that g_a take values between $-N_s + 1$ and $N_s - 1$.

In the representation (3.21) one has $d \equiv z(d/dz)$. Therefore if $[Q_s, X] = x_s X$, with Q_s given by (3.1) one has $[Q_s, zX] = (x_s + N_s)zX$. Now, if $b^{(N)}$ is an element of the center of $\text{Ker}(\text{ad } E)$, so is $z b^{(N)}$. We shall denote $b^{(N+N_s)} \equiv z b^{(N)}$. Therefore, $z U^{-1} b^{(N)} U = U^{-1} b^{(N+N_s)} U$, and so

$$z(U^{-1} b^{(N)} U)_{\geq 0} + z(U^{-1} b^{(N)} U)_{< 0} = (U^{-1} b^{(N+N_s)} U)_{\geq 0} + (U^{-1} b^{(N+N_s)} U)_{< 0}, \quad (3.24)$$

where $(\cdot)_{\geq 0}$ and $(\cdot)_{< 0}$ mean the non-negative and negative s -grade components, respectively. But since multiplication by z increases the s -grade by N_s we have that the positive part of (3.24) leads to

$$B_{b(N+N_s)} = zB_{b(N)} + z \sum_{j=1}^{N_s} (U^{-1}b^{(N)}U)_{-j}. \quad (3.25)$$

Notice the second term on the r.h.s. of (3.25) have components with \mathbf{s} -grades varying from 0 to $N_s - 1$. But, analyzing (3.1), one concludes that any generator having positive grade w.r.t. d , has necessarily \mathbf{s} -grade greater or equal to N_s . Therefore, we conclude that the quantity $z \sum_{j=1}^{N_s} (U^{-1}b^{(N)}U)_{-j}$ is z independent in the representation (3.21).

A. The case of $\widehat{sl}(M+K+1)$

We now apply the above formalism to the example of the affine Kac–Moody algebra $\hat{\mathcal{G}} = \widehat{sl}(M+K+1)$, $(A_{M+K}^{(1)})$ without a central term (i.e., a loop algebra), furnished with the following gradation \mathbf{s} and corresponding grading operator Q_s :

$$\mathbf{s} = (1, \underbrace{0, \dots, 0}_M, \underbrace{1, \dots, 1}_K); \quad Q_s = \sum_{j=M+1}^{M+K} \lambda_j \cdot H^{(0)} + (K+1)d. \quad (3.26)$$

We will denote the simple roots of $\widehat{sl}(M+K+1)$ by α_j , $j=0, 1, \dots, M+K$, with $\alpha_0 \equiv -\psi$, and ψ being the highest positive root of $\mathcal{S} = sl(M+K+1)$, which is the subalgebra of $\widehat{sl}(M+K+1)$ commuting with d . Since $\widehat{sl}(M+K+1)$ is simply laced we normalize the roots such that $\alpha_j^2 = 2$. The ordering of the roots is such that for $j \neq k$, $\alpha_j \cdot \alpha_k = -\delta_{j,k \pm 1 \pmod{M+K+1}}$, $j, k = 0, 1, \dots, M+K$. The fundamental weights λ_j satisfy $\lambda_j \cdot \alpha_k = \delta_{j,k}$. We choose the semi-simple element E to be

$$E = \sum_{j=M+1}^{M+K} E_{\alpha_j}^{(0)} + E_{-(\alpha_{M+1} + \dots + \alpha_{M+K})}^{(1)}. \quad (3.27)$$

One can check that each generator in E has grade one w.r.t. Q_s . Hence

$$[Q_s, E] = E. \quad (3.28)$$

The zero grade subalgebra is

$$\hat{\mathcal{G}}_0 = \{\mathcal{S}_0^{(0)} \equiv sl(M+1); \quad \alpha_j \cdot H^{(0)}, \quad j = M+1, M+2, \dots, M+K\}, \quad (3.29)$$

where $\mathcal{S}_0^{(0)}$ is the $sl(M+1)$ subalgebra of $\mathcal{S} = sl(M+K+1)$, with simple roots $\alpha_1, \dots, \alpha_M$.

For E defined in (3.27) we have

$$\text{Ker}(\text{ad } E) = \{\hat{K}_0 \equiv \widehat{sl}(M) \oplus \hat{U}(1), \hat{\mathcal{H}}_K\}, \quad (3.30)$$

where $\widehat{sl}(M)$ is the affine Kac–Moody subalgebra of $\hat{\mathcal{G}} = \widehat{sl}(M+K+1)$ with simple roots α_j , $j=1, 2, \dots, M-1$, and $\alpha_0 = -(\alpha_1 + \alpha_2 + \dots + \alpha_{M-1})$. The Kac–Moody algebra $\hat{U}(1)$ is generated by $\lambda_M \cdot H^{(k)}$, $k \in \mathbb{Z}$. In addition $\hat{\mathcal{H}}_K$ is the principal Heisenberg subalgebra of $\widehat{sl}(K+1) \in \widehat{sl}(M+K+1)$ containing E . We can denote its generators by E_N , where N contains the integers $1, 2, 3, \dots, K$ (modulo $K+1$) or in other words, the integers without the multiples of $(K+1)$. In this notation we have

$$\begin{aligned}
E_{l+(K+1)n} = & E_{\alpha_{M+1}+\alpha_{M+2}+\dots+\alpha_{M+l}}^{(n)} + E_{\alpha_{M+2}+\alpha_{M+3}+\dots+\alpha_{M+l+1}}^{(n)} + \dots \\
& + E_{\alpha_{M+K-l+1}+\alpha_{M+K-l+2}+\dots+\alpha_{M+K-1}+\alpha_{M+K}}^{(n)} + E_{-(\alpha_{M+1}+\alpha_{M+2}+\dots+\alpha_{M+K-l+1})}^{(n+1)} \\
& + E_{-(\alpha_{M+2}+\alpha_{M+3}+\dots+\alpha_{M+K-l})}^{(n+1)} + \dots + E_{-(\alpha_{M+l}+\alpha_{M+3}+\dots+\alpha_{M+K})}^{(n+1)}
\end{aligned} \quad (3.31)$$

with $l=1,2,3,\dots,K$, and so

$$[Q_s, E_{l+(K+1)n}] = (l+(K+1)n)E_{l+(K+1)n}. \quad (3.32)$$

Notice that $E_1 \equiv E$. In addition, since we are working here with the case of loop algebra ($c=0$),

$$[E_N, E_{N'}] = 0. \quad (3.33)$$

In fact, in the representation (3.22) of the loop algebra one has

$$E = \lambda \tilde{E} = \lambda \left(\sum_{j=M+1}^{M+K} E_{\alpha_j} + E_{-(\alpha_{M+1}+\dots+\alpha_{M+K})} \right), \quad (3.34)$$

and in the defining representation of $sl(M+K+1)$, one has

$$\tilde{E} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{e} \end{pmatrix}; \quad \tilde{e} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad (3.35)$$

where \tilde{E} and \tilde{e} are $(M+K+1) \times (M+K+1)$ and $(K+1) \times (K+1)$ matrices, respectively. Elements of this type are among the generators of the nonequivalent Heisenberg subalgebras, see, e.g., Appendix of Ref. 13 for the cases of $sl(3)$ and $sl(4)$.

In addition, one has

$$E_{l+(K+1)n} = \lambda^{l+(K+1)n} (\tilde{E})^l. \quad (3.36)$$

Also for $c=0$ we have

$$\text{center Ker(ad } E) = \{\hat{U}(1), \hat{\mathcal{H}}_K\}, \quad (3.37)$$

where $\hat{U}(1)$ is generated by $\lambda_M \cdot H^{(k)}$, $k \in \mathbb{Z}$. Notice that $[Q_s, \lambda_M \cdot H^{(k)}] = k(K+1)\lambda_M \cdot H^{(k)}$. Therefore the center of $\text{Ker(ad } E)$, while having generators of arbitrary grade, has one and only one generator of a given grade. Then the choices we have for the elements $b^{(N)}$, introduced in (3.7), are

$$b^{(N)} = E_N, \quad N = 1, 2, \dots, K \bmod (K+1), \quad (3.38)$$

$$b^{(k(K+1))} = \lambda_M \cdot H^{(k)}, \quad k \in \mathbb{Z}. \quad (3.39)$$

According to (3.9) each of the generators from the center of $\text{Ker(ad } E)$ appearing in (3.38)–(3.39) will give rise to the corresponding flow with times $t_{b^{(N)}}, t_{b^{(k(K+1))}}$. In particular the element $E_1 \equiv E$ will generate the flow corresponding to $\partial/\partial t_1 = \partial/\partial x$.

The gauge symmetries of the model are then given by the transformations (3.17), where S belongs to the subalgebra

$$\mathcal{K} \equiv \hat{\mathcal{G}}_0 \cap \text{Ker}(\text{ad } E) = \{sl(M), \lambda_M \cdot H^{(0)}\}, \quad (3.40)$$

where $sl(M)$ is the subalgebra of $\mathcal{G} = sl(M+K+1)$ with simple roots $\alpha_1, \alpha_2, \dots, \alpha_{M-1}$.

The generators of the complement \mathcal{M} of \mathcal{K} in $\hat{\mathcal{G}}_0$ are

$$\mathcal{M} = \{P_{\pm i} = E_{\pm(\alpha_i + \alpha_{i+1} + \dots + \alpha_M)}, i = 1, 2, \dots, M, \text{ and } \alpha_a \cdot H^{(0)}, a = M+1, M+2, \dots, M+K\}. \quad (3.41)$$

We then parametrize A_0 , defined in (3.19), as follows:

$$A_0 = \sum_{i=1}^M (q_i P_i + r_i P_{-i}) + \sum_{a=M+1}^{M+K} U_a (\alpha_a \cdot H^{(0)}), \quad (3.42)$$

where q_i , r_i and U_a are fields of the model.

As we have shown in (3.18), $A^{\mathcal{K}}$ is constant in time. Therefore, we will work with the submodel defined by

$$A^{\mathcal{K}} = 0. \quad (3.43)$$

The flow equations (3.9), in this case, become

$$\frac{dA_0}{dt_{b(N)}} - \partial_x B_{b(N)}^0 = [E + A_0, B_{b(N)}^0], \quad (3.44)$$

where $B_{b(N)}^0$ is the constrained $B_{b(N)}$, i.e.,

$$B_{b(N)}^0 = B_{b(N)}|_{A^{\mathcal{K}}=0}. \quad (3.45)$$

The effective potential of our submodel lies therefore, on the tangent plane of the coset space $\hat{\mathcal{G}}_0/\mathcal{K} \equiv (sl(M+1) \oplus U(1)^K)/(sl(M) \oplus U(1)_M)$. $U(1)_M$ is generated by $\lambda_M \cdot H^{(0)}$, and consequently is a linear combination of the generators of the Cartan subalgebra of $sl(M+1)$ and also of all the generators of $U(1)^K$. Remember that $\lambda_i = K_{ij}^{-1} \alpha_j$, and the inverse of the Cartan matrix K_{ij}^{-1} of A_n has no vanishing elements. Those facts prevent $\hat{\mathcal{G}}_0/\mathcal{K}$ from being a symmetric space. Indeed, one can verify that

$$[P_j, P_{-j}] = \alpha_M \cdot H^{(0)} + \sum_{i=j}^{M-1} \alpha_i \cdot H^{(0)}, \quad \in \mathcal{K} + \mathcal{M}, \quad P_j, P_{-j} \in \mathcal{M}, \quad (3.46)$$

since $\alpha_M \cdot H^{(0)}$ have components on both \mathcal{K} and \mathcal{M} .

However, one has

$$[P_j, P_{-k}] \in \mathcal{K} \quad \text{for } j \neq k, \quad [P_j, P_k] = [P_{-j}, P_{-k}] = 0 \quad \text{for any } j, k. \quad (3.47)$$

1. The case $K=0$

In this case we have $\hat{\mathcal{G}} = \widehat{sl}(M+1)$. The relevant gradation is the homogeneous one, $s = (1, 0, 0, \dots, 0)$, and so $Q_s = d$. The semisimple element E is now given by

$$E = \lambda_M \cdot H^{(1)}, \quad [d, E] = E. \quad (3.48)$$

This example was discussed in detail in Ref. 14, and here we just give a brief description of it to make contact with the model discussed above.

The grade zero subalgebra is

$$\hat{\mathcal{G}}_0 = \{\mathcal{G} \equiv sl(M+1)\} \quad (3.49)$$

and

$$\text{Ker}(\text{ad } E) = \{\hat{sl}(M) \oplus \hat{U}(1)\} \quad (3.50)$$

with $\hat{U}(1)$ being generated by $\lambda_M \cdot H^{(k)}$, $k \in \mathbb{Z}$.

The center of $\text{Ker}(\text{ad } E)$ is just the homogeneous Heisenberg subalgebra of $\hat{\mathcal{G}} = \hat{sl}(M+1)$, namely,

$$\text{center Ker}(\text{ad } E) = \{\lambda_i \cdot H^{(k)}, i=1, 2, \dots, M, k \in \mathbb{Z}\}. \quad (3.51)$$

Therefore, we can introduce a flow for each element [see (3.7)]

$$b_i^{(k)} \equiv \lambda_i \cdot H^{(k)}, \quad k \text{ being a positive integer.} \quad (3.52)$$

The gauge subalgebra \mathcal{K} introduced in (3.17) is

$$\mathcal{K} = \{sl(M) \oplus U(1)\}, \quad (3.53)$$

where $sl(M)$ is the subalgebra of $\mathcal{G} = sl(M+1)$ with simple roots $\alpha_1, \alpha_2, \dots, \alpha_{M-1}$, and $U(1)$ is generated by $\lambda_M \cdot H^{(0)}$. We write the potential $A \in \hat{\mathcal{G}}_0$ as $A = A^{\mathcal{K}} + A_0$, with $A^{\mathcal{K}} \in \mathcal{K}$ and A_0 lying in the complement \mathcal{M} of \mathcal{K} in $\hat{\mathcal{G}}_0$. Then, we parametrize A_0 as

$$A_0 = \sum_{i=1}^M (q_i P_i + r_i P_{-i}), \quad (3.54)$$

where $P_{\pm i}$ were introduced in (3.41). Comparing with (3.42), we notice an absence of U_a 's fields in this special example. Also $\hat{\mathcal{G}}_0/\mathcal{K}$ is now a symmetric space.

IV. ZAKHAROV-SHABAT EQUATION AND RECURSION OPERATOR

Recall that within the models considered here the semisimple element E is given for $\mathcal{G} = sl(M+K+1)$ by (3.34). We first notice that E commutes with its conjugated counterpart E^\dagger and therefore, although not Hermitian, may be diagonalized.²² As a consequence, the Lie algebra \mathcal{G} under consideration can be decomposed into graded subspaces, i.e.,

$$\mathcal{G} = \oplus_s \mathcal{G}_s; \quad [E, \mathcal{G}_s] = s \mathcal{G}_s, \quad (4.1)$$

where s can be in general a complex number ($s \in \mathbb{C}$). This is a crucial property allowing to solve the Zakharov-Shabat (ZS) equation in the manner shown below. Consider, namely, vs the ZS equation

$$\partial_m A_0 - \partial B_m + \lambda [E, B_m] + [A_0, B_m] = 0, \quad (4.2)$$

where A_0 defined in Section III lies, as described there, in subspace orthogonal to $\text{Ker}(\text{ad } E)$.

Decomposing $B_m = \sum_s B_m^{(s)}$ and $A_0 = \sum_s A^{(s)}$ into components according to the gradation defined by E induces a natural decomposition of the Z-S equation (4.2) into the zero and nonzero components:

$$-\partial B_m^{(0)} + \sum_{x+y=0} [A^{(x)}, B_m^{(y)}] = 0, \quad (4.3)$$

$$\partial_m A^{(s)} - \partial B_m^{(s)} + \lambda s B_m^{(s)} + \sum_{x+y=s} [A^{(x)}, B_m^{(y)}] = 0, \quad (4.4)$$

where in the last equation the summation is over $x, y \in \mathbb{C}$ and includes $y=0, x=s$. Equations (4.3) and (4.4) contain components of (4.2) in $\text{Ker}(\text{ad } E)$ and $\text{Im}(\text{ad } E) = \bigoplus_{s \in \mathbb{C} - \{0\}} \mathcal{S}_s$. We now assume the following expansions:

$$B_m^{(s)} = \sum_{i=0}^{m-1} B_m^{(s)}(i) \lambda^i, \quad s \neq 0, \quad (4.5)$$

for all nonzero gradation components of B_m , while for the zero component we find from (4.3) by integration:

$$B_m^{(0)} = D^{-1} \left(\sum_{x, y \neq 0}^{x+y=0} [A^{(x)}, B_m^{(y)}] \right) + \lambda^m \Lambda_m, \quad (4.6)$$

where the last (integration constant) term on the right hand side of (4.6) is of higher order than those in (4.5). Its presence is allowed by the structure of (4.2) as long as $\Lambda_m \in \text{Ker}(\text{ad } E)$.

Inserting (4.5) in (4.4) we find by collecting the coefficients of λ^{k-1}

$$l B_m^{(l)}(k-1) = -\partial B_m^{(l)}(k) - \sum_{x, y \neq 0}^{x+y=0} [A^{(l)}, D^{-1} [A^{(x)}, B_m^{(y)}(k)]] - \sum_{x, y \neq l}^{x+y=l} [A^{(l)}, B_m^{(y)}(k)]; \quad (4.7)$$

for $k=m$ we obtain

$$l B_m^{(l)}(m-1) = -[A^{(l)}, \Lambda_m]. \quad (4.8)$$

From (4.7) we find that the general solution can be rewritten as

$$B_m^{(l)}(k-1) = \sum_{y \neq 0} \Phi_{l,y} B_m^{(y)}(k) \quad (4.9)$$

with

$$\Phi_{l,y} = \frac{1}{l} \left(D \delta_{l,y} - \sum_{x \in \text{Grad}(\mathcal{M})} [ad_{A^{(l)}} D^{-1} ad_{A^{(x)}} \delta_{x,-y} + ad_{A^{(x)}} \delta_{l-x,y}] \right). \quad (4.10)$$

Using (4.9) repeatedly we are led to

$$B_m^{(l)}(0) = \sum_{y_1, \dots, y_{m-1}} \Phi_{l,y_1} \Phi_{y_1,y_2} \cdots \Phi_{y_{m-2},y_{m-1}} B_m^{(y_{m-1})}(m-1), \quad (4.11)$$

and after taking into account (4.8)

$$B_m^{(l)}(0) = \sum_{y_1, \dots, y_{m-1}} \Phi_{l,y_1} \Phi_{y_1,y_2} \cdots \Phi_{y_{m-2},y_{m-1}} \frac{1}{y_{m-1}} ad_{\Lambda_m} A^{(y_{m-1})}. \quad (4.12)$$

Projecting (4.4) on the λ -independent component we find

$$\partial_m A^{(l)} = \partial B_m^{(l)}(0) - \sum_{x+y=l} [A^{(x)}, B_m^{(y)}(0)] = l \sum_y \Phi_{l,y} B_m^{(y)}(0). \quad (4.13)$$

Substituting the solution of $B_m^{(l)}(0)$ in terms of $A^{(y)}$ from (4.12) we arrive at

$$\frac{1}{l} \partial_m A^{(l)} = \sum_{y_1, \dots, y_m} \Phi_{l,y_1} \Phi_{y_1,y_2} \cdots \Phi_{y_{m-1},y_m} \text{ad}_{\Lambda_m} \left(\frac{A^{(y_m)}}{y_m} \right). \quad (4.14)$$

This expression leads to a recursion operator relating consecutive flows belonging the same family of flows generated by the specific element $\Lambda_m = (\tilde{E}^l, \lambda_M \cdot H^{(0)})$ from the center of $\text{Ker}(\text{ad } E)$ as explained in discussion around Eqs. (3.38)–(3.39). Therefore these consecutive times have indices modulo $K+1$ for the case of $sl(M+K+1)$.

To get the closed expression for the recursion operator we compare Eq. (4.14) to the corresponding expression for $\partial_{m-K-1} A^{(l)}$. These flows are related through:

$$\frac{1}{l} \partial_m A^{(l)} = \sum_{y_1, \dots, y_{K+1}} \Phi_{l,y_1} \cdots \Phi_{y_K,y_{K+1}} \partial_{m-K-1} \left(\frac{A^{(y_{K+1})}}{y_{K+1}} \right) \equiv \mathcal{R}_{l,K+1} \partial_{m-K-1} A^{(y_{K+1})}/y_{K+1}. \quad (4.15)$$

This yields an expression for the recurrence operator \mathcal{R} as $\mathcal{R} = \Phi^{K+1}$ for the $sl(M+K+1)$ -matrix hierarchy.

V. THE SECOND BRACKET STRUCTURE

The potential A introduced in (3.3) is an element of the subalgebra $\hat{\mathcal{G}}_0$, and so we shall denote it as

$$A = \eta^{ab} \varphi(T_a) T_b, \quad \varphi(T_a) \equiv \text{Tr}(T_a A), \quad (5.1)$$

where η^{ab} is the inverse of the trace form of $\hat{\mathcal{G}}_0$, $\eta_{ab} \equiv \text{Tr}(T_a T_b)$, $a, b = 1, 2, \dots, \dim \hat{\mathcal{G}}_0$.

There is a natural Poisson bracket structure for the manifold spanned by φ 's components of A , induced by the \mathcal{G}_0 -KM current algebra:

$$\{\varphi(T)(x), \varphi(T')(y)\}_{PB} = \varphi([T, T'])(x) \delta(x-y) + \text{Tr}(TT') \delta'(x-y), \quad T, T' \in \hat{\mathcal{G}}_0. \quad (5.2)$$

The model we are interested in is a constrained system where the components of A in $\text{Ker}(\text{ad } E)$ are set to zero [see (3.18)]. The bracket structure of such submodel is then given by the Dirac bracket associated to (5.2). We denote by \mathcal{K}_i , $i = 1, 2, \dots, \dim \mathcal{K}$, and M_r , $r = 1, 2, \dots, \dim \hat{\mathcal{G}}_0 - \dim \mathcal{K}$, the generators of the subalgebra \mathcal{K} , defined in (3.17), and of its complement \mathcal{M} in $\hat{\mathcal{G}}_0$, respectively.

The Dirac matrix is given by

$$\Delta_{ij}(x, y) \equiv \{\varphi(\mathcal{K}_i)(x), \varphi(\mathcal{K}_j)(y)\} \approx \eta_{ij} \delta'(x-y), \quad (5.3)$$

where $\eta_{ij} \equiv \text{Tr}(\mathcal{K}_i \mathcal{K}_j)$, and \approx means equality after the constraints are imposed. Therefore

$$\Delta_{ij}^{-1}(x, y) \approx \eta^{ij} \partial_x^{-1} \delta(x-y), \quad i, j = 1, 2, \dots, \dim \mathcal{K}. \quad (5.4)$$

Consequently the Dirac bracket is

$$\begin{aligned} \{\varphi(M_r)(x), \varphi(M_s)(y)\}_{DB} &= \varphi([M_r, M_s])(x) \delta(x-y) + \text{Tr}(M_r M_s) \delta'(x-y) \\ &+ \eta^{ij} \varphi([\mathcal{K}_i, M_r])(x) \varphi([\mathcal{K}_j, M_s])(y) \partial_x^{-1} \delta(x-y). \end{aligned} \quad (5.5)$$

The subspace \mathcal{M} constitutes a representation of the subalgebra \mathcal{K} ,

$$[\mathcal{K}_i, M_r] = R_{rs}(\mathcal{K}_i) M_s. \quad (5.6)$$

Therefore, the second term on the r.h.s. of the bracket (5.5) can be calculated using representation theory. The relevant representation here is the tensor product of the representation, $R \otimes R$, defined by the linear functionals $\varphi(M_r)(x)$. We then write

$$\begin{aligned} X_{rs}(x, y) &\equiv \eta^{ij} \varphi([\mathcal{K}_i, M_r])(x) \varphi([\mathcal{K}_j, M_s])(y) \partial_x^{-1} \delta(x-y) \\ &= \eta^{ij} R_{ri}(\mathcal{K}_i) R_{sj}(\mathcal{K}_j) \varphi(M_t)(x) \varphi(M_u)(y) \partial_x^{-1} \delta(x-y) \\ &\equiv C |M_r\rangle_x \otimes |M_s\rangle_y \partial_x^{-1} \delta(x-y), \end{aligned} \quad (5.7)$$

where

$$C \equiv \eta^{ij} \mathcal{K}_i \otimes \mathcal{K}_j \quad (5.8)$$

and where we have denoted states of the representation $R \otimes R$ as $\varphi(M_r)(x) \varphi(M_s)(y) \equiv |M_r\rangle_x \otimes |M_s\rangle_y$. So, the space variables x and y define the left and right entries, respectively, of the tensor product.

The operator (5.8) commutes with any generator

$$[C, 1 \otimes \mathcal{K}_i + \mathcal{K}_i \otimes 1] = 0, \quad (5.9)$$

and, according to Schur's lemma, it is proportional to the identity in each irreducible component of $R \otimes R$. That fact simplifies substantially the evaluation of (5.7). Notice that C is not the quadratic Casimir operator in $R \otimes R$. That operator is given by $C_{R \otimes R} \equiv \eta^{ij} (1 \otimes \mathcal{K}_i + \mathcal{K}_i \otimes 1) \times (1 \otimes \mathcal{K}_j + \mathcal{K}_j \otimes 1)$, and therefore we have

$$C = \frac{1}{2} (C_{R \otimes R} - 1 \otimes C - C \otimes 1), \quad (5.10)$$

where $C = \eta^{ij} \mathcal{K}_i \mathcal{K}_j$ is the quadratic Casimir in R .

Decomposing the representation $R \otimes R$ in its irreducible components, one can evaluate (5.7), using (5.10) and the fact that the value of the quadratic Casimir operator in an irreducible representation is $\lambda(\lambda + 2\delta)$, where λ is the highest weight, and $\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{a=1}^{\text{rank}} \lambda_a$, with α being the positive roots, and λ_a the fundamental weights of \mathcal{K} .

A. The case of $\widehat{sl}(M+K+1)$ -matrix integrable hierarchy

We now consider the example of the affine Kac–Moody algebra $sl(M+K+1)$ discussed in subsection III A. The subalgebra \mathcal{K} and subspace \mathcal{M} are defined in (3.40) and (3.41), respectively. We shall denote by $\tilde{\mathcal{K}}_i$, $i=1, 2, \dots, M^2-1$, the generators of the subalgebra $sl(M)$ of \mathcal{K} .

One can easily verify that P_j and P_{-j} , $j=1, 2, \dots, M$, transform under the representations M and \bar{M} of $sl(M)$, respectively. The highest weights and highest weight states of these representations are (λ_1, P_1) and (λ_{M-1}, P_{-M}) , respectively. The remaining generators of \mathcal{M} (3.41), namely $\alpha_a \cdot H^{(0)}$, $a=M+1, M=2, \dots, M+K$, are scalars under $sl(M)$. The charges of the $U(1)$ factor of \mathcal{K} (3.40), generated by $\lambda_M \cdot H^{(0)}$, are 1 for P_j , -1 for P_{-j} , and 0 for $\alpha_a \cdot H^{(0)}$. Therefore, the representation R of $\mathcal{K} = sl(M) \oplus U(1)$, defined by \mathcal{M} (5.6), is $R = (M, 1) + (\bar{M}, -1) + (0, 0)^K$.

We denote the operator (5.8) as

$$\begin{aligned} \mathbb{C} &= \tilde{\eta}^{ij} \tilde{\mathcal{H}}_i \otimes \tilde{\mathcal{H}}_j + \frac{1}{\text{Tr}(\lambda_M \cdot H^{(0)})^2} \lambda_M \cdot H^{(0)} \otimes \lambda_M \cdot H^{(0)} \\ &\equiv \tilde{\mathbb{C}} + \frac{M+K+1}{M(K+1)} \lambda_M \cdot H^{(0)} \otimes \lambda_M \cdot H^{(0)}, \end{aligned} \quad (5.11)$$

where $\tilde{\eta}^{ij}$ is the inverse of $\tilde{\eta}_{ij} = \text{Tr}(\tilde{\mathcal{H}}_i \tilde{\mathcal{H}}_j)$. Notice that $\text{Tr}(\cdot \cdot)$, as introduced in (5.1), is the trace form of the subalgebra $\tilde{\mathcal{G}}_0$.

Let us then analyze the various irreducible components of the representation $R \otimes R$. The states $|P_i\rangle \otimes |P_j\rangle$ decompose into the symmetric and antisymmetric parts, which are the $M(M+1)/2$ and $M(M-1)/2$ irreducible representations of $sl(M)$, respectively. The highest weights of these representations are $2\lambda_1$ and $2\lambda_1 - \alpha_1$, respectively. Therefore, using (5.11), and then (5.10) for $\tilde{\mathbb{C}}$, one gets

$$\begin{aligned} \mathbb{C}|P_i\rangle \otimes |P_j\rangle &= \left(\frac{M+K+1}{M(K+1)} - \lambda_1(\lambda_1 + 2\delta) \right) |P_i\rangle \otimes |P_j\rangle \\ &\quad + (\lambda_1(\lambda_1 + \delta))(|P_i\rangle \otimes |P_j\rangle + |P_j\rangle \otimes |P_i\rangle) \\ &\quad + \frac{1}{4}((2\lambda_1 - \alpha_1)(2\lambda_1 - \alpha_1 + 2\delta))(|P_i\rangle \otimes |P_j\rangle - |P_j\rangle \otimes |P_i\rangle). \end{aligned} \quad (5.12)$$

Denoting the simple roots of $sl(M)$ as $\alpha_j = e_j - e_{j+1}$, $j = 1, 2, \dots, M-1$, $e_j \cdot e_k = \delta_{jk}$, one has $\lambda_j = \sum_{k=1}^j e_k - (j/M) \sum_{k=1}^M e_k$, and therefore $\delta = \sum_{k=1}^{M-1} \lambda_k = \frac{1}{2} \sum_{k=1}^M (M-2k+1)e_k$. Consequently

$$\mathbb{C}|P_i\rangle \otimes |P_j\rangle = \frac{1}{K+1} |P_i\rangle \otimes |P_j\rangle + |P_j\rangle \otimes |P_i\rangle. \quad (5.13)$$

By the same arguments, one gets

$$\mathbb{C}|P_{-i}\rangle \otimes |P_{-j}\rangle = \frac{1}{K+1} |P_{-i}\rangle \otimes |P_{-j}\rangle + |P_{-j}\rangle \otimes |P_{-i}\rangle. \quad (5.14)$$

As for the states $|P_i\rangle \otimes |P_{-j}\rangle$, we use the fact that the tensor product of the M and \bar{M} representations of $sl(M)$ produces an adjoint and a singlet, i.e., $M \otimes \bar{M} = \text{Adj} + 1$. The singlet is the state $|S\rangle \equiv \sum_{j=1}^M |P_j\rangle \otimes |P_{-j}\rangle$. The states of the adjoint are $|P_i\rangle \otimes |P_{-j}\rangle$ for $i \neq j$, and $|P_j\rangle \otimes |P_{-j}\rangle - |P_{j+1}\rangle \otimes |P_{-(j+1)}\rangle$. The highest weight of the adjoint is the highest positive root $\psi = \alpha_1 + \alpha_1 + \dots + \alpha_{M-1} = e_1 - e_M$. Therefore, using (5.11), and then (5.10) for $\tilde{\mathbb{C}}$, one gets for $i \neq j$

$$\begin{aligned} \mathbb{C}|P_i\rangle \otimes |P_{-j}\rangle &= -\frac{M+K+1}{M(K+1)} |P_i\rangle \otimes |P_{-j}\rangle \\ &\quad + \frac{1}{2}(\psi(\psi + 2\delta) - \lambda_1(\lambda_1 + 2\delta) - \lambda_{M-1}(\lambda_{M-1} + 2\delta)) |P_i\rangle \otimes |P_{-j}\rangle \\ &= -\frac{1}{K+1} |P_i\rangle \otimes |P_{-j}\rangle. \end{aligned} \quad (5.15)$$

One can easily check that

$$|P_j\rangle \otimes |P_{-j}\rangle = \frac{1}{M} |S\rangle + |X_j\rangle, \quad (5.16)$$

where

$$|X_j\rangle = -\frac{1}{M} \left(\sum_{k=1}^{j-1} k |v_k\rangle - \sum_{k=j}^{M-1} (M-k) |v_k\rangle \right) \quad (5.17)$$

and where we have denoted $|v_k\rangle \equiv (|P_k\rangle \otimes |P_{-k}\rangle - |P_{k+1}\rangle \otimes |P_{-(k+1)}\rangle)$.

Therefore

$$\begin{aligned} \mathbb{C}[P_j] \otimes |P_{-j}\rangle &= \frac{1}{M} \left(-\frac{M+K+1}{M(K+1)} - \frac{1}{2} (\lambda_1(\lambda_1+2\delta) + \lambda_{M-1}(\lambda_{M-1}+2\delta)) \right) |S\rangle \\ &\quad + \left(-\frac{M+K+1}{M(K+1)} + \frac{1}{2} (\psi(\psi+2\delta) - \lambda_1(\lambda_1+2\delta) - \lambda_{M-1}(\lambda_{M-1}+2\delta)) \right) |X_j\rangle \\ &= -\frac{1}{K+1} |P_j\rangle \otimes |P_{-j}\rangle - |S\rangle. \end{aligned} \quad (5.18)$$

Consequently, from (5.15) and (5.18),

$$\mathbb{C}[P_i] \otimes |P_{-j}\rangle = -\frac{1}{K+1} |P_i\rangle \otimes |P_{-j}\rangle - \delta_{ij} \sum_{k=1}^M |P_k\rangle \otimes |P_{-k}\rangle. \quad (5.19)$$

In addition one has

$$\mathbb{C}[P_{\pm i}] \otimes |\alpha_a \cdot H^{(0)}\rangle = \mathbb{C}[\alpha_a \cdot H^{(0)}] \otimes |\alpha_b \cdot H^{(0)}\rangle = 0 \quad (5.20)$$

with $a, b = M+1, M+2, \dots, M+K$, $i = 1, 2, \dots, M$.

From (3.42), (5.1) $\text{Tr}(E_a E_{-a}) = 1$, and $\text{Tr}(\alpha_a \cdot H^{(0)} \alpha_b \cdot H^{(0)}) = \alpha_a \cdot \alpha_b$, one gets that

$$q_i = \varphi(P_{-i}), \quad r_i = \varphi(P_i), \quad U_a = K_{ab}^{-1} \varphi(\alpha_b \cdot H^{(0)}), \quad (5.21)$$

where K_{ab}^{-1} is the inverse of $K_{ab} = \alpha_a \cdot \alpha_b$, $a, b = M+1, M+2, \dots, M+K$.

Therefore from (5.5), (5.7), (5.13), (5.14), (5.19), and (5.20) one gets the Dirac bracket for $sl(M+K+1)$, which reproduces (after an appropriate Miura transformation) the second bracket of $\text{cKP}_{(K+1, M)}$ hierarchy:

$$\begin{aligned} \{r_i(x), q_j(y)\} &= \left(\partial_x - U_{M+1}(x) - \sum_{s=1}^M r_s(x) \partial_x^{-1} q_s(x) \right) \delta_{ij} \delta(x-y) \\ &\quad - \frac{1}{K+1} r_i(x) \partial_x^{-1} q_j(x) \delta(x-y), \end{aligned} \quad (5.22)$$

$$\{r_i(x), r_j(y)\} = \frac{1}{K+1} r_i(x) \partial_x^{-1} r_j(x) \delta(x-y) + r_j(x) \partial_x^{-1} r_i(x) \delta(x-y), \quad (5.23)$$

$$\{q_i(x), q_j(y)\} = \frac{1}{K+1} q_i(x) \partial_x^{-1} q_j(x) \delta(x-y) + q_j(x) \partial_x^{-1} q_i(x) \delta(x-y), \quad (5.24)$$

$$\{q_i(x), U_b(y)\} = -\frac{1}{2} q_i(x) \delta_{bM+1} \delta(x-y); \{r_i(x), U_b(y)\} = \frac{1}{2} r_i(x) \delta_{bM+1} \delta(x-y), \quad (5.25)$$

$$\{U_a(x), U_b(y)\} = \frac{1}{4} K_{ab} \partial_x \delta(x-y). \quad (5.26)$$

Note that for $K=0$ (and $U_a=0$) we recover from the above bracket structure the second bracket of the NLS- $sl(M+1)$ model.^{14,15} This can also be checked directly by applying the same technique as above to the model described in subsection III A 1. Calculation shows that the equation (5.11) in this case is replaced by

$$C = \tilde{C} + \frac{M+1}{M} \lambda_M \cdot H^{(0)} \otimes \lambda_M \cdot H^{(0)} \quad (5.27)$$

and Eqs. (5.13), (5.14) and (5.19) hold with $K=0$.

For the purpose of illustration let us consider the example of $sl(3)$ with $M=K=1$ corresponding to $\text{cKP}_{2,1}$ with the Lax operator as in (2.5):

$$L_1 = (D - v_2)(D - v_1 - rD^{-1}q) = D^2 + u + \Phi D^{-1}\Psi, \quad (5.28)$$

where $u = -v^2 - v' - rq$, $\Phi = -r' - rv$, $\Psi = q$ with $v = v_2 = -v_1$. The second bracket for u, Φ, Ψ obtained from (5.22)–(5.26) indeed reproduces the standard $\text{cKP}_{2,1}$ second bracket (see e.g., Ref. 2). One checks easily that the following equations:

$$\begin{aligned} \partial_{t_2} q &= -q'' + v'q + rq^2 + v^2q, \\ \partial_{t_2} r &= r'' + v'r + r^2q + v^2r, \\ \partial_{t_2} v &= (rq)', \end{aligned} \quad (5.29)$$

following from the Dirac bracket (5.22)–(5.26) and the Hamiltonian $H_1 = \int \Phi \Psi$ reproduce the correct flows for u, Φ, Ψ :

$$\frac{\partial \Phi}{\partial t_2} = \frac{\partial^2 \Phi}{\partial x^2} + 2u_0 \Phi, \quad \frac{\partial \Psi}{\partial t_2} = -\frac{\partial^2 \Psi}{\partial x^2} - 2u_0 \Psi, \quad \frac{\partial u_0}{\partial t_2} = \partial_x(\Phi \Psi). \quad (5.30)$$

Equation (5.30) agrees with the second flow equation of the so-called Yajima–Oikawa hierarchy.^{1,2} Note that in this calculation of the t_2 flow the consistency of the ZS problem required use of $b^{(2)} = \lambda^2(\lambda_1 \cdot H)$ in (4.6) according to the discussion of Section III. Generally for $sl(3)$ we need to take $b^{(2k)} = \lambda^{2k}(\lambda_1 \cdot H)$ and $b^{(2k+1)} = \lambda^{2k+1}E$.

VI. DISCUSSION AND OUTLOOK

In the formalism based on the pseudo-differential Lax operator, the cKP hierarchy is obtained by constraining the complete KP hierarchy with the symmetry constraints expressed in terms of the eigenfunctions Φ_i, Ψ_i from (1.1) and imposed on the isospectral flows.

In this paper we have obtained an alternative derivation of the cKP hierarchy as an integrable $\widehat{sl}(M+K+1)$ -matrix hierarchy generalizing the Drinfeld–Sokolov hierarchy. The main ingredients of this construction were the semisimple graded, nonregular element E of $sl(M+K+1)$ and the potential A belonging to the grade zero subalgebra $\hat{\mathcal{S}}_0$. Both the Lax matrix operator as well as the underlying recurrence operator were constructed in terms of these basic elements. The matrix hierarchy exhibited a gauge symmetry related to $\text{Ker}(\text{ad } E)$. Due to presence of this gauge

symmetry the relevant phase space turned out to be the quotient space $\hat{\mathcal{G}}_0/(\text{Ker}(\text{ad } E) \cap \hat{\mathcal{G}}_0)$. The structure of the flows of the hierarchy was shown to be related to the center of $\text{Ker}(\text{ad } E)$.⁹ The algebraic approach allowed us to write down in closed form a very simple expression for the second Hamiltonian structure with respect to which the flows are Hamiltonian. This bracket structure was explicitly calculated as a Dirac bracket emerging from reduction of $\hat{\mathcal{G}}_0$ to the effective phase space of $\hat{\mathcal{G}}_0/(\text{Ker}(\text{ad } E) \cap \hat{\mathcal{G}}_0)$. Alternative approaches to calculate this bracket structure in various parametrizations of the constrained KP hierarchy have been proposed in the past and include e.g., applications of the r -matrix formulation developed in a discrete context^{23,24} and applications of the generalized Kupershmidt–Wilson theorem.^{3,25}

One expects that several aspects of the cKP formalism will gain substantially by being treated within the algebraic formalism proposed in this paper. Work is in fact in progress regarding the following issues. Possible extensions of the cKP scalar Lax examples by going beyond the algebraic construction based on the $sl(M)$ algebra by employing different algebras. Calculation of the tau-function, following the dressing transformation method²⁶ and Darboux–Bäcklund methods⁷ will help to further establish the connection between the pseudo-differential and matrix techniques. One also expects that the use of the matrix hierarchy will be essential for describing additional symmetries of the cKP models.

Note added: After this paper had been completed we received a copy of the paper *Extensions of the matrix Gelfand–Dickey hierarchy from generalized Drinfeld–Sokolov reduction*, by L. Fehér and I. Marshall (SWAT-95-61, hep-th/9503217), which contains related results. We thank Dr. L. Fehér for sending us a hard copy of his paper.

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